Continuous Matrix Rings and Self-Injective Rings

Rajashekhar Upase and Asha Saraswathi B.

Rajashekhar, **Upase** Faculty Member's, Upase Education Institute, Jayanagar, Dharwad-580001, Karnataka.

Asha Saraswathi B. Professor and H.O.D, Department of Mathematics, Srinivas University, Srinivas Nagar, Mukka. Surathkal. Mangalore – 574146. India

ABSTRACT: This paper investigating the relationship between the proof of Utumi's Theorem that a ring R (with identity) is self-injective if and only if the matrix ring $M_n(R)$ (for a fixed $n \in \mathbb{N}$) is self-injective. Throughout this proof, for convenience, we denote the matrix ring $M_n(R)$ by S and the matrix units in $M_n(R)$ by e_{ij} instead of customary E_{ij}.

Keywords: Ring R, Matrix ring, continuous ring, .R- module, R_n - modules and isomorphism.

1. INTRODUCTION: In all modules are unitary ideal of R respectively. A $R-$ module M is said to be principally injective or simply $P-$ injective. If for any principal ideal P of R and any $R-$ module homomorphism $f: P \to M$ can be extended to R . A ring R is said to be $P-$ injective, if R is $P-$ injective as a $R-$ module. A self-injective ring is clearly $P-$ injective and von Neumann regular ring is also left $P-$ injective. However, in general the converse does not holds in either case. The connection between von Neumann regular ring and Utumi's Theorem, self-injective rings and $P-$ injective rings are studied a several papers, for example Utumi [13], than Chan Young Hang, Jin Young Kim and Nam Kyun Kim have proved in self –injective rings which are either semi prime PI or semi prime rings of all essential ideals are two-sided are von Neumann regular [6], However Hirano [4] showed that there exists a semi prime PI left injective rings but not von Neumann regular. It is well known that reduces or semi prime left duo injective rings are von Neumann regular [10, 14].

2. MOTIVATION: From these facts, we may ask the following question. In particular question was also raised by Utumi's Theorem that a ring R (with identity) is self-injective if and only if the matrix ring $M_n(R)$ (for a fixed $n \in \mathbb{N}$) is self-injective.. Yue Chi Ming [15, 16]. Is a semi prime injective ring, all of whose essential ideals are two sided von Neumann regular and is a factor ring module the Jacobson radical of a injective ring over von Neumann regular.

For most ring R, R_R is simply not injective but do exist rings for which R_R is injective, we say that such rings are right self-injective. For example, the ring Z is clearly not self-injective in fact $f: 2Z \to Z$ defined by $f(2n) = n$ for every $n \in Z$,clearly we say that it can't be extended to a homomorphism $f^{'}: Z \to Z$. For example, let R be the ring of $n \times n$ upper triangular matrices over a ring $k \neq 0$ where $n \geq 2$ then R is not self –injective. To simplify the notation's , we work in the case $n = 2$. Now we consider the ideal $A = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}$ and define map $f: A \to R$ by $f\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$, this is easily check to be a right $R-$ homomorphism, if f can extended to R there would exist a matrix $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$ in R such that $f\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & xa \\ 0 & 0 \end{pmatrix}$ for all a in k , which is clearly impossible. This shows that R_R is not injective.

3. LATTICE ISOMORPHISM OF A SIMPLE MATRIX RINGS

3.1. Definition: i) An $R-$ module M is said to be **injective** if every $R-$ homomorphism from a sub module A of an $R-$ module B into M can be extended to B.

ii) An $R-$ module M is said to be **quasi-injective** if every $R-$ homomorphism from a sub module A of M into M can be extended to M

iii) An $R-$ module M is said to be **continuous** if the following conditions are satisfied

(C1): Every sub module of M is essential in a direct summand of M.

(C2): If A is a sub module of M isomorphic to a direct summand of M, A itself is a direct summand of M.

iv) A ring R is said to be **right self-(quasi)injective (resp. continuous)**, if R, as a right $R-$ module, injective (resp. continuous). We state the Baer's criterion of injectivity whose is available in all standard texts in module theory.

Baer's criterion of infectivity: An $R-$ module M is injective if and only if every $R-$ homomorphism from a right ideal of R into M can be extended to M. Equivalently, every $R-$ homomorphism $f: I \to M$ from a right ideal I of R into M is given by multiplication by an element of M. That is, there exists an element m in M such that such that $f(x) = mx$ for each $x \in M$.

We note, by the Baer's criterion of injectivity, that a ring R is right self-injective if and only if R is right quasi-injective.

Proposition: 3.2. Every quasi-injective module is continuous.

Proof: For the proof we refer to Proposition 2.1 of [19].

Corollary: 3.3. Every right self-injective ring is right continuous.

We now derive a sufficient condition for a right continuous ring to be right self-injective. For this, we prove some preliminary results.

Lemma: 3.4. Let M be a continuous module and let N_1, N_2 be direct summands of M such that $N_1 \cap N_2 = 0$, then $N_1 \oplus N_2$ is also a direct summand of M.

