Tuijin Jishu/Journal of Propulsion Technology
ISSN: 1001-4055
Vol. 46 No. 2 (2025)

Perfect Degree Support Graphs

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Abstract: In a graph $G(V,E)$, the support $s(v)$ of a vertex v is defined as the sum of degrees of its neighbours. Let k be any positive integer. In this paper, we introduce two new concepts in graph theory, namely, k – perfect degree support graph and (k,c) – linear degree support graph. A graph G is said to be a k – perfect degree support graph (k – *pds* graph) if for any vertex v in G, the ratio of its support to its degree is the constant k. A graph G is called a (k, c) – linear degree support graph ((k,c) – *lds* graph) if, the support of any vertex is k times its degree with a constant integer c added to it. Some families of (k,c) – *lds* graphs and k – pds graphs have been constructed in this paper. In addition, an interesting relationship between the eigen values of the adjacency matrix of a k – pds graph with the degree sequence of G as its eigen vector has been studied.
Keywords: *Support of a vertex, balanced graph, highly unbalanced graph, k – perfect degree support graph, (k,c) – linear degree support graph, eigen graph.*
AMS Subject Classification code (2010): 05C (Primary)

1 Introduction

Graphs under consideration in this paper are finite, simple, undirected and connected. For notations and terminology, we follow [5]. A graph G is said to be r- *regular*, if every vertex of G has degree r. For $r \neq l$, a graph G is said to be (r, l) – *biregular* if the degree $d(v)$ is either r or l for any vertex v in G. Let $D(G)$ denote the set of all degrees of vertices in G. Let δ and Δ denote the minimum and maximum degree respectively in a graph G. Throughout this paper, let k denote a positive integer and let c denote an integer.

Let G_1 and G_2 be any two graphs. The graph $G_1 \circ G_2$ obtained from one copy of G_1 and $|V(G_1)|$ copies of G_2 by joining each vertex in the i^{th} copy of G_2 to the i^{th} vertex of G_1 is called the *corona* of G_1 and G_2. A subset S of V is called a *dominating set* of G if every vertex in $V – S$ is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality taken over all dominating sets in G. Let χ denote the *chromatic number* of a graph and let ω denote its clique number.

In a graph G, deleting an edge uv and introducing a new vertex w and two new edges uw and vw is called the *subdivision of the edge uv*. The *edge subdivision graph* denoted by $S_1(G)$ is obtained from the graph G by subdividing every edge of G.

The term arithmetic progression is not new to graph theory. Arithmetic progressions in cycle lengths and colouring have won the interest of many researchers till today. For further details on these topics, one can refer [6],[7],[8] and [9]. In this paper, we study a similar property comparing degree sequences and a related parameter in a graph.

It is quite obvious that the degree of a vertex is an important parameter in a graph which decides most of the graph theoretical properties. But it is not sufficient to judge the entire importance of a vertex in a graph. Even two vertices of same degrees will be not of equal weightage in the same graph unless one is an isomorphic image of the other. The degrees of its neighbours contribute much in determining the weightage of a vertex in a graph. Hence it becomes essential to study about the degrees of neighbour vertices also.

The concepts of support of a vertex, balanced graphs and highly unbalanced graphs have been introduced and studied by Selvam Avadayappan and G. Mahadevan [1].

The *support* s(v) of a vertex v is the sum of degrees of its neighbours. That is, $s(v) = \sum_{u \in N(v)} d(u)$. Note that the support of any vertex in an r – regular graph is r^2.

A graph G is said to be a *balanced graph,* if any two vertices in G have the same support. It is easy to observe that the complete bipartite graphs $K_{m,n}$ and any regular graphs are balanced graphs. A graph G is said to be *highly unbalanced,* if distinct vertices of G have distinct supports. For example, a highly unbalanced graph is shown in Figure 1.

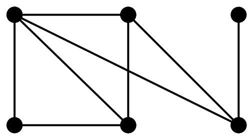


Figure 1 A highly unbalanced graph

The following results have been proved in [1]:

Result 1 $\sum_{v \in V} s(v) = \sum_{v \in V} d(v)^2$.

Result 2 For any balanced graph G, $\delta(G) = 1$ if and only if $G \cong K_{1,n}$, $n \geq 1$.

Result 3 For any $n \geq 6$, there is a highly unbalanced graph of order n.

Result 4[4] A graph G is a balanced graph if and only if G is regular or biregular bipartite with each partition having vertices of same degree.

Balanced graphs are nothing but support – regular graphs with no restriction on the degrees of their vertices. Regular graphs are the only graphs with all vertices of same degree and same support. Though it seems that the parameters degree and support of a vertex are related to each other, normally these two parameters are not dependent on each other. Two vertices of same degree need not have same support in a graph.

Even a pendant vertex in a graph may have same or more or less support compared to that of a vertex of degree two in the same graph. More surprisingly, in $K_{1,n}$, the pendant vertex as well as the center vertex have the same support n. What happens if the support of a vertex varies proportionately with its degree in a graph? Does there exist a graph in which the ratio of support of any vertex to its degree is always a constant?

To answer for these questions affirmatively we introduce the concepts of *k – perfect degree support graph* and *(k,c) – linear degree support graph* in this paper.

A graph *G* is said to be a *k – perfect degree support graph* (or simply a *k – pds graph*), if for any vertex *v* in *G*, $\frac{s(v)}{d(v)} = k$. For example, the graph $C_4 \circ K_2$ shown in Figure 2 is a *3 – pds* graph. In general, $C_n \circ K_2$ is a *3 – pds* graph for any $n \geq 3$.

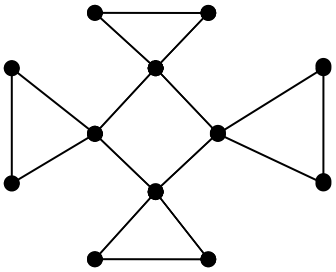


Figure 2 $C_4 \circ K_2$

A graph *G* is said to be a *(k,c) – linear degree support graph* (or simply a *(k,c) – lds graph*), if $s(u) = k\, d(u) + c$, for every vertex *u* in *G*. It is obvious that if *G* is *(k,c) – lds*, then for any two vertices *u* and *v* in *G*, $s(u) – s(v) = k(d(u) – d(v))$. For example, the graph shown in Figure 3 is a *(3, -1) – lds* graph.

