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The Minimum Dominating Hub Energy of a Graph

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Abstract: - In this research paper, we introduce the concept of minimum dominating hub energy of a graph, denoted as $E_{DH}(G)$ and also computed this value for some standard graphs and well known families of graphs. Also, we are discussing the bounds for this hub energy.

Keywords: *Minimum Dominating Hub Set, Minimum Dominating Hub Matrix, Minimum Dominating Hub Energy.*

1. Introduction

Let G be a non-empty simple graph which is finite, having no loops, no multiple and directed edges. Let n and m be the number of its vertices and edges respectively. The symbols $\Delta(G)$ and $\delta(G)$ denotes the maximum and minimum degree of graph G respectively. In this case, [11] likely refers to a research paper that provides the accepted definitions and terminology of a graph.

M. Walsh [17] introduced the concept theory of hub numbers in the year 2006. A hub set $H \subseteq V(G)$ which is a subset of vertices in a graph G, where for any two vertices x and y which are not in H, there exist a H-path between them. A H-path is a path in G where all the intermediate vertices (vertices between x and y) belong to the hub set H. I f $x = y$, then H-path is just x which doesn't require any intermediate vertices. Similarly, if x and y are directly connected, the path from x to y is the direct edge (x, y) which is called as trivial H-path. A set $H \subseteq V(G)$ is a hub set of G which has the property that, for any $(x, y) \in V(G) - H$ there is a H-path in G between x and y. The minimum number of elements in a hub set is called hub number of G and is denoted by $h(G)$[16]. Reference [5] is likely a detailed work that expands the concept of hub number.

A set $S \subseteq G$ is called a dominating set of G if each vertex of $V - S$ is adjacent to atleast one vertex in S. The minimum cardinality of a dominating set in G is called domination number and is denoted by $\gamma(G)$ [12]. The energy of a graph G is defined to be the sum of the absolute values of eigenvalues of its adjacency matrix.

$$E(G) = \sum_{i=1}^{n} |\lambda_i|$$

The energy concept of a graph was introduced by I. Gutmann in 1978 [7]. Let $G = (n, m)$ be a graph and let $A(G) = (a_{ij})$ be the adjacency matrix of graph G. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $A(G)$ which are arranged in the non-increasing order. As the adjacency matrix $A(G)$ is real, symmetric and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the distinct eigen values of G with multiplicity $\alpha_1, \alpha_2, \ldots, \alpha_n$ respectively. The multi-set

$$\text{spec}(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_n \\ \alpha_1 & \alpha_2 & \ldots & \alpha_n \end{pmatrix}$$

of eigenvalues of $A(G)$ is called adjacency $\text{spec}(G)$ and the eigenvalues are real with sum equal to zero. The research work of Coulson [4] shows that there is a continuous interest towards the general mathematical properties of the total $\pi -$ electron energy as calculated within the framework of the Huckel molecular orbital (HMO) model. The properties of this energy are discussed in detail in $[2, 8, 9, 10, 15]$.

In this research paper, we introduce the concept of minimum dominating hub energy, denoted by $E_{DH}(G)$. The minimum dominating hub energy likely incorporates both the structure of the graph and the concept of domination, which adds layers of complexity to its eigenvalues and energy calculations. Initially, the concept might have seemed niche, but its implications have broadened significantly. Researchers have explored not only graph energy but also analogous quantities in various types of matrices, leading to applications in fields like chemistry, physics and network theory. The interplay between graph theory and linear algebra has opened new avenues for understanding complex systems. When we study the eigenvalues of this matrix, we can uncover properties related to the graph's connectivity, bounds and other characteristics. In chemistry for instance, the energy of a graph can be related to the stability of molecular structures where vertices represent atoms and edges represent bonds. The minimum dominating hub energy could provide insights into how molecular configurations impact stability or reactivity. The graphs we are considering are assumed to be finite, simple and undirected having no isolated vertices and of order at least two.