Proof: Since, by hypothesis, N_1 is a direct summand of M, $M = N_1 \oplus K_1$ for some sub module K_1 of M. Let $\pi: M \to K_1$ be the canonical projection. **Claim:** $N_1 \oplus N_2 = N_1 \oplus \pi(N_2)$. Clearly $N_1 \subseteq N_1 \oplus \pi(N_2)$. Let $x \in N_2 \subseteq M = N_1 \oplus K_1$. Then $x = y + z$ for some $y \in N_1, z \in K_1$, Clearly $z = \pi(x) \in \pi(N_2)$, and hence, $x \in N_1 \oplus \pi(N_2)$. So $N_1 \oplus N_2 \subseteq N_1 \oplus \pi(N_2)$. For the reverse inclusion, we need only prove that $\pi(N_2) \subseteq N_1 \oplus N_2$. Let $x \in \pi(N_2)$ then $x = \pi(y)$ for some $y \in N_2$. So $y = n + x$ for some $n \in N_1$. So, $x = -n + y \in N_1 \oplus N_2$. This proves our claim. Since ($Ker(\pi) = N_1$ and by hypothesis, $N_1 \cap N_2 = 0$, it follows that $\pi | N_2$ is a monomorphism . So, by condition (C2) of continuous module, it follows that $\pi(N_2)$ is a direct summand of M. Since $\pi(N_2) \subseteq K_1$, it follows that $\pi(N_2)$ is a direct summand of K_1 and hence, $N_1 \oplus N_2 = N_1 \oplus \pi(N_2)$ is a direct summand of $N_1 \oplus K_1 = M$. This completes the proof of the lemma.

Lemma: 3 .5. Let R be right continuous ring.

i) Let e, f be idempotent's in R such that $eR \cap (1-f)R = 0$. If eR is an essential extension of a right ideal A in R. Then feR is generated by an idempotent and is an essential extension of fA.

ii) If e, f idempotent's in R such that $eR \cap fR = 0$, then $eR + fR = hR$ for some idempotent h in R.

iii) If A is right ideal in R contained in eR (for some idempotent e in R), then there exist an idempotent $f \in eR$, such that A is essential in fR .

Proof: i) Since, by hypothesis, $eR \bigcap (1-f)R = 0$, the left multiplication by f gives an isomorphism of eR onto feR . Hence feR is an essential extension of fA . By the condition (C2), feR is generated by an idempotent.

ii) Now $eR \cap fR = 0 \Rightarrow (1-f)R$ contains an isomorphic copy of $eR.$

(Consider the map $\phi : eR \to (1-f)R$ defined by $\phi(x) = (1-f)x$ for each $x \in eR$. $x \in Ker(\phi)$ implies $(1-f)x = 0$, implies $x = fx \in eR \cap fR = 0 \Rightarrow x = 0$). Since, $B = \phi(eR) \simeq eR$, $B = hR$ for some idempotent h in R (by condition (C2)).

Now, $fR + B = fR + hR$ and $h \in B \subseteq (1-f)R \Rightarrow fh = 0$, where e, f are idempotent's, $ef = 0$. $eR + fR = (e + f - fe)R$.

Because, $(e+f-fe)e = e$, $(e+f-fe)f = f$, $(e+f-fe)fe = fe$, $(e+f-fe)(e+f-fe) = e+f-ef$)

$a \in eR \Leftrightarrow ea = a$. So, $f + h - hf$ is an idempotent and $fR + hR = (f + h - hf)R$ implies $fR \oplus B = (f + h - hf)R$ (because $fR \cap hR = 0$). Now, $eR \simeq B \Rightarrow eR \oplus fR \simeq B \oplus fR = kR$, where $k = f + h - hf$ implies $eR \oplus f R$ is generated by an idempotent.

iii) By condition (C1), there exist an idempotent g in R such that A is essential gR . Since, by hypothesis, $A \subseteq eR$, $A \cap (1-e)R = 0$, and hence, $gR \cap (1-e)R = 0$ .Now, by (i) above, egR is generated by an idempotent, say, h and hR is an essential extension of $eA = A$.

Lemma: 3.6. Let R be a right continuous ring, let A be right ideal of R and let eR be an essential extension of A where $(e = e^2 \in R)$ . .Let $v : A \to R$ be an $R-$ homomorphism. Suppose, f is an idempotent in R such that $eR \cap f R = 0$ and $v(A) \subseteq fR$ then v extended to a homomorphism $w : eR \to fR.$

Proof: At the outset, we note that a right ideal I of R is a direct summand of r if and only if $I = eR$, for some idempotent e in R . By hypothesis, $eR \cap f R = 0$ and hence, by Lemma **3.4,** then $eR \oplus f R$ is generated by an idempotent say h in R . We may assume that e, f are orthogonal. (For, $R = hR \oplus (1-h)R = eR \oplus fR \oplus (1-h)R$. Then $1 = e_1 + e_2 + e_3$ for $e_1 \in eR, e_2 \in fR, e_3 \in (1-h)R$. Clearly, e_1, e_2, e_3 are mutually orthogonal idempotent's and $eR = e_1R, fR = e_2R, (1-h)R = e_3R$). Now define, $G = \{x \in R : x = a + v(a), \text{for some}\, a \in A\}$. Clearly G is a right ideal of R . **Claim:** $G \subseteq (e+f)R$. Since e, f are mutually orthogonal idempotent's, $(e+f)$ is an idempotent. It is enough to prove that $(e+f)g = g$ for all $g \in G$. Let $g \in G$ then $g = a + v(a)$ for some $a \in A$ then $(e+f)g = (e+f)(a+v(a))$ $= ea + fa + ev(a) + f\,v(a) = a + o + ef\,v(a) + f\,v(a) = a + o + o + fv(a) = a + f\,v(a) = g$.