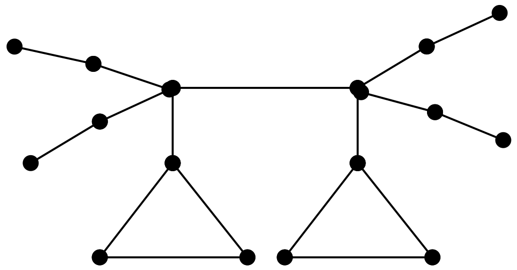


Figure 3 A (3, -1) – lds graph

The following facts can easily be verified:

Fact 1 Any *k – pds* graph is nothing but a *(k,0) – lds* graph.

Fact 2 In a *(k,c) – lds* graph, $k = 0$ if and only if it is a balanced graph.

Fact 3 *r* – regular graphs are *r – pds* graphs and hence *(r,0) – lds* graphs.

Fact 4 $K_{m,n}$ is a *k – pds* graph if and only if $m = n = k$.

Fact 5 In a *(k,c) – lds* graph, two vertices of same degree have the same support. That is, $d(u) = d(v)$ implies that $s(u) = s(v)$.

Note that the converse of Fact 5 is not true. For example, in $K_{1,n}$, $n \geq 2$, the vertices of same degree have the same support. But it is not a *(k,c) – lds* graph for any *k* and *c*.

Eigen value of a *k – pds* graph

Recall that for an $n \times n$ matrix A, a number λ is called an eigen value of A if there exists a non zero vector $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ such that $AX = \lambda X$ and X is called an eigen vector corresponding to the eigen value λ. If X is an eigen vector corresponding to the eigen value λ of A, then αX, where α is any non zero number, is also an eigen vector corresponding to λ.

In this paper, we first prove that for any *k - pds* graph *G*, *k* is an eigen value for the adjacency matrix of *G* and the corresponding eigen vector is the degree sequence of *G*. We also obtain some bounds for the minimum and maximum degree of a *k – pds* graph. In addition we construct a few families of *k – pds* graphs and *(k,c) – lds* graphs with some constraints. All *k – pds* trees have been characterized.

Further studies on *k – pds* and *(k,c) – lds product graphs* are in [2]. Generating new families of *k – pds* and *(k,c) – lds* graphs from given *k – pds* and *(k,c) – lds* graphs and still more characterizations of *k – pds* graphs are in [3].

2 Eigen value of a *k – pds* graph

Vibration analysis of structures from buildings to bridges is done at the time of designing using eigen values and eigen vectors. Eigen values can also be used to test for cracks or deformities in structural components used for construction. In addition, the eigen values can be used to determine if a structure has deformed under the application of a particular force. In control theory, the eigen values of the system matrix of a linear system tell us information about the stability and response of our system. A lot more applications of eigen values can be found in literature.

Based on the enormous applications of eigen values and eigen vectors to the society and importance of degrees of nodes of a network, we naturally think of methods of linking degree sequence and eigen vectors of the network models.

In other words, the networks could be designed in such a way to have desired eigen values to optimize the favourable results obtained out of it. As we have mentioned earlier, graph models are the most appropriate model to any network of known and unknown fields in daily life.

Does there exist any graph in which its degree sequence itself serves as an eigen vector? Or for any eigen vector fixed, could we develop a graph model with its degree sequence as the given eigen vector?

Thinking of this line it is of great interest to spot out that *k - pds* graphs have eigen value to be *k* and its degree sequence itself as an eigen vector.

This existence encourages us to develop the concept of eigen graphs. In any graph G of order ν, let $M_D(G)$ be an $\nu \times 1$ matrix such that its entries are degrees of vertices of G. That is,

$$M_D(G) = \begin{pmatrix} d(v_1) \\ d(v_2) \\ \vdots \\ d(v_\nu) \end{pmatrix}.$$

Now if λ is an eigen value of $A(G)$ with respect to eigen vector equivalent to degree sequence of G, then $A(G)M_D(G) = \lambda M_D(G)$. We call such graphs as λ – eigen graphs. That is, λ - *eigen graphs* are graphs with its degree sequence as an eigen vector for the eigen value λ with respect to its adjacency matrix.

Theorem 2.1 For any positive integer λ, a graph G is a λ - eigen graph if and only if it is a λ - pds graph.

Proof Let G be a λ - eigen graph for some positive integer λ. Then $A(G)M_D(G) = \lambda M_D(G) = (\gamma_i)$ where $\gamma_i = \sum_{k=1}^{\nu} \beta_{ik}\alpha_k$ which is equivalent to $\sum_{v_k \in N(v_i)} d(v_k)$. That is, $\sum_{v_k \in N(v_i)} d(v_k) = \gamma_i = \lambda d(v_i)$. We know that $s(v) = \sum_{u \in N(v)} d(u)$. Therefore we have $s(v_i) = \lambda d(v_i)$ for every vertex v_i in G which implies that G is a λ - pds graph. Conversely, suppose that G is a λ - pds graph. Retracing the above steps, we get G to be a λ - eigen graph. ■

All k – pds graphs constructed in Section 3 and 4 are k – eigen graphs.

3 Results on *k – pds* and *(k,c) – lds* graphs

The following theorem proves the existence of a k – *pds* graph for any $k \geq 1$.

Theorem 3.1 For any $k \geq 1$, there exists a k – *pds* graph H_k.