2. The Minimum Hub Energy of a Graph

Let $G = (V, E)$ be a graph with vertex set $V(G) = \{v_1, v_2, v_3, \dots, v_n\}$ and the edge set $E(G)$. Any hub set with minimum cardinality is called minimum hub number. Let H be a minimum hub set of G. The minimum hub matrix of G is the $n \times n$ matrix $A_H(G) = (a_{ij})$ where

$$\mathbf{a_{ij}} = \begin{cases} \mathbf{1}, & \mathbf{v_i v_j \in E} \\ \mathbf{1}, & \mathbf{if\ i = j\ and\ v_i \in H} \\ \mathbf{0}, & \mathbf{otherwise} \end{cases}$$

The characteristic polynomial of $A_H(G)$ denoted by $f_n(G,\lambda)$ is defined as $f_n(G,\lambda) = \det(\lambda I - A_H(G))$. The minimum hub eigenvalues of the graph G are the eigenvalues of $A_H(G)$. Since $A_H(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The minimum hub energy of G is defined as

$$E_H(G) = \sum_{i=1}^{n} |\lambda_i|$$

3. The Minimum Dominating Hub Energy of a Graph

Let G be a graph of order n with vertex set $V(G) = \{v_1, v_2, v_3, \dots, v_n\}$ and the edge set $E(G)$. Any dominating hub set with minimum number of elements is called minimum dominating hub number. Let D be a minimum dominating hub set of G. The minimum dominating hub matrix of G is the $n \times n$ matrix $A_{DH}(G) = (a_{ij})$ where

$$\mathbf{a_{ij}} = \begin{cases} \mathbf{1}, & \mathbf{v_i \sim v_j\ for\ i \neq j} \\ \mathbf{1}, & \mathbf{if\ v_i = v_j\ and\ v_i \in D\ for\ i = j} \\ \mathbf{0}, & \mathbf{otherwise} \end{cases}$$

The characteristic polynomial of $A_{DH}(G)$ denoted by $f_n(G,\lambda)$ is defined as $f_n(G,\lambda) = \det(\lambda I - A_{DH}(G))$. The minimum hub eigenvalues of the graph G are the eigenvalues of $A_{DH}(G)$. Since $A_{DH}(G)$ is real and symmetric, its eigenvalues are real numbers and are arranged in non-increasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The minimum dominating hub energy of G is defined as

$$E_{DH}(G) = \sum_{i=1}^{n} |\lambda_i|$$

4. Example

Let $\mathbf{G = P_4}$ be a path graph with the vertex set $\mathbf{V(G) = \{v_1, v_2, v_3, v_4\}}$. Let $\mathbf{D_1 = \{v_1, v_3\}}$ be the minimum dominating hub set then the minimum dominating hub matrix of $\mathbf{G}$ is

$$\mathbf{A_{D_1H}(G)} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The characteristic polynomial of $A_{D_1H}(G)$ is $f_n(P_4, \lambda) = \lambda^4 - 2\lambda^3 - 2\lambda^2 + 3\lambda + 1$ and the minimum dominating eigenvalues are $\lambda_1 = 2.19352$, $\lambda_2 = 1.29496$, $\lambda_3 = -0.29496$ and $\lambda_4 = -1.19352$. Hence the minimum dominating hub energy of G is $E_{D_1H}(G) = 4.97696$.

Similarly, if we consider another minimum dominating hub set of G namely $D_2 = \{v_2, v_3\}$ then the minimum dominating hub matrix of G is

$$\mathbf{A_{D_2H}(G)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The characteristic polynomial of $A_{D_2H}(G)$ is $f_n(P_4, \lambda) = \lambda^4 - 2\lambda^3 - 2\lambda^2 + 2\lambda + 1$ and the minimum dominating eigenvalues are $\lambda_1 = 2.41421, \lambda_2 = 1, \lambda_3 = -0.41421$ and $\lambda_4 = -1$. Hence the minimum dominating hub energy of G is $E_{D_2H}(G) = 4.82842$.

The above example illustrates that the minimum dominating hub energy of a graph G depends on the choice of the minimum dominating hub set. i.e., the minimum dominating hub energy is not a graph invariant. We need the following to prove main results.