Hence $G \subseteq (e+f)R$. Then, by Lemma 3.5, there is an idempotent h in $(e+f)R$ such that $G \subseteq' hR \subseteq (e+f)R$.**Claim:** $G \cap (1-e)R = 0$. Let $x \in G \cap (1-e)R \Rightarrow x = (1-e)x$ then $ex = 0$. Let $x = a + v(a)$ for some $a \in A$

$$\Rightarrow ea + ev(a) = 0$$

$$\Rightarrow ea = 0, \qquad (\because ev(a) = efv(a) = o)$$

$$\Rightarrow a = o$$

$$\Rightarrow x = 0$$

Since, $G \subseteq' hR$, it follows that $hR \cap (1-e)R = 0$. (by an earlier lemma $A \subseteq' eR$, $(1-f)R \cap eR = 0$ $\Rightarrow feR = hR$ $\Rightarrow feR\ '\supseteq f\ A\ ) \Rightarrow ehR = gR$ for idempotent g and $ehR\ '\supseteq eG = A$.

Claim: $ehR = eR$ .Now, $ehR \subseteq eR$ and $ehR = gR$, direct summand of R implies ehR is a direct summand of eR. Also, $A \subseteq ehR \subseteq eR$ and $A \subseteq' eR \Rightarrow ehR \subseteq' eR \Rightarrow ehR = eR$. Let, $e = ehx$ for some $x \in R$ and $t = fhx \in fR$. Consider the map, $w: eR \to f\,R$ defined by $w(y) = ty$ for all $y \in eR$. Now, $y \in fR$ implies $ty \in fR$ for all $y \in eR$. So, w is a well- defined map and is clearly $R-$ isomorphism.

Claim: $v = w \mid A$ .Let $a \in$ A then $a + ta = ea + ta = ehxa + fhxa = (e+f)hxa = hxa \in hR$ and $a + v(a) \in G \subseteq hR$ implies $v(a) - ta = v(a) + a - (a + ta) \in hR \cap f\,R \subseteq hR \cap (1-e)R = 0$ because $(1-e)f = f$, $f \in (1-e)R$ implies $v(a) - ta = 0$ .Hence, $v(a) = ta$.

Consider the following condition for a sub module N of an $R-$ module M .

$(*)$ Every homomorphism from a sub module K of N into M can be extended to N .

Lemma: 3.7. If N_1, N_2 are sub modules of an $R-$ module M , satisfying the condition $(*)$ above, and if $N_1 \cap N_2 = 0$ then $N_1 + N_2$ also satisfies $(*)$.

Proof: Let $f : K \to M$ be a homomorphism from a sub module K of $N_1 + N_2$ into M . Let $g = f \mid (K \cap N_1)$. By hypothesis, there exists a homomorphism $h: N_1 \to M$ extending 'g'. Define a map $\alpha : (K + N_1) \cap N_2 \to M$ as follows. Let $x \in (K + N_1) \cap N_2$ then $x = k + y \in N_2$ for some $k \in K, y \in N_1$. Define $\alpha(x) = f(k) + h(n_1)$. **Claim:** α is well-defined map. Suppose, $x = k + y = k' + y'$ for some $k, k' \in K, y, y' \in N_1$ then $k - k' = y' - y \in K \cap N_1$ implies $h(k-k') = g(k-k') = f(k-k') = f(k) - f(k')$ .Also, $h(y'-y) = h(y') - h(y)$ implies $f(k) - f(k') = h(y') - h(y)$ therefore

$f(k) + h(y) = f(k') + h(y')$ .Thus α is a well-defined map. Clearly, it is an $R-$ homomorphism. Then, by hypothesis, there exists $\beta : N_2 \to M$, extending α . Let $p: N_1 \oplus N_2 \to N_1$, $q: N_1 \oplus N_2 \to N_2$ be projection maps and let $\aleph = h \circ p + \beta \circ q : N_1 + N_2 \to M$. **Claim:** $\aleph \mid K = f$. Let $x \in K \subseteq N_1 + N_2$ then $x = p(x) + q(x)$ .So, $q(x) = x - p(x)$ in $(K + N_1) \cap N_2$ So, $\beta(q(x)) = \beta(x - p(x)) = \alpha(x - p(x))$ $= f(x) - h(p(x))$, by definition of α . So, $f(x) = h(p(x)) + \beta(q(x)) = (h \circ p)(x) + (\beta \circ q)(x)$ $= ((h \circ p) + (\beta \circ q))(x) = \aleph(x)$. So, γ extends f , and hence, $N_1 + N_2$ satisfies (*).

We now derive sufficient conditions for a right continuous ring to be right self-injective.

Theorem: 3.8. Let R be a right continuous ring and let $1 = e_1 + e_2 + \ldots + e_n$ (for some $n > 1$) for some mutually orthogonal idempotent $e_1 e_2 \ldots\ldots e_n$ in R. Suppose further that for each $i = 1,..,n$, $(1-e_i)R$ contains an isomorphic copy of $e_i R$ then R is a right self- injective ring.

Proof: By hypothesis, $R = e_1 R \oplus e_2 R \oplus \ldots\ldots \oplus e_n R$ .So, by Lemma 3.7 and induction, we need only prove the following, If $1 \le i \le n$ and if $\alpha : A \to R$ is an $R-$ homomorphism from a right ideal A of R contained in $e_i R$ into R, then α extends to an $R-$ homomorphism $\beta : e_i R \to R$. Let $1 \le i \le n$ and let $\alpha : A \to R$ be an $R-$ homomorphism from a right ideal A of R contained in $e_i R$ into R. Since, by hypothesis, R is right continuous, then by Lemma 3.5(iii), there exists an idempotent e in R such that $A \subseteq' eR \subseteq e_i R$ .Clearly, $e_i e = e$ .Let $e' = ee_i$. We note the following,

i) $(e')^2 = e'e' = ee_i ee_i = eee_i = ee_i = e'$

ii) $e_i e' = e_i ee_i = ee_i = e'$

iii) $e'e_i = ee_i e_i = ee_i = e'$

iv) $A \subseteq e'R \subseteq eR$, (for, let, $a \in A \subseteq eR \Rightarrow a = ea = ee_i a = e'a \in e'R$ $(\because A \subseteq e_i R)$ .so, $A \subseteq' e'R$).