Proof Construct the graph H_k with vertex set $V(H_k) = \{x\} \cup \{ u_{ij} / 1 \leq i \leq 3, 1 \leq j \leq k-1\} \cup \{v_{ij} / 1 \leq i \leq 3, 1 \leq j \leq k-1\} \cup \{w_{ij} / 1 \leq i \leq 3, 1 \leq j \leq k-2\}$ and the edge set $E(H_k) = \{ u_{ij}v_{ij} / 1 \leq i \leq 3, 1 \leq j \leq k-1\} \cup \{u_{ij}w_{il} / 1 \leq i \leq 3, 1 \leq j \leq k-1, 1 \leq l \leq k-2\} \cup \{xu_{ij} / 1 \leq i \leq 3, 1 \leq j \leq k-1\}$. Clearly x is of degree $3(k-1)$ and of support $3k(k-1)$, u_{ij} is of degree k and support k^2, v_{ij} is a pendant vertex with support k and w_{ij} is of degree $k-1$ and support $k(k-1)$, for $1 \leq i \leq 3$ and $1 \leq j \leq k-1$. This gives that H_k is a k – *pds* graph.

For example the 5 – *pds* graph H_5 is shown in Figure 4. ■

The existence of a (k, c) – *lds* graph for any $k \geq 1$ has been established in the following theorem.

Theorem 3.2 For any $k \geq 1$, there exists a (k,c) – *lds graph* G_k for some integer c.

Proof Let G_k be the graph with vertex set $V(G_k) = \{ u_{ij} / 1 \leq i \leq 2, 1 \leq j \leq k-1\} \cup \{ v_{ij} / 1 \leq i \leq 2, 1 \leq j \leq k-1\} \cup \{w_i / 1 \leq i \leq k-1\} \cup \{ x_{ij} / 1 \leq i \leq k-1, 1 \leq j \leq k-1\} \cup \{ y_i / 1 \leq i \leq k-1\}$; *And the edge set* $E(G_k) = \{ u_{ij}w_j / 1 \leq i \leq 2, 1 \leq j \leq k-1\} \cup \{ u_{ij}v_{ij} / 1 \leq i \leq 2,$

$1 \le j \le k-1\} \cup \{w_iw_j \,/\, 1 \le i,j \le k-1,\ i \ne j\} \cup \{x_{ij}y_i \,/\, 1 \le i \le k-1,\ 1 \le j \le k-1\} \cup \{y_iw_i \,/\, 1 \le i \le k-1\} \cup \{x_{ij}x_{im} \,/\, 1 \le i, j, m \le k-1, j \ne m\}$.

Clearly if the degree of a vertex v in G_k is $1, 2, k-1, k$ or $k+1$, then the support of v is $2, 2+k, 1+(k-1)^2, 2+k(k-1)$ or $2+k^2$ respectively. G_k is easily verified to be a $(k,c)-lds$ graph with $c = 2-k$. For example G_7 is shown in Figure 5. ■

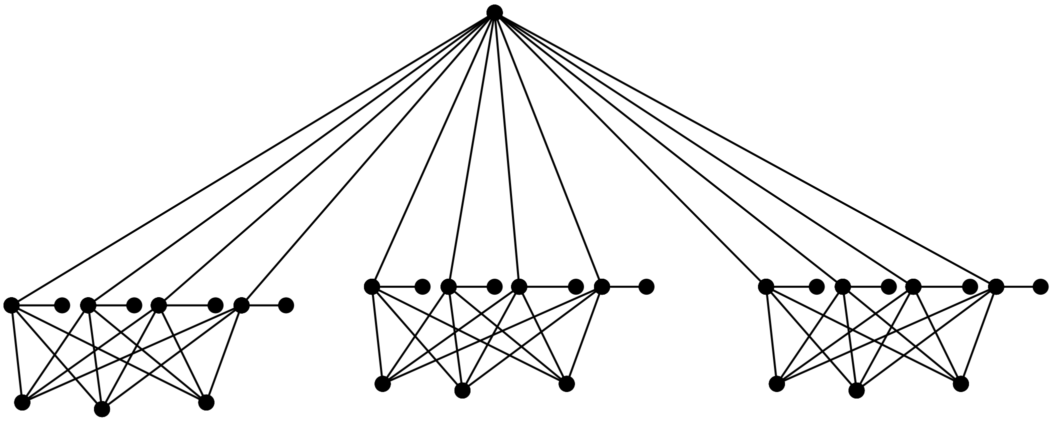


Figure 4 The 5 – pds graph H_5

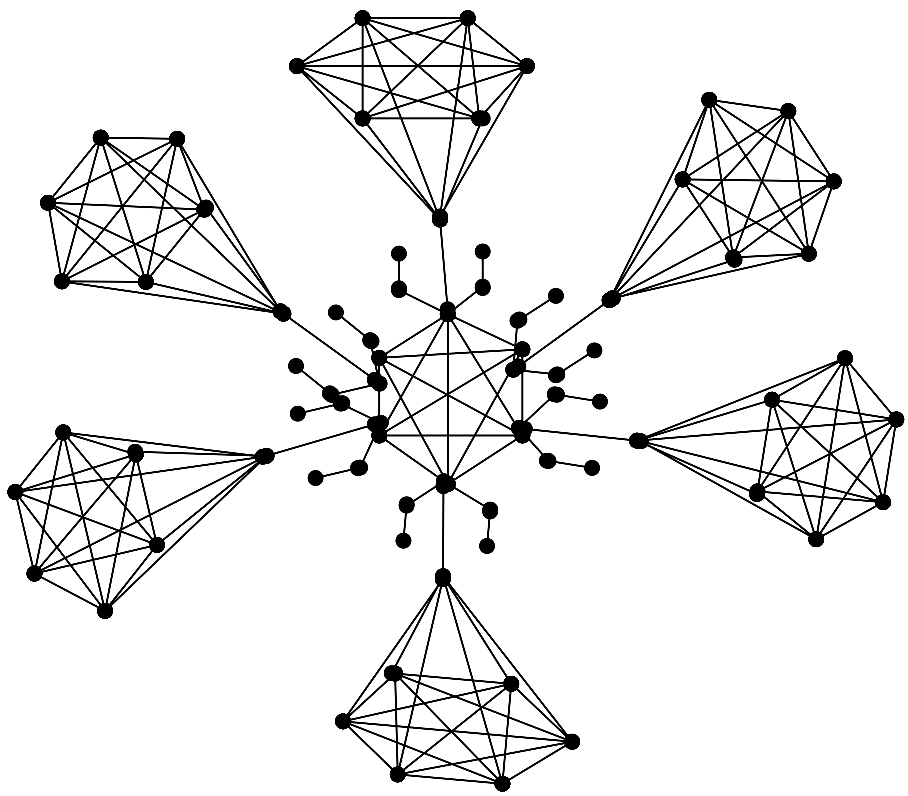


Figure 5 The graph G_7

Next we discuss about the bounds for minimum and maximum degrees in a $k - pds$ graph.