Theorem 4.1. For any (n, m) graph G, $n - m \leq \gamma(G)$. Furthermore $\gamma(G) = n - m$ if and only if each component of G is a star.

Lemma 4.1. For any graph G we have, $\gamma(G) \leq D(G) + 1$

Theorem 4.2. If G is a connected graph, then $D(G) \leq |V(G)| - \Delta(G)$ and the inequality is sharp.

5. Minimum Dominating Hub Energy of some Standard Graphs

In this section, we investigate the exact values of the minimum hub energy of some standard graphs.

Theorem 5.1. The minimum dominating hub energy of a complete graph K_n for $n \geq 2$ is $(2n - 3)$

Proof: Let K_n be a complete graph with vertex set $V(G) = \{v_1, v_2, v_3, \dots, v_n\}$. The minimum dominating hub set is $D = \{v_1\}$ and the minimum dominating hub number is 1. Then the minimum dominating hub matrix is as follows:

$$\mathbf{A_{DH}(K_n)} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 0 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & 0 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 0 \end{pmatrix}_{n \times n}$$

The characteristic polynomial of $\mathbf{A_{DH}(K_n)}$ is

$$\mathbf{f_n(K_n, \lambda)} = \begin{pmatrix} \lambda - 1 & -1 & -1 & -1 & \cdots & -1 & -1 \\ -1 & \lambda & -1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & \lambda & -1 & \cdots & -1 & -1 \\ -1 & -1 & -1 & \lambda & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & -1 & \cdots & \lambda & -1 \\ -1 & -1 & -1 & -1 & \cdots & -1 & \lambda \end{pmatrix}_{n \times n}$$

$$= (\lambda+1)^{n-2}\cdot(\lambda^2-(n-1)\lambda-1)$$

The eigenvalues of $\mathbf{K_n}$ are $\boldsymbol{\lambda = -1}$ with multiplicity $(\mathbf{n-2})$ and $\boldsymbol{\lambda} = \frac{(\mathbf{n-1})\pm\sqrt{\mathbf{n^2-2n+5}}}{\mathbf{2}}$ with multiplicity 1 each. The spectrum of $\mathbf{K_n}$ is as follows:

$$\mathrm{spec}(K_n) = \begin{pmatrix} \frac{(n-1)+\sqrt{n^2-2n+5}}{2} & \frac{(n-1)-\sqrt{n^2-2n+5}}{2} & -1 \\ 1 & 1 & (n-2) \end{pmatrix}$$

The minimum dominating hub energy of K_n is

$$E_{DH}(K_n) = \left|\frac{(n-1)+\sqrt{n^2-2n+5}}{2}\right|(1) + \left|\frac{(n-1)-\sqrt{n^2-2n+5}}{2}\right|(1) + |-1|(n-2)$$

$$= \frac{(n-1)+\sqrt{n^2-2n+5}+(n-1)-\sqrt{n^2-2n+5}}{2} + (n-2)$$

$$= \frac{2(n-2)}{2} + (n-2) = 2n-3$$

$$\therefore E_{DH}(K_n) = 2n-3$$

Theorem 5.2. The minimum dominating hub energy of a complete bipartite graph $K_{n,n}$ for $n \geq 3$ is $(n+1)+(n-1)\sqrt{n}$

Proof: Let $K_{n,n}$ be a complete bipartite graph with vertex set $V(G) = \{u_1, u_2, u_3, \dots, u_n, v_1, v_2, v_3, \dots, v_n\}$. The minimum dominating hub set is $D = \{u_1, v_1\}$ and the minimum dominating hub number is 2. Then the minimum dominating hub matrix is as follows:

$$\mathbf{A_{DH}(K_{n,n})} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 1 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}_{\mathbf{2n\times 2n}}$$