Because (i) to (iv) we may assume that $e' = e$ and hence $e = ee_i$ .By hypothesis $(1-e_i)R$ contains an isomorphic copy of $e_i R$. Clearly $(1-e)R \supseteq (1-e_i)R$ and $e_i R \supseteq eR$ hence $(1-e_i)R$ contains an isomorphic copy of eR .

Let $\phi : eR \to (1-e)R$ be a monomorphism from eR into $(1-e)R$. So, ϕ induces an isomorphism $\theta = \phi^* : eR \to B$, from eR onto a right ideal B of R. So, by condition (C2) of right continuous ring, $B = fR$, for some idempotent f in R. So, $\theta : eR \to fR$ is an $R-$ homomorphism. Then, there exist $p, q \in R$ such that $p = fp$ and $qp = e$ because($e = e^2 \in R$ and $a \in R$, $ae = ea = a$, where e is idempotent in R implies $pf = fp = p$, where f is idempotent in R).

Let $p = \theta(e)$, $q = \theta^{-1}(f)$ implies $f = \theta(q)$, because θ is isomorphic. Consider the map, $\theta : eR \to fR$.

Let $p = \theta(e) \Rightarrow fp = \theta(e)$, $(\because p = fp)$

$\Rightarrow \theta(q)p = \theta(e)$

$\Rightarrow \theta(qp) = \theta(e)$, $\because \theta$ is module isomorphism.

$\Rightarrow qp = e$ $\quad \because \theta$ is one-one

Define a map $\beta : A \to fR$ by $\beta(a) = p\alpha(a)$ for all $a \in A$. $eR \cap fR \subseteq eR \cap (1-e)R = 0$ because $A \subseteq' eR$ implies their exist $\beta^* : eR \to fR$ extending β . Next define $\aleph(a) = (1-e)\alpha(a)$ for all $a \in A$ implies their exist $\gamma^* : eR \to (1-e)R$ extending to γ . Define $\alpha^* : e_i R \to R$ by $\alpha^*(x) = (q\beta^*(e) + (1-e)\gamma^*(e))(x)$ for all $x \in e_i R$. Clearly α^* is R- homomorphism.

Claim: $\alpha^* | A = \alpha$ .Let $a \in A$ then $\alpha^*(a) = \left(q\beta^*(e) + (1-e)\aleph^*(e)\right)a$

$$= q\beta^*(e)a + (1-e)\aleph^*(e)a$$

$$= q\beta^*(ea) + (1-e)\aleph^*(ea)$$

$$= q\beta^*(ea) + (1-e)\aleph^*(ea)$$

$$= q\beta^*(a) + (1-e)\aleph^*(a)$$

$$= qp\alpha(a) + (1-e)(\alpha(a))$$

$$= (qp)\alpha(a) + (1-e)\alpha(a)$$

$$= e\alpha(a) + (1-e)\alpha(a)$$

$$= e\alpha(a) + \alpha(a) - e\alpha(a)$$

$=\alpha(a)$. Hence, the proof of right self -continuity of R.

A ring R is said to be **of order** n **(for some** $n > 1$**)** if the identity element 1 of R is the sum of mutually orthogonal idempotent's $e_1, \ldots, e_n$ such that $e_i R \cong e_j R$ (as $R-$ modules) for each i, j = 1,..,n.

Corollary: 3.9. A ring R of order $n > 1$ is right continuous if and only if it is right self-injective.

Proof: Let R is a self-injective. Claim; R_n is right continuous. Since $e_{11}R_n e_{11} \approx R$, $e_{11}R_n e_{11}$ is a right self-injective. Claim $\left(R_n e_{11}\right)_{e_{11}R_n e_{11}}$ is an injective. Claim $R_n e_{11} = e_{11}R_n e_{11} \oplus (1-e)R_n e_{11}$ is a right $e_{11}R_n e_{11}$ -modules.so it clearly say that $R_n e_{11} = e_{11}R_n e_{11} + (1-e_{11})R_n e_{11}$. Now let $x \in e_{11}R_n e_{11} \cap (1-e_{11})R_n e_{11}$. So $x = e_{11}x = (1-e_{11})x$ implies $x = 0$. So $e_{11}R_n e_{11} \cap (1-e_{11})R_n e_{11} = 0$. Claim $(1-e_{11})R_n e_{11} \approx e_{11}R_n e_{11}$ as right $e_{11}R_n e_{11} -$ modules. Let $e_{11}R_n e_{11} = \{ae_{11} / a \in R\}$.

Let $A = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}e_{ij}$ in R then $(1-e_{11})Ae_{11} = \left(\sum_{i=2}^{n} e_{ii}\right)\left(\sum_{r=1}^{n}\sum_{s=1}^{n} a_{rs}e_{rs}\right)e_{11} = \sum_{i=2}^{n} e_{ii}\left(\sum_{r=1}^{n} a_{r1}e_{r1}\right) = \sum_{i=2}^{n}\sum_{r=1}^{n} a_{r1}\left(e_{ii}e_{r1}\right) = \sum_{r=2}^{n} a_{r1}e_{r1}$.