Theorem 3.3 In any non trivial $k - pds$ graph G, $1 \leq \delta \leq k \leq \Delta \leq k^2 - k + 1$, for any $k \geq 1$.

Proof Since we consider only connected graphs, obviously $\delta \geq 1$. Let G be a $k - pds$ graph, then for any vertex $v \in V(G)$, $s(v) / d(v) = k$. If possible, assume that $k < \delta$. Then one can easily note that for any vertex v, $s(v) \geq \delta d(v) > kd(v)$ which implies $s(v) / d(v) > k$, which is a contradiction. Hence we conclude that $k \geq \delta$. Similarly if $k > \Delta$, then $s(v) \leq \Delta d(v) < kd(v)$, which is again a contradiction. Hence we can conclude that $\delta \leq k \leq \Delta$.

Now it remains to show that $\Delta \leq k^2 - k + 1$. Let, if possible, G contain a vertex, say v of degree $k^2 - k + 1+i$, for some $i \geq 1$. If there exists a neighbour w of v in G such that $d(w) < k$, then we take $d(w) = k - m$, where $m \geq 0$. Now $s(w) \geq k^2 - k + 1 + i + (k - m - 1)\delta$. But $\delta \geq 1$ and so $s(w) \geq k^2 - k + 1+i + k - m - 1 = k^2 +i - m > k(k - m)$. This forces that $\frac{s(w)}{d(w)} > k$, which is a contradiction to the fact that G is $k - pds$. Hence the degree of every neighbour of v exceeds k. That is, $d(w) > k$, for every $w \in N(v)$. Therefore $s(v) \geq d(v)\ (k+1)$, which is a contradiction. Therefore $d(v) \leq k^2 - k +1$. Hence we proved. ■

Note that the above inequalities are all strict. As an illustration, one can verify that any $r - regular$ *graph* is an $r - pds$ *graph* with $\Delta = \delta = r$. Also a $3 - pds$ *graph* with $\Delta = 7$ is shown in Figure 6. The graph shown in Figure 6 stands as an example of a graph with $1 < \delta < k < \Delta < k^2 - k + 1$.

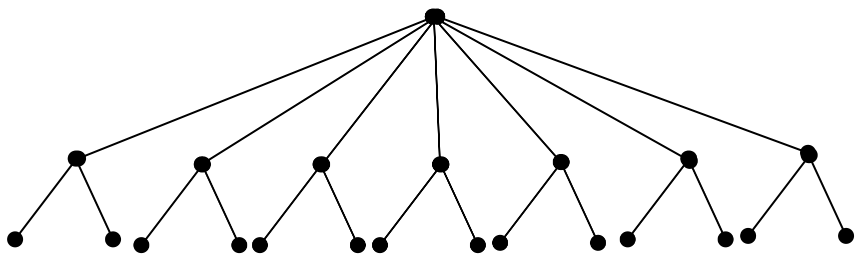


Figure 6 A graph for which $1 < \delta < k < \Delta < k^2 - k + 1$.

In the case of $(k,c) - lds$ *graphs* the above inequality does not hold good. In fact, we cannot fix any bounds for minimum and maximum degrees of a $(k,c) - lds$ *graph*. Supporting the above argument, we present the next theorem in case of $(1,c) - lds$ *graph.*

Theorem 3.4 For any two positive integers m, n such that $m > n \geq 1$, there exists a $(1,c) - lds$ *graph* G with $\delta(G) = n$ and $\Delta(G) = m$, for some integer c.

Proof Let m and n be two positive integers such that $m > n$. Now we construct a $(1,c) - lds$ *graph* of minimum degree n and maximum degree m.

Case 1 *m* is even.

Construct *G* with vertex set $V = \{u_1, u_2, \dots, u_{n-1}, v_1, v_2, \dots, v_m\}$ and edge set $E = \{u_iv_j, 1 \le i \le n-1 \text{ and } 1 \le j \le m; v_{2i-1}v_{2i}, 1 \le i \le m/2\}$. Here all u_i's are of degree *m* and support *mn* and all v_j's are of degree *n* and support $mn - m + n$, where $1 \le i \le n-1$ and $1 \le j \le m$. And so for every vertex *v* in *G*, $s(v) = d(v) + m - n$. Hence *G* is a $(1,m\text{-}n)$ – *lds graph.*

For example, the case when $n = 3$ and $m = 6$ is illustrated in Figure 7.

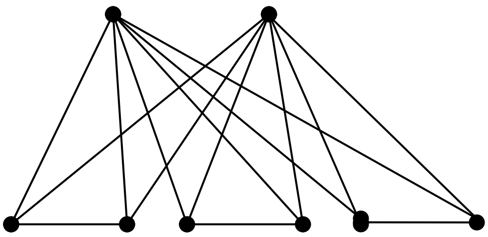


Figure 7

Case 2 *m* is odd.