The characteristic polynomial of $\mathbf{A_{DH}(K_{n,n})}$ is

$$\mathbf{f_n(K_{n,n},\lambda)} = \begin{pmatrix} \lambda-1 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 & -1 \\ 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & \cdots & -1 & -1 & \lambda-1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & \cdots & -1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \cdots & -1 & -1 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}_{\mathbf{2n\times 2n}}$$

$$= \lambda^{2n-4}\cdot\left(\lambda^2-(n+1)\lambda+(n-1)\right)\cdot\left(\lambda^2+(n-1)\lambda-(n-1)\right)$$

The eigenvalues of $K_{n,n}$ are as follows:

$\boldsymbol{\lambda = 0}$ with multiplicity $(\mathbf{2n-4})$, $\boldsymbol{\lambda} = \frac{(\mathbf{n+1})\pm\sqrt{\mathbf{n^2-2n+5}}}{\mathbf{2}}$ and $\boldsymbol{\lambda} = \frac{(\mathbf{1-n})\pm\sqrt{\mathbf{n^2+2n-3}}}{\mathbf{2}}$ with multiplicity 1 each.

The spectrum of $\mathbf{K_{n,n}}$ is as follows:

$$\mathrm{spec}(K_{n,n}) = \begin{pmatrix} \frac{(n+1)+\sqrt{n^2-2n+5}}{2} & \frac{(1-n)+\sqrt{n^2+2n-3}}{2} & \frac{(n+1)-\sqrt{n^2-2n+5}}{2} & 0 & \frac{(1-n)-\sqrt{n^2+2n-3}}{2} \\ 1 & 1 & 1 & (2n-4) & 1 \end{pmatrix}$$

The minimum dominating hub energy of $K_{n,n}$ is

$$E_{DH}(K_{n,n}) = \left|\frac{(n+1)+\sqrt{n^2-2n+5}}{2}\right| + \left|\frac{(1-n)+\sqrt{n^2+2n-3}}{2}\right| + \left|\frac{(n+1)-\sqrt{n^2-2n+5}}{2}\right| + 0 + \left|\frac{(1-n)-\sqrt{n^2+2n-3}}{2}\right|$$

$$= \frac{2(n+1)}{2} + \frac{2(n-1)\sqrt{n}}{2}$$

$$= (n+1) + (n-1)\sqrt{n}$$

$$\therefore E_{DH}(K_{n,n}) = (n+1) + (n-1)\sqrt{n}$$

Theorem 5.3. The minimum dominating hub energy of a star graph $K_{1,n-1}$ for $n \geq 2$ is equal to $\sqrt{4n-3}$

Proof: Let $K_{1,n-1}$ be a star graph with vertex set $V(G) = \{v_0, v_1, v_2, v_3, \dots, v_{n-1}\}$ where v_0 is the center vertex. The minimum dominating hub set is $D = \{v_0\}$ and the minimum dominating hub number is 1. Then the minimum dominating hub matrix is as follows:

$$\mathbf{A_{DH}(K_{1,n-1})} = \begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \cdots & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix}_{\mathbf{n \times n}}$$

The characteristic polynomial of $\mathbf{A_{DH}(K_{1,n-1})}$ is

$$\mathbf{f_n(K_{1,n-1}, \lambda)} = \begin{pmatrix} \boldsymbol{\lambda - 1} & \mathbf{-1} & \mathbf{-1} & \cdots & \mathbf{-1} \\ \mathbf{-1} & \boldsymbol{\lambda} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{-1} & \mathbf{0} & \boldsymbol{\lambda} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{-1} & \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{\lambda} \end{pmatrix}_{\mathbf{n \times n}}$$

$$= \boldsymbol{\lambda^{n-2}} \cdot \left(\boldsymbol{\lambda^2 - \lambda - (n-1)}\right)$$

The eigenvalues of $K_{1,n-1}$ are $\lambda = 0$ with multiplicity $(n-2)$ and $\lambda = \frac{1 \pm \sqrt{4n-3}}{2}$ with multiplicity 1 each.