(because $R_3 e_{11} = e_{11}R_3 e_{11} \oplus e_{22}R_3 e_{11} \oplus e_{33}R_3 e_{11} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a & b & a \\ d & e & f \\ g & h & k \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ d & e & f \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ d & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Map $\phi : ae_{11} \to ae_{21}$ defined by $\phi(ae_{11}) = ae_{21}$. So $\phi((ae_{11})(be_{11})) = \phi((ab)e_{11}) = (ab)e_{21}$ and $(\phi(ae_{11}))be_{11} = (ae_{21})be_{11} = (ab)e_{21}e_{11} = (ab)e_{21}$). So $R_n e_{11} = \sum_{i=1}^{n} \oplus e_{ii}R_n e_{11}$ as $e_{11}R_n e_{11} -$ module and $e_{ii}R_n e_{11} \approx e_{11}R_n e_{11}$ for all $i = 1, 2, 3, \ldots, n$ as $e_{11}R_n e_{11} -$ module implies $\left(R_n e_{11}\right)_{e_{11}R_n e_{11}}$ is an injective.

Claim: R_n is right continuous

Case(i) : Let A be any right ideal of R_n then $\psi(A) = Ae_{11}$ is an $e_{11}R_n e_{11}$ sub-module of $R_n e_{11}$. Since $\left(R_n e_{11}\right)_{e_{11}R_n e_{11}}$ is injective then there exist a direct summand say G of $R_n e_{11}$ such that Ae_{11} is essentially in G. let $\pi : R_n e_{11} \to G$ is the projection map, clearly $\pi \in End_{e_{11}R_n e_{11}}\left(R_n e_{11}\right)$ is idempotent. Since $End_{e_{11}R_n e_{11}}\left(R_n e_{11}\right) \approx R_n$, there exist an idempotent say e in R_n such that $G = eR_n e_{11}$.Now $Ae_{11} \subseteq' G \Rightarrow \phi(Ae_{11}) \subseteq' \phi(G)$ but $\phi(Ae_{11}) = \phi(\psi(A)) = A$ and $\phi(G) = \phi(eR_n e_{11}) = \phi(\psi(eR_n)) = \phi\psi(eR_n) = eR_n$ implies $A \subseteq' eR_n$.

Case(ii) : Let $e = e^2$ in R_n and let A be any right idempotent of R_n is isomorphism to eR_n . Claim $A = fR_n$, $f = f^2$. Let $\phi : eR_n \to A$ be an R_n is isomorphism. Let $\phi(e) = x$ in A then $x = \phi(e) = \phi(ee) = \phi(e)e = xe$ and $A = \phi(eR_n) = \phi(e)R_n = xeR_n$ implies $Ae_{11} = xeR_n e_{11}$ then the map $\theta : eR_n e_{11} \to Ae_{11}$ defined by $\theta(y) = xy$ for all y in $eR_n e_{11}$ is an $e_{11}R_n$ is isomorphism now we have seen above that $R_n e_{11}$ is injective right $e_{11}R_n e_{11} -$ module. Hence Ae_{11} is also injective as right $e_{11}R_n e_{11} -$ module.. So Ae_{11} is direct sub-module of $R_n e_{11}$ as $e_{11}R_n e_{11} -$ module. Then $\Phi(Ae_{11})$ is summand of $\Phi(R_n e_{11})$. But $\Phi(Ae_{11}) = \Phi\Psi(A) = A$ and $\Phi(R_n e_{11}) = R_n e_{11}R_n$. Clearly $eR_n e_{11}$ is summand of the injective module $\left(R_n e_{11}\right)_{e_{11}R_n e_{11}}$. Hence $eR_n e_{11}$ is injective as right $e_{11}R_n e_{11} -$ module. So Ae_{11} is injective as right $e_{11}R_n e_{11} -$ module of $R_n e_{11}$. Let $P : R_n e_{11} \to Ae_{11}$ be the projection map, so

$P \in End_{e_{11}R_ne_{11}}\left(R_ne_{11}\right) \approx R_n$ and $P\left(R_ne_{11}\right) = Ae_{11}$. So there exist an idempotent f in R_n such that $Ae_{11} = fR_ne_{11}$ and $A = \Phi\Psi\left(A\right) = \Phi\left(\Psi\left(A\right)\right) = \Phi\left(Ae_{11}\right) = \Phi\left(fR_ne_{11}\right) = fR_n$. So R_n is right continuous.

Proposition: 3.11. If R is a ring with identity and $n > 1$, the matrix ring $M_n(R) = S$ is of order n.

Proof: Clearly the matrix units e_{ii}'s $M_n(R)$ are mutually orthogonal idempotent's and add upto the identity matrix. We need only prove that $e_{ii}S \cong e_{jj}S$ (as $S-$ modules) for each $i, j = 1,2,3,\cdots,n$. For this, we first note that, if $1 \le p \le n$, $A = ((a_{ij})) \in S$, $e_{pp}A = e_{pp}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}e_{ij} = \sum_{j=1}^{n} a_{pj}e_{pj}$. Thus, $e_{pp}S = \{e_{pp}A / A \in S\} = \{\sum_{j=}^{n} a_je_{pj} / a_j \in R, 1 \le j \le n\}$. Thus elements of $e_{pp}S$ has all rows, except p^{th}, are zero. So, there is a canonical $S-$ isomorphism from $e_{pp}S$ onto $e_{qq}S$ for any $p,q = 1,2,3,\cdots,n$. Thus S is a ring of order 'n'. So, we have the following result.