We now construct *G* with vertex set $V = \{u_1, u_2, \dots, u_{n-1}, v_1, v_2, \dots, v_m, x_1, x_2, \dots, x_{n-1}, y_1, y_2, \dots, y_m\}$ and edge set $E = \{u_iv_j \text{ and } x_iy_j, 1 \le i \le n-1 \text{ and } 1 \le j \le m; v_{2i-1}v_{2i} \text{ and } y_{2i-1}y_{2i}\ 1 \le i \le (m-1)/2; v_my_m\}$. Here all u_i's and x_i's are of degree *m* and support *mn* and all v_j's and y_j's are of degree *n* and support $mn - m + n$, where $1 \le i \le n-1$ and $1 \le j \le m$. Here also for every vertex *v* in G, $s(v) = d(v) + m - n$. Therefore *G* is a $(1,m\text{-}n)$ – *lds graph.* Such a $(1,c)$ – *lds graph G* constructed with $m = 5$ and $n = 3$ is given in Figure 8. ■

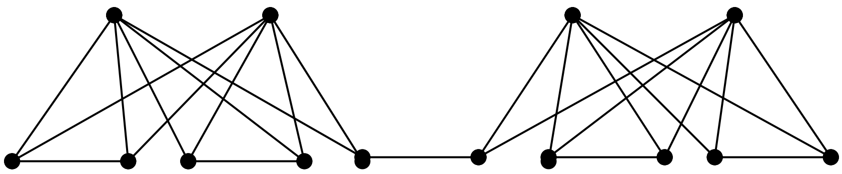


Figure 8 A (1,c) lds graph with Δ = 5 and δ = 3.

It is clear that K_2 is the only 1 – *pds* graph. For any $k \ge 1$, we have proved the existence of a *k* – *pds* graph. Next we characterize all 2 – *pds* graphs in the next theorem.

Theorem 3.5 A connected graph *G* is 2 – *pds* if and only if $G \cong C_n, n \ge 3$, or $S_1(K_{1,3})$.

Proof Let *G* be a 2 – *pds* graph. Then by *Theorem 3.3*, minimum degree in *G* is either *1 or 2*.

Case 1 *G* contains a pendant vertex *v*.

Since *G* is 2 – *pds*, *v* must be adjacent to a vertex *u* of degree *2*. This means that $s(u) = 4$ and hence *u* is adjacent to a vertex w of degree *3*. We also conclude that no neighbour of *w* is a pendant vertex and $s(w) = 6$. This forces that, all neighbours of *w* are of degree *2*. Let *N(w)*

$= \{u,x,y\}$. Then we have $d(u) = d(v) = d(x) = 2$. Now $s(x) = 4$. Hence the remaining neighbour of x is a pendant vertex. Similarly y is adjacent to a pendant vertex. Thus $G \cong S_1(K_{1,3})$.

Case 2 $\delta(G) = 2$.

That is, G contains no pendant vertex. Let v be a vertex of degree 2. Then clearly $s(v) = 4$. This forces that any vertex in $N(v)$ can have degree at most 3. In other words, $N(v)$ can have vertices of degree *2 or 3*. If v is adjacent to a vertex of degree 3, then the other neighbour of v is a pendant vertex, which leads to a contradiction. Thus any vertex in G of degree 2 can be adjacent only to vertices of degree 2. This forces that, G is a $2 -$ *regular* connected graph. This means that G is isomorphic to C_n, for $n \geq 3$. And the converse is easy to verify.

■

It is interesting to characterize (k,c) – *lds graphs* also. As an initialization, we characterize $(1,c)$ – *lds graphs* with $\delta = 1$ in the next theorem.

The Bistar graph, $B_{n,n}$, $n \geq 0$ is obtained from two stars $K_{1,n}$ by adding an edge between their central vertices. For example, one can refer $B_{4,4}$ in Figure 9.

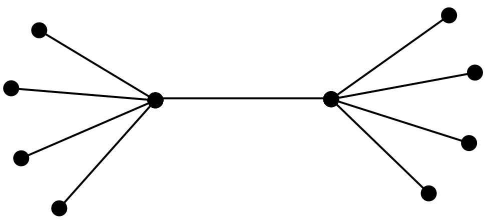


Figure 9 The Bistar graph $B_{4,4}$

Theorem 3.6 A graph with at least one pendant vertex is $(1,c)$ – *lds* if and only if it is a bistar $B_{c+1,c+1}$.

Proof Let G be a graph with $\delta(G) = 1$. Assume that G is a $(1,c)$ – *lds* graph. Then for every vertex v in G, $s(v) = d(v) + c$, for some constant c. In particular, we have $c \geq 0$. Otherwise, support of any pendant vertex is less than or equal to 0, which is a contradiction.

When $c = 0$, $G \cong K_2$, which is nothing but $B_{0,0}$. Now let $c > 0$. Let u be a pendant vertex in G and let v be its neighbour. Then $s(u) = c+1$ and hence $d(v) = c+1$. This implies that $s(v) = 2c+1$. If there exists a vertex $w \in N(v)$, with degree $m > 1$, then $s(w) = m + c$. Hence all the neighbours of w except v must be pendant. Also we know that pendant vertices can be adjacent only to vertices of degree $c + 1$. Hence $m = c + 1$. That is, $d(w) = c + 1$. Therefore the remaining $c - 1$ neighbours of v are pendant. Hence $G \cong B_{c+1,c+1}$.

The converse is obvious. ■

Recall that mG denotes the union of m copies of G. Let $< K_{1,n}, m >$ denote the graph obtained from $mK_{1,n}$ by adding a new vertex and joining it with the center vertex of each copy of $K_{1,n}$. For example, the graph depicted in Figure 6 is nothing but $< K_{1,2}, 7 >$. One can easily note that $< K_{1,\, k-1}, k^2 - k + 1 >$ is a $k - pds$ tree.

Theorem 3.7 For $k \geq 2$, a tree G is $k - pds$ tree if and only if $G \cong < K_{1,\, k-1}, k^2 - k + 1 >$.

Proof Let G be a $k - pds$ tree. Then G contains at least two pendant vertices. Also all the pendant vertices of G are adjacent to vertices of degree k in G. That is, G contains a vertex of degree k. In fact, G contains at least two such vertices of degree k, say v_1 and v_2, for which every neighbour except one is a pendant vertex. Otherwise, the graph obtained by removing all pendant vertices from G is not a tree, which is a contradiction.

Since $s(v_1) = k^2$, the only non pendant neighbour of v_1 and v_2 is of degree $k^2 - k + 1$. The same is true for v_2 also.