The spectrum of a star graph $\mathbf{K_{1,n-1}}$ is as follows:

$$\mathrm{spec}(K_{1,n-1}) = \begin{pmatrix} \frac{1+\sqrt{4n-3}}{2} & 0 & \frac{1+\sqrt{4n-3}}{2} \\ 1 & (n-2) & 1 \end{pmatrix}$$

The minimum dominating hub energy of $K_{1,n-1}$ is

$$E_{DH}(K_{1,n-1}) = \left|\frac{1+\sqrt{4n-3}}{2}\right|(1) + 0 + \left|\frac{1-\sqrt{4n-3}}{2}\right|(1)$$

$$= \frac{2\sqrt{4n-3}}{2} = \sqrt{4n-3}$$

$$\therefore E_{DH}(K_{1,n-1}) = \sqrt{4n-3}$$

Definition 5.1. The double star graph $S_{n,m}$ is the graph constructed from $K_{1,n-1}$ and $K_{1,m-1}$ by joining their centers v_0 and u_0. A vertex set $V(S_{n,m}) = V(K_{1,n-1}) \cup V(K_{1,m-1}) = \{v_0, v_1, \dots, v_{n-1}, u_0, u_1, \dots, u_{m-1}\}$ and the edge set $E(S_{n,m}) = \{v_0u_0, v_0v_i, u_0u_j / 1 \leq i \leq n-1, 1 \leq$

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$j \leq m-1\}$

Theorem 5.4. The minimum dominating hub energy of the double star graph $S_{n,n}$ for $n \geq 3$ is equal to $2(\sqrt{n-1}+\sqrt{n})$

Proof: Let $S_{n,n}$ be a double star graph with the vertex set $V(G) = \{v_0, v_1, v_2, \dots, v_{n-1}, u_0, u_1, \dots, u_{n-1}\}$. The minimum dominating hub set is $D = \{v_0, u_0\}$ and the minimum dominating hub number is 1. Then the minimum dominating hub matrix is as follows:

$$\mathbf{A_{DH}(S_{n,n})} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \dots & 0 \end{pmatrix}_{2n\times 2n}$$

The characteristic polynomial of $\mathbf{A_{DH}(S_{n,n})}$ is

$$\mathbf{f_n(S_{n,n},\lambda)} = \begin{pmatrix} \lambda-1 & -1 & -1 & \cdots & -1 & -1 & 0 & 0 & \cdots & 0 \\ -1 & \lambda & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & \lambda & 0 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 & \lambda-1 & -1 & -1 & \cdots & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & \lambda & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 & \dots & \lambda \end{pmatrix}_{2n\times 2n}$$

$$= \boldsymbol{\lambda^{2n-4} \cdot (\lambda^2-(n-1)) \cdot (\lambda^2-2\lambda-(n-1))}$$

The eigenvalues of $S_{n,n}$ are as follows:

$\boldsymbol{\lambda = 0}$ with multiplicity $\mathbf{(2n-4)}$, $\boldsymbol{\lambda = \pm\sqrt{n-1}}$ and $\boldsymbol{\lambda = 1 \pm \sqrt{n}}$ with multiplicity 1 each.

The spectrum of $\mathbf{S_{n,n}}$ is as follows:

$$\operatorname{spec}(S_{n,n}) = \begin{pmatrix} 1+\sqrt{n} & \sqrt{n-1} & 0 & 1-\sqrt{n} & -\sqrt{n-1} \\ 1 & 1 & (2n-4) & 1 & 1 \end{pmatrix}$$

The minimum dominating hub energy of $S_{n,n}$ is

$$E_{DH}(S_{n,n}) = |1+\sqrt{n}|(1) + |\sqrt{n-1}|(1) + 0 + |1-\sqrt{n}|(1) + |-\sqrt{n-1}|(1)$$

$$= 2\sqrt{n-1} + 2\sqrt{n} = 2(\sqrt{n-1}+\sqrt{n})$$

$$\therefore E_{DH}(S_{n,n}) = 2(\sqrt{n-1}+\sqrt{n})$$

6. Properties on Minimum Dominating Hub Energy of Graphs

In this section, we introduce some properties of characteristic polynomials of minimum dominating hub matrix of a graph G and some properties of minimum dominating hub eigenvalues.