Proposition: 3.12. If R is a ring with identity and $n > 1$, the matrix ring $M_n(R) = S$ is right continuous if and only if it is right self-injective.

UTUMI'S THEOREM: A ring R (with identity) is self-injective if and only if the matrix ring $M_n(R) = S$ (for a fixed n ∈) is self-injective

Proof: We first prove the **"If"** part which is fairly simple. By hypothesis, S is right self-injective. Since $e_{11}Se_{11} \cong R$. We need only prove that $e_{11}Se_{11}$ is right self-injective. Let Ω be any right ideal of $e_{11}Se_{11}$ and let $f : \Omega \to e_{11}Se_{11}$ be an $e_{11}Se_{11}-$homomorphism. Define a map $f^* : \Omega S \to S$ as follows. Let $x \in \Omega S$ then $x = \sum_{i=1}^{k} a_is_i$ for $a_i \in \Omega, s_i \in S, k \in$ .We define $f^*(x) = \sum_{i=1}^{k} f(a_i)s_i$. Claim f^* **is a well-defined map:** Suppose, $x = \sum_{i=1}^{k} a_ir_i = \sum_{j=1}^{l} b_js_j$, for some $a_i, b_j \in \mathbf{\Omega}, r_i, s_j \in S, k, l \in$. **Claim:** $\sum_{i=1}^{k} f(a_i)r_i = \sum_{j=1}^{l} f(b_j)s_j$.

Enough to prove that: $\sum_{i=1}^{k} f(a_i)r_i - \sum_{j=1}^{l} f(b_j)s_j = 0$. Our problem is reduced to the following:

if $\sum_{i=1}^{k} a_ir_i = 0$, then $\sum_{i=1}^{k} f(a_i)r_i = 0$. So, let $\sum_{i=1}^{k} a_ir_i = 0$ and let $1 \le p \le n$ then $\left(\sum_{i=1}^{k} a_ir_i\right)e_{p1} = 0$.

$$\Rightarrow \sum_{i=1}^{k} a_ir_ie_{p1} = 0$$

$$\Rightarrow \sum_{i=1}^{k} a_ie_{11}r_ie_{p1}e_{11} = 0$$

$$\Rightarrow f\left(\sum_{i=1}^{k} a_ie_{11}\left(r_ie_{p1}\right)e_{11}\right) = 0$$

$$\Rightarrow \sum_{i=1}^{k} f\left(a_i\right)e_{11}\left(r_i\, e_{p1}\right)\, e_{11} = 0$$, because $f : \Omega \to e_{11}Se_{11}$ is in $e_{11}Se_{11}$- map

$$\Rightarrow \left(\sum_{i=1}^{k} f\left(a_i\right)r_i\right)e_{p1} = 0$$

$$\Rightarrow\left(\sum_{i=1}^{k} f(a_i)r_i\right)e_{p1}e_{1p}=0$$

$$\Rightarrow\left(\sum_{i=1}^{k} f(a_i)r_i\right)e_{pp}=0 \quad \text{, because this is true for each } p\in\{1,\ldots,n\}$$

$$\Rightarrow\sum_{i=1}^{k} f(a_i)r_i=0 \quad \text{,Because, } \sum_{p=1}^{n} e_{pp}=1.$$

Thus f^* is well defined map. Clearly f^* is $S-$ homomorphism. Since by hypothesis S is right self-injective, there exist $x_0\in S$ such that $f^*(x)=x_0x$ for all $x\in \Omega S$.

Now, let $y_o=e_{11}x_oe_{11}\in e_{11}Se_{11}$. **Claim:** $f(a)=y_0a$ for all $a\in\Omega$. Let $a\in\Omega$ then $f(a)=e_{11}f(a)=e_{11}f^*(a)$ $=e_{11}x_oe_{11}a=(e_{11}x_0e_{11})a=y_0a$. This proves that $e_{11}Se_{11}$ and hence, R is right self- injective.

Before proving the "Only if" part, we prove the following results.

Proposition: 3.13. If S is isomorphic to the endomorphism ring of the right $e_{11}Se_{11}-$-module Se_{11}.

Proof: Define a map $\phi:S\to \underset{e_{11}Se_{11}}{End}(Se_{11})$ as Let $A\in S$. Consider the map $\phi(A):Se_{11}\to Se_{11}$ defined by $\phi(A)(B)=AB,\forall B\in Se_{11}$. Note that $A\in S, B\in Se_{11}\Rightarrow AB\in Se_{11}$. Thus $\phi(A)$ is a well-defined map. Clearly $\phi(A)$ is a $e_{11}Se_{11}-$map and $\phi(A)\in \underset{e_{11}\Re_ne_{11}}{End}(\Re_ne_{11})$ .Thus $\phi:S\to \underset{e_{11}Se_{11}}{End}(Se_{11})$ is a well-defined map. ϕ is a ring homomorphism:

Clearly ϕ is additive. Now let $A,B\in S$ and let $C\in Se_{11}$. **Claim:** $\phi(AB)C=(\phi(A)\circ\phi(B))(C)$.