Let us first assume that v_1 and v_2 have a common neighbor, say w, of degree $k^2 - k + 1$. Now we think of the remaining neighbours of w. One can note that no neighbour of w can be of degree less than k. For, if a neighbour of w is of degree $k - 1$, then its support will be at least $k^2 + k - 3$ which is always greater than the required support $k^2 - k$ since $k \geq 2$. Therefore no neighbour of w is of degree less than k. But $s(w) = k(k^2 - k + 1)$ and so every neighbour of w should be of degree k. Consequently, except w, every negihbour of vertices in $N(w)$ is a pendant vertex. Thus $G \cong < K_{1,\, k-1}, k^2 - k + 1 >$.

If suppose v_1 and v_2 have two different neighbours w_1 and w_2 of degree $k^2 - k + 1$, then as in the above discussion, G is disconnected with each component isomorphic to $< K_{1,\, k-1}, k^2 - k + 1 >$, which is a contradiction since G is a tree. Hence we conclude that if G is a $k - pds$ tree then $G \cong < K_{1,\, k-1}, k^2 - k + 1 >$. The converse follows easily. ■

Corollary 3.8 For $k \geq 2$, an acyclic graph G is a $k - pds$ graph if and only if $G \cong m < K_{1,\, k-1}, k^2 - k + 1 >$, for some $m \geq 1$.

Tuijin Jishu/Journal of Propulsion Technology
ISSN: 1001-4055
Vol. 46 No. 2 (2025)

4 Construction of *k – pds* and *(k,c) – lds* graphs

In this section, we construct few families of $k - pds$ and $(k,c) - lds$ graphs. The next theorem gives the construction of a $k - pds$ family with all odd degrees from 1 to $2k - 1$ together with k.

Theorem 4.1 For any $k \geq 1$, there exists a $k - pds$ graph G with $D(G) = \{1,3,5,\ldots,2k-1, k\}$.

Proof Let $m = lcm(1,3,5,\ldots,2k-1)$. Let us construct a graph G_k with vertex set $V(G_k) = \{v_i / 1 \leq i \leq m\} \cup \{u_{ij} / 1 \leq i \leq k,\ 1 \leq j \leq m/(2i-1)\}$ and the edge set $E(G_k) = \{ u_{ij}v_l / 1 \leq i \leq k,\ 1 \leq j \leq m/(2i-1),\ (2i-1)(j-1)+1 \leq l \leq (2i-1)j\}$.

In the constructed graph G_k, for $1 \leq i \leq m$, $d(v_i) = k$ and $s(v_i) = k^2$. And u_{ij}, $1 \leq i \leq k$, $1 \leq j \leq m/(2i-1)$ is of degree $2i-1$ and of support $(2i-1)k$. Hence G_k is a $k - pds$ *graph.*

For example, the constructed $3 - pds$ graph G_3 is given in Figure 10. ■

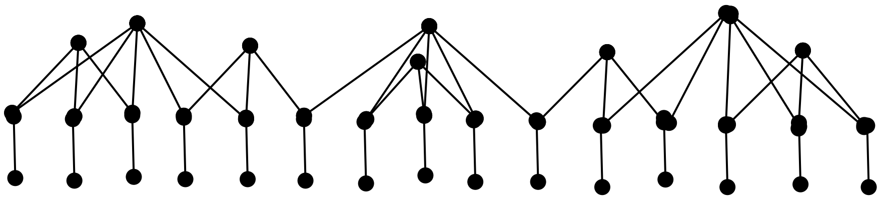


Figure 10 A 3 – pds graph G$_3$

Next we construct another family of $k - pds$ graphs with domination number $k + 1$ for any $k \geq 2$.

Theorem 4.2 For any given $k \geq 2$, there exists a k - pds graph with $D(G) = \{2,4,6,\ldots,2k-2, k\}$.

Proof Let $m = lcm(2,4,\ldots,2k\text{-}2)$. Let us construct a graph G_k with vertex set $V(G_k) = \{v_i / 1 \leq i \leq m\} \cup \{u_{ij} / 1 \leq i \leq k-1,\ 1 \leq j \leq m/(2i)\}$ and the edge set $E(G_k) = \{ u_{ij}v_l / 1 \leq i \leq \lambda\text{-}1,\ 1 \leq j \leq m/2i,\ 2i(j-1)+1 \leq l \leq 2ij\} \cup \{v_{2i-1}v_{2i} / 1 \leq i \leq m/2\}$.

In G_k, for $1 \leq i \leq m$, $d(v_i) = k$ and $s(v_i) = k^2$. And u_{ij}, $1 \leq i \leq k\text{-}1$, $1 \leq j \leq m/(2i)$ is of degree $2i$ and of support $2ik$. Hence G_k is a $k - pds$ graph.

For example, a $4 - pds$ graph constructed as above is shown in Figure 11. ■

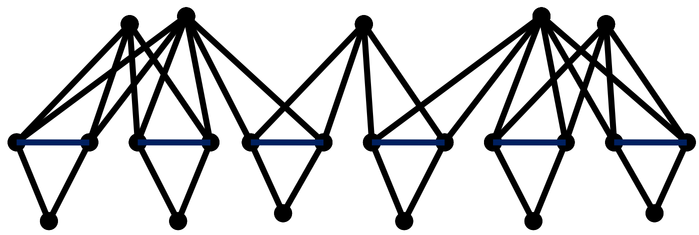


Figure 11 A 4 – pds graph

Tuijin Jishu/Journal of Propulsion Technology
ISSN: 1001-4055
Vol. 46 No. 2 (2025)

Theorem 4.3 For any $k \geq 2$, there exists a $k - pds$ bipartite graph with $\gamma(G) = k + 1$.

Proof Let us construct a graph G_k with vertex set $V(G_k) = \{u_i / 1 \leq i \leq k-1\} \cup \{v_i / 1 \leq i \leq k+1\} \cup \{w_i / 1 \leq i \leq k+1\}$ and the edge set $E(G_k) = \{u_i v_j / 1 \leq i \leq k - 1, 1 \leq j \leq k+1\} \cup \{v_i w_i / 1 \leq i \leq k+1\}$. In G_k, u_i's are of degree $k+1$ and support $k(k+1)$, v_j's are of degree k and support k^2 whereas w_k's are pendant vertices with support k.