Theorem 6.1. Let $\mathbf{G = (n, m)}$ be a simple graph of order $\mathbf{n}$, degree $\mathbf{m}$ and dominating hub number $\mathbf{d(G)}$. Let $\mathbf{f_n(G,\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n}$ be the characteristic equation of minimum dominating hub eigenvalues. Then

(i) $\mathbf{a_0 = 1}$

(ii) $\mathbf{a_1 = -d(G)}$

(iii) $\mathbf{a_2 = \binom{d(G)}{2} - m}$

Proof: (i) Proof is followed by the definition of $f_n(G,\lambda)$.

(ii) W.K.T the sum of principal diagonal elements of $A_{DH}(G)$ is equal to $|D| = d(G)$, the sum of determinants of all 1×1 principal sub matrices of $A_{DH}(G)$ is the trace of $A_{DH}(G)$ which is equal to $d(G)$. Thus, we conclude $(-1)^1 a_1 = d(G)$ and hence $a_1 = -d(G)$.

(iii) $(-1)^2 a_2$ is equal to the sum of determinants of all 2×2 principal sub matrices of $A_{DH}(G)$. We have,

$$a_2 = \sum_{1\le i\le j\le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix}$$

$$= \sum_{1\le i\le j\le n} (a_{ii}a_{jj} - a_{ij}a_{ji})$$

$$= \sum_{1\le i\le j\le n} a_{ii}a_{jj} - \sum_{1\le i\le j\le n} a_{ij}^2$$

$$\therefore \mathbf{a_2 = \binom{d(G)}{2} - m}$$

Theorem 6.2. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of $A_{DH}(G)$ then

(i) $\sum_{i=1}^{n} \lambda_i = d(G)$

(ii) $\sum_{i=1}^{n} \lambda_i^2 = d(G) + 2m$

Proof: (i) W.K.T the sum of eigenvalues of $A_{DH}(G)$ is equal to the trace of $A_{DH}(G)$ then we have,

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii} = |D| = d(G) \ \ \forall \ i$$

$$\therefore \sum_{i=1}^{n} \lambda_i = d(G)$$

(ii) W.K.T the sum of squares of the eigenvalues of $A_{DH}(G)$ is the trace of $\left(A_{DH}(G)\right)^2$ then we have,

$$\sum_{i=1}^{n} \lambda_i^2 = \sum_{i=1}^{n} \lambda_i \cdot \sum_{j=1}^{n} \lambda_j = \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i \lambda_j$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}a_{ji} = \sum_{i=1}^{n}\sum_{j=1}^{n} (a_{ij}a_{ji} + a_{ji}a_{ij} + a_{ii}a_{jj})$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} (2a_{ij}a_{ji} + a_{ii}^2) = \sum_{i=1}^{n}\sum_{j=1}^{n} (2a_{ij}^2 + a_{ii}^2)$$

$$= 2\sum_{i<j}^{n} a_{ij}^2 + \sum_{j=1}^{n} a_{ii}^2 = 2m + |D|$$

$$\therefore \sum_{i=1}^{n} \lambda_i^2 = d(G) + 2m$$

Theorem 6.3. Let $G = (n, m)$ be a graph and let $\lambda_1(G)$ be the largest minimum dominating hub eigenvalues of $A_{DH}(G)$ then,

$$\lambda_1(G) \geq \frac{2m + d(G)}{n}$$

Proof: Let G be a graph and let λ_1 be the minimum dominating hub eigenvalues of $A_{DH}(G)$. Then from [2] we have, $\lambda_1 = \max_{X \neq 0} \left(\frac{X^T AX}{X^T X}\right)$ where X is any non-zero vector and X^T is its transpose and A is a matrix. If we take $X = J = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ then we get $\lambda_1 \geq \frac{J^T A_{DH}(G)\, J}{J^T\, J} = \frac{2m+d(G)}{n}$

7. Bounds on Minimum Dominating Hub Energy

In this section, we shall investigate some bounds for minimum dominating hub energy of graphs.