Now, $\phi(AB)(C)=\phi(AB)(C)=\phi(A)(BC)=\phi(A)(\phi(B)(C))=(\phi(A)\circ\phi(B))(C)$. Thus $\phi(AB)(C)=(\phi(A)\circ\phi(B))(C)$ for each $C\in Se_{11}$. Hence, $\phi(AB)=\phi(A)\circ\phi(B)$. ϕ **is one-one : it is e**nough to prove $\ker\phi=\{0\}$. Let $A\in\ker\varphi$ implies $\phi(A)=0\Rightarrow\phi(A)(B)=0$ for all $B\in Se_{11}\Rightarrow AB=0$ for all $B\in Se_{11}$. **Claim:** $A=0$ it is enough to prove that: $a_{ij}=0$ for all $i,j=1,2,\ldots,n$. Let $1\le r,\ s\le n$. Since, $e_{s1}\in Se_{11}$, $Ae_{s1}=0$ implies $(Ae_{s1})_{r1}=0$. But $(Ae_{s1})_{r1}=\sum_{p=1}^{n}a_{rp}(e_{s1})_{p1}=a_{rs}$. Hence $a_{rs}=0$ for all $r,s=1,2,\ldots,n$, thus proving that $A=0$. So, $\ker\phi=\{0\}$ and ϕ is one-one. Claim ϕ **is onto:** Let $\alpha\in\underset{e_{11}Se_{11}}{End}(Se_{11})$. It is easy to check that Se_{11} is generated by $e_{11},e_{21},e_{31},\ldots,e_{n1}$ over $e_{11}Se_{11}$ (as a right module). For $1\le p\le n$, $\alpha(e_{p1})=K^{(p)}\in Se_{11}$. Let $K^{(p)}=\left(k_{ij}^{(p)}\right)$. So $k_{ij}^{(p)}=0$ for $j>1$. Define, $K=\left(k_{ij}\right)$ in S, by $k_{ij}=k_{(i1)}^{(j)}$ for all $i,j=1,2,\ldots,n$. So $K=\begin{pmatrix}k_{11}^{(1)} & k_{11}^{(2)}\cdots k_{11}^{(n)}\\ k_{21}^{(1)} & k_{21}^{(2)}\cdots k_{21}^{(n)}\\ k_{r1}^{(1)} & k_{r1}^{(2)}\cdots k_{r1}^{(2)}\\ k_{n1}^{(1)} & k_{n1}^{(2)}\ldots k_{n1}^{(2)}\end{pmatrix}$

Claim: $\alpha(A)=KA$ for all $A\in Se_{11}$. Let $A=(a_{ij})$ in Se_{11} then $a_{ij}=0$ for all $j>1$. So, $A=\sum_{i=1}^{n}a_{i1}e_{i1}=\sum_{i=1}^{n}a_{i1}e_{i1}e_{11}$ $=\sum_{i=1}^{n}e_{i1}(a_{i1}e_{11})$, where $e_{i1}\in Se_{11},a_{i1}e_{11}\in e_{11}Se_{11}$.So, $\alpha(A)=\sum_{p=1}^{n}\alpha(e_{p1})(a_{p1}e_{11})=\sum_{p=1}^{n}K^{(p)}(a_{p1}e_{11})$. Since $\alpha(A)\in Se_{11}$

$,\alpha(A)_{ij}=0$ for $j>1$. Now for $1\le r\le n$ $,\alpha(A)_{r1}=\left(\sum_{p=1}^{n}K^{(p)}a_{p1}e_{11}\right)_{r1}=\sum_{p=1}^{n}\sum_{q=1}^{n}K_{rq}^{(p)}(a_{p1}e_{11})_{q1}=\sum_{p=1}^{n}K_{r1}^{(p)}a_{p1}$

implies $\sum_{p=1}^{n}k_{rp}a_{p1}=(KA)_{r1}$

Also, $(KA)_{ij}=0$ for all $j>1$ because $KA\in R_n e_{11}$. Thus $(\alpha A)_{r1}=(KA)_{r1}$ for all $r=1,2,\ldots,n$ implies $(\alpha A)=KA$. Hence, $\alpha=\phi(K)$. Hence $\phi:S\to \underset{e_{11}Se_{11}}{End}(Se_{11})$ is a ring isomorphism.

Proposition:3.14. The lattice $\mathfrak{I}$ of all sub modules the right $e_{11}Se_{11}-$ module Se_{11} and the lattice R of all right ideals of S are isomorphic under the following mutually reciprocal mappings: $p:A(\in\mathfrak{I})\to AS(\in R)$ *and* $q:B(\in R)\to Be_{11}(\in\mathfrak{I})$

Proof: Define map $p:\mathfrak{I}\to R$ *by* $p(N)=NS$ for all $N\in\mathfrak{I}$ and define a map $q:R\to\mathfrak{I}$ *by* $q(A)=Ae_{11}$ for all $A\in R$. Clearly p *and* q are well defined maps. **Claim:** (i) $q\circ p=Id_{\mathfrak{I}}$, (ii) $p\circ q=Id_R$

(i) Let $N\in\mathfrak{I}$.We need only prove that $N=NSe_{11}$. Now N is a sub module of the unitary right $e_{11}Se_{11}-$ module Se_{11} So

$N=N(e_{11}Se_{11})=Ne_{11}Se_{11}=NSe_{11}$ implies $(q\circ p)(N)=q(p(N))=q(NS)=NSe_{11}=N$. Hence $q\circ p=Id_{\mathfrak{I}}$

(ii) let $A\in R$. Claim: $(Ae_{11})S=A$. Clearly $Ae_{11}S\subseteq A$ because A is right ideal of S. For the reverse inclusion we note that $A=AI=A(e_{11}+(I-e_{11}))\subseteq Ae_{11}+A(I-e_{11})=Ae_{11}+A\left(\sum_{i=2}^{n}e_{ii}\right)\subseteq Ae_{11}+\sum_{i=2}^{n}Ae_{ii}=Ae_{11}+\sum_{i=2}^{n}Ae_{i1}e_{11}e_{1i}$ $\subseteq Ae_{11}S+Ae_{11}S$. Thus, $A\subseteq Ae_{11}S$ and hence $A=Ae_{11}S$. So, $(p\circ q)(A)=p(q(A))=p(Ae_{11})=Ae_{11}S=A$. Hence, $p\circ q=Id_R$.