Thus G_k is a $k - pds$ bipartite graph with $\gamma(G) = k + 1$. As an example, the $8 - pds$ graph G_8 is shown in Figure 12. ∎

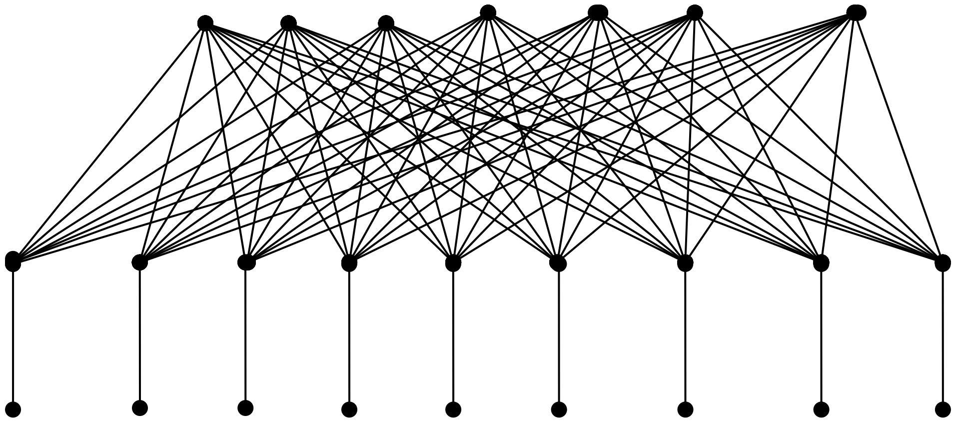


Figure 12 An 8 – pds graph G_8

Note that K_2 is the only $1 - pds$ graph with domination number one.

The next theorem gives yet another construction for a family of $k - pds$ graphs without pendant vertices.

Theorem 4.4 For any $k \geq 3$, there exists a bipartite $k - pds$ *graph* G_k with $\delta(G) = 2$.

Proof To prove the existence of such a $k - pds$ graph, we give two different constructions depending on the parity of k.

Case (i) Let k be even.

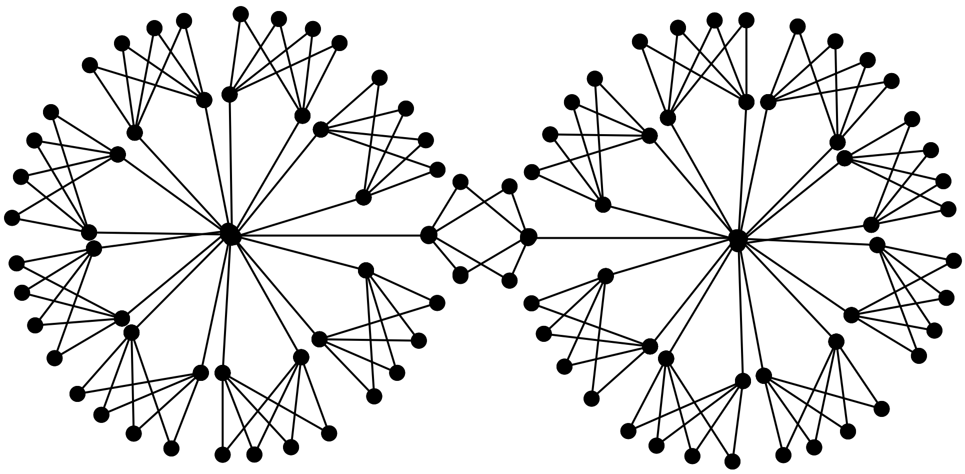
Let the vertex set of G_k be $V(G_k) = \{u_{ij} / 1 \leq i \leq \frac{k^2 - 2k+2}{2}, 1 \leq j \leq k-1\} \cup \{v_{ij} / 1 \leq i \leq \frac{k^2 - 2k+2}{2}, 1 \leq j \leq 2\} \cup \{w\}$ and the edge set be $E(G_k) = \{ u_{ij} v_{ir} / 1 \leq i \leq \frac{k^2 - 2k+2}{2}, 1 \leq j \leq k - 1, 1 \leq r \leq 2\} \cup \{v_{ij} w / 1 \leq i \leq \frac{k^2 - 2k+2}{2}, 1 \leq j \leq 2\}$.

In G_k, u_{ij}'s are of degree 2 and of support $2k$, v_{ij}'s are of degree k and support k^2 whereas w is of degree k^2-$2k+2$ and of support $k(k^2$-$2k+2)$. Therefore G_k is a $k - pds$ graph.

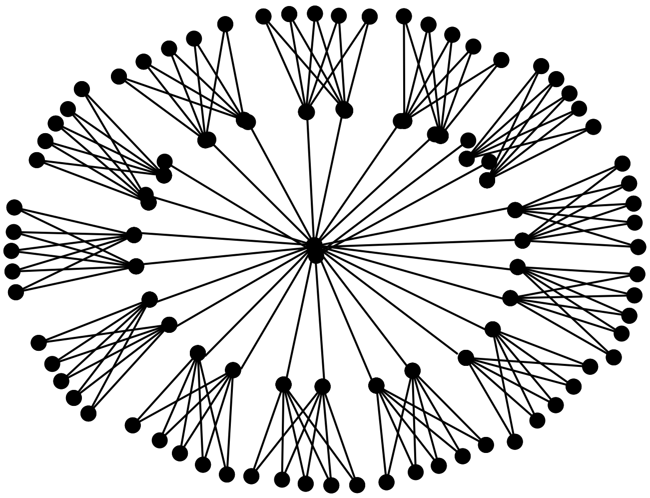
Case (ii) Let k be odd.