Theorem 7.1. Let G be a connected graph of order n and degree m. Then

$$\sqrt{2m + d(G)} \leq E_{DH}(G) \leq \sqrt{n\big(2m + d(G)\big)}$$

Proof: The Cauchy's-Schwarz inequality is

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right) \cdot \left(\sum_{i=1}^{n} b_i^2\right)$$

Put $a_i = 1$ and $b_i = |\lambda_i|$ in the above expression

$$\left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \leq \left(\sum_{i=1}^{n} 1\right) \cdot \left(\sum_{i=1}^{n} |\lambda_i|^2\right)$$

$$\big(E_{DH}(G)\big)^2 \leq n \cdot \left(\sum_{i=1}^{n} |\lambda_i|^2\right)$$

$$\big(E_{DH}(G)\big)^2 \leq n \cdot \left(\sum_{i=1}^{n} {\lambda_i}^2\right)$$

$\big(E_{DH}(G)\big)^2 \leq n\big(2m + d(G)\big)$ (Theorem 6.2. (ii))

$E_{DH}(G) \leq \sqrt{n\big(2m + d(G)\big)}$ ------------ (1)

Now consider,

$$\big(E_{DH}(G)\big)^2 = \sum_{i=1}^{n} |\lambda_i|^{\,2}$$

$\big(E_{DH}(G)\big)^2 \geq 2m + d(G)$ (Theorem 6.2. (ii))

$E_{DH}(G) \geq \sqrt{2m + d(G)}$ ------------- (2)

From (1) and (2) we conclude that

$$\sqrt{2m + d(G)} \leq E_{DH}(G) \leq \sqrt{n\big(2m + d(G)\big)}$$

Theorem 7.2. Let $G = (n, m)$ be a connected graph then

$$\sqrt{2n - m - 1} \leq E_{DH}(G) \leq n\sqrt{n - \frac{\Delta(G)}{n}}$$

Proof: By Lemma 4.1. we have, $\gamma(G) \leq D(G) + 1$ and by Theorem 4.2. we have, $D(G) \leq |V(G)| - \Delta(G)$. Thus, from both these results

$\gamma(G) - 1 \leq D(G) \leq n - \Delta(G)$--------------(*)

For any connected graph G we have, $2m \leq (n^2 - n)$ it follows by Theorem 7.1, that

$$E_{DH}(G) \leq \sqrt{n\big(2m + d(G)\big)}$$

$$\leq \sqrt{n[(n^2 - n) + n - \Delta(G)]}$$

$$\leq \sqrt{n(n^2 - \Delta)}$$

$$\leq \sqrt{n^2\left(n - \frac{\Delta(G)}{n}\right)}$$

$E_{DH}(G) \leq n\sqrt{n - \frac{\Delta(G)}{n}}$ -------------------(1)

For any connected graph G we have, $n \leq 2m$ (lower bound) Also by Theorem 7.1., Equation (*) and Theorem 4.1. we have,

$$E_{DH}(G) \geq \sqrt{2m + d(G)}$$

$$\geq \sqrt{n + \gamma(G) - 1}$$

$$\geq \sqrt{n + n - m - 1}$$

$E_{DH}(G) \geq \sqrt{2n - m - 1}$ -----------------(2)

From (1) and (2) we conclude that

$$\sqrt{2n - m - 1} \leq E_{DH}(G) \leq n\sqrt{n - \frac{\Delta(G)}{n}}$$

Theorem 7.3. Let $G = (n, m)$ be a graph then

$$E_{DH}(G) \leq \left(\frac{2m + d(G)}{n}\right) + \sqrt{(n-1)\left[2m + d(G) - \left(\frac{2m + d(G)}{n}\right)^2\right]}$$

Proof: The Cauchy's-Schwarz inequality is

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right) \cdot \left(\sum_{i=1}^{n} b_i^2\right)$$