Proposition: 3.15. $Se_{11}=e_{11}Se_{11}\oplus e_{22}Se_{11}\oplus\cdots\cdots\oplus e_{nn}Se_{11}$ as $e_{11}Se_{11}-$module and $e_{ii}Se_{11}\cong e_{11}Se_{11}$ (as $e_{11}Se_{11}-$ module) for each $i=1,2,3,\cdots,n$.

Proof: We first note If $A=((a_{ij}))$ in S then $Ae_{11}=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}e_{ij}e_{11}=\sum_{i=1}^{n}a_{i1}e_{i1}$.So $Se_{11}=\{Ae_{11}/A\in S\}$ $=\{\sum_{i=1}^{n}a_ie_{i1}/a_i\in R, for 1\le i\le n\}$ ..And if $1\le p\le n$ and $A=((a_{ij}))\in S$ then $e_{pp}Ae_{11}=\sum_{i=1}^{n}\sum_{j=1}^{n}e_{pp}a_{ij}e_{ij}e_{11}=\sum_{i=1}^{n}e_{pp}a_{i1}e_{i1}=a_{p1}e_{p1}$ .So, for $1\le p\le n$, $e_{pp}Se_{11}=\{e_{pp}Ae_{11}/A\in S\}=\{ae_{p1}/a\in R\}$. Thus the map $\phi:Se_{11}\to e_{11}Se_{11}\oplus e_{22}Se_{11}\oplus\cdots\cdots\oplus e_{nn}Se_{11}$ defined by $\phi(\sum_{i=1}^{n}a_ie_{i1})=(a_1e_{11},a_2e_{21},\cdots,a_{n1})$ for $\sum_{i=1}^{n}a_ie_{i1}\in Se_{11}$ is clearly a well-defined bijective map and is additive, we need only prove that it is an $e_{11}Se_{11}-$map. Let $A=\sum_{i=1}^{n}a_ie_{i1}\in Se_{11},B=be_{11}\in e_{11}Se_{11}$. Then $\phi(AB)=\sum_{i=1}^{n}a_ibe_{i1}=(a_1be_{11},..,a_nbe_{n1})=((a_1e_{11})(be_{11}),...,(a_ne_{n1})(be_{11}))=(a_1e_{11},..,a_ne_{n1})be_{11}=\phi(A)B$. Thus ϕ is an $e_{11}Se_{11}-$map and hence an $e_{11}Se_{11}-$isomorphism.

We next prove that $e_{ii}Se_{11}\cong e_{11}Se_{11}$ (as $e_{11}Se_{11}-$module) for each $i=1,2,3,\cdots,n$. So, let $1\le p\le n$. Consider the map $\theta:e_{pp}Se_{11}\to e_{11}Se_{11}$ defined by $\theta(ae_{p1})=ae_{11}$ for each $a\in R$. Clearly θ is a well-defined bisection and is additive. We now prove that it is an $e_{11}Se_{11}-$map. Let $ae_{p1} \in e_{pp}Se_{11}, be_{11} \in e_{11}Se_{11}$ then $\theta((ae_{p1})(be_{11})) = \theta(abe_{p1}) = abe_{11} = (ae_{11})(be_{11}) = \theta(ae_{11})be_{11}$.Thus $\theta : e_{pp}Se_{11} \to e_{11}Se_{11}$ is an $e_{11}Se_{11}-$ isomorphism.

We now prove the “Only if” part of Utumi’ theorem. Because of Proposition 3.11, we need only prove that S is right continuous. For this, we need to prove that S satisfies both the conditions (C1) and (C2) of a continuous module.

(C1): By hypothesis, $e_{11}Se_{11} \cong R$ is right self-injective. So, by Proposition 3.14, the right $e_{11}Se_{11}-$module Se_{11} is injective. Now, let B be any right ideal of S. Then Be_{11} has an essential extension, say, G which is a direct summand of Se_{11}. Now, by Proposition 4.11, there is an idempotent ‘ e ’ such that $G = eSe_{11}$. Then, by Proposition 3.13, $eS(= p(G))$ is an essential extension of $B(= pq(B) = p(Be_{11}))$. This proves the condition (C1).

(C2): Let C be a right ideal of S isomorphic to fR for some idempotent f in S. Then there exists $x \in S$ such that $C = xfS$ and $r(x) \cap fS = 0$ .Thus, the left multiplication of x gives an isomorphism of fSe_{11} onto Ce_{11} .Since fSe_{11} is a direct summand of the injective module Se_{11} , it is also injective which implies that Ce_{11} is also injective. Therefore Ce_{11} is a direct summand of Se_{11} , whence, $Ce_{11} = gSe_{11}$ for some idempotent g in S. It then follows that $C = pq(C) = p(Ce_{11}) = p(gSe_{11}) = gS$. Thus S satisfies the condition (C2). Hence S is right continuous, and hence, right self-injective.

Conclusion: we investigating the relationship between continuous matrix rings and the proof of Utumi’s Theorem that a ring R (with identity) is self-injective if and only if the matrix ring $M_n(R)$ (for a fixed $n \in$) is self-injective.

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