We construct G_k as follows: $V(G_k) = \{ u_{ij}, a_{ij} / 1 \le i \le \frac{k^2-2k+1}{2}, 1 \le j \le k-1\} \cup \{v_{ij}, b_{ij} / 1 \le i \le \frac{k^2-2k+1}{2}, 1 \le j \le 2\} \cup \{ w_i / 1 \le i \le 2\} \cup \{c_i, d_j / 1 \le i \le k-1, 1 \le j \le 2\}$ and the edge set $E(G_k) = \{ u_{ij}v_{ir} / 1 \le i \le \frac{k^2-2k+1}{2}, 1 \le j \le k-1, 1 \le r \le 2\} \cup \{a_{ij}b_{ir} / 1 \le i \le \frac{k^2-2k+1}{2}, 1 \le j \le k-1, 1 \le r \le 2\} \cup \{v_{ij}w_1 / 1 \le i \le \frac{k^2-2k+1}{2}, 1 \le j \le 2\} \cup \{b_{ij}w_2 / 1 \le i \le \frac{k^2-2k+1}{2}, 1 \le j \le \} \cup \{w_id_i / 1 \le i \le 2\} \cup \{c_jd_i / 1 \le i \le 2, 1 \le j \le k-1\}$. In G_k, u_{ij}, a_{ij}, c_{ij}*'s* are of degree 2 and support $2k$, v_{ij}, b_{ij}, d_{ij}*'s* are of degree k and support k^2, w_1 and w_2 are of degree k^2-2k+2 and support $k(k^2-2k+2)$. Hence G_k is a $k-pds$ graph.

For example, the graphs G_5 and G_6 are illustrated in Figure 13. ■



The graph G_5



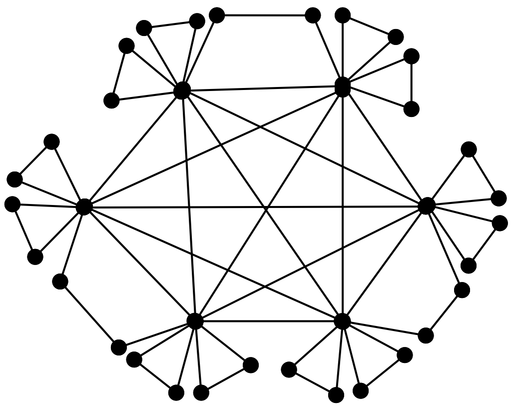
The graph G_6

Figure 13

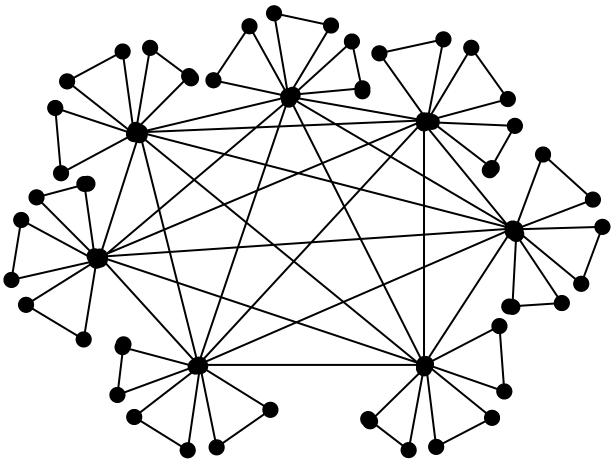
There exists a family of $k - pds$ graphs with clique number k which is proved in the next theorem.

Theorem 4.5 For any $k \geq 2$, there exists a $k - pds$ biregular graph H with $\gamma(H) = \chi(H) = \omega(H) = k$.

Proof Let $k \geq 2$ be any integer. Construct the graph H_k with vertex set, $V(H_k) = \{u_i \,/\, 1 \leq i \leq k\} \cup \{ v_{ij} \,/\, 1 \leq i \leq k,\ 1 \leq j \leq k-1\}$. Let k be even. Then the edge set of H_k is given as, $E(H_k) = \{u_iu_j \,/\, 1 \leq i \leq k,\ 1 \leq j \leq k,\ i \neq j\} \cup \{ v_{i,2j-1}v_{i,2j} \,/\, 1 \leq i \leq k,\ 1 \leq j \leq (k/2)-1\} \cup \{v_{2i-1,k-1}v_{2i,k-1} \,/ 1 \leq i \leq k/2\} \cup \{ u_iv_{ij} \,/\, 1 \leq i \leq k;\ 1 \leq j \leq k-1\}$. For odd integer k, $E(H_k) = \{ u_iu_j \,/\, 1 \leq i \leq k,\ 1 \leq j \leq k,\ i \neq j\} \cup \{v_{i,2j-1}v_{i,2j} \,/\, 1 \leq i \leq k,\ \ 1 \leq j \leq (k-1)/2\} \cup \{ u_iv_{ij} \,/\ \ 1 \leq i \leq k;\ 1 \leq j \leq k-1\}$. The vertices u_i's are of degree $2(k-1)$ and support $2k(k-1)$ whereas v_{ij}'s are of degree 2 and support $2k$. Thus, H_k is a $k - pds$ biregular graph. Also clique number of H_k is k and it is easy to note that $\gamma(H_k) = \chi(H_k) = k$. For example, H_6 and H_7 are shown in Figure 14. ■



The graph H_6



The graph H_7

Figure 14

Theorem 4.6 For any $k \geq 1$ and $c \geq 1$, there exists a $(k,c) - lds$ graph G with $s(v) = k\, d(v) + c$, for any vertex $v \in V(G)$.

Proof Consider the graph $G = K_{k+1} \circ K_c^c$. In this graph, pendant vertices are of support $k + c$ and the other vertices are of degree $k + c$ and support $k^2 + kc + c$. Hence G is the required $(k,c) - lds$ graph.

For example, the (k,c) – lds graph with $k = 3$ and $c = 3$ is shown in Figure 15. ■

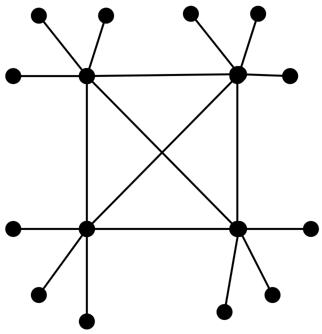


Figure 15

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