Put $a_i = 1$ and $b_i = |\lambda_i|$ in the above inequality

$$\left(\sum_{i=2}^{n}|\lambda_i|\right)^2 \leq \left(\sum_{i=2}^{n}1\right)\cdot\left(\sum_{i=2}^{n}|\lambda_i|^2\right)$$

$$(E_{DH}(G)-\lambda_i)^2 \leq (n-1)(2m+d(G)-\lambda_1^2)$$

$$(E_{DH}(G)-\lambda_1) \leq \sqrt{(n-1)(2m+d(G)-\lambda_1^2)}$$

$$E_{DH}(G) \leq \lambda_1 + \sqrt{(n-1)(2m+d(G)-\lambda_1^2)}$$

Let $f(x) = x + \sqrt{(n-1)(2m+d(G)-x^2)}$

For decreasing function $f'(x) \leq 0$ we have,

$\Rightarrow 1 + \frac{1}{2\sqrt{(n-1)(2m+d(G)-x^2)}}(n-1)(-2x) \leq 0$

$1 - \frac{x(n-1)}{\sqrt{(n-1)(2m+d(G)-x^2)}} \leq 0$

$1 \leq \frac{x\sqrt{n-1}}{\sqrt{(2m+d(G)-x^2)}}$

$\frac{\sqrt{(2m+d(G)-x^2)}}{\sqrt{n-1}} \leq x$

$x \geq \frac{\sqrt{2m+d(G)}}{\sqrt{n-1}}$

Since $2m+d(G) \geq n$ we have,

$\sqrt{\frac{2m+d(G)}{n}} \leq \frac{2m+d(G)}{n} \leq \lambda_1$

$\Rightarrow f(\lambda_1) \leq f\left(\frac{2m+d(G)}{n}\right)$

$$\therefore E_{DH}(G) \leq \left(\frac{2m+d(G)}{n}\right) + \sqrt{(n-1)\left[2m+d(G)-\left(\frac{2m+d(G)}{n}\right)^2\right]}$$

Theorem 7.4. Let $G = (n, m)$ be a connected graph. If $\Delta = \det(A_{DH}(G))$ then

$$E_{DH}(G) \geq \sqrt{2m+d(G)+n(n-1)\Delta^{\frac{2}{n}}}$$

Proof: Consider

$$\left(E_{DH}(G)\right)^2 = \left(\sum_{i=1}^{n}|\lambda_i|^2\right) = \left(\sum_{i=1}^{n}|\lambda_i|\right)\cdot\left(\sum_{i=1}^{n}|\lambda_i|\right)$$

$$= \sum_{i=1}^{n}|\lambda_i|^2 + \sum_{i\neq j}|\lambda_i|\,|\lambda_j|$$

$$\left(E_{DH}(G)\right)^2 - \sum_{i=1}^{n}|\lambda_i|^2 = \sum_{i\neq j}|\lambda_i|\,|\lambda_j|$$

The inequality between the arithmetic and geometric means (AM-GM inequality) is

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$$\frac{1}{n(n-1)}\sum_{i\neq j}|\lambda_i|\,|\lambda_j| \geq \left(\prod_{i\neq j}|\lambda_i||\lambda_j|\right)^{\frac{1}{n(n-1)}}$$

$$\sum_{i\neq j}|\lambda_i|\,|\lambda_j| \geq n(n-1)\left(\prod_{i\neq j}|\lambda_i||\lambda_j|\right)^{\frac{1}{n(n-1)}}$$

Thus

$$\left(E_{DH}(G)\right)^2 - \sum_{i=1}^{n}|\lambda_i|^2 \geq n(n-1)\left(\prod_{i\neq j}|\lambda_i||\lambda_j|\right)^{\frac{1}{n(n-1)}}$$

$$\left(E_{DH}(G)\right)^2 \geq \sum_{i=1}^{n}|\lambda_i|^2 + n(n-1)\left(\prod_{i\neq j}|\lambda_i||\lambda_j|\right)^{\frac{1}{n(n-1)}}$$

$$= \sum_{i=1}^{n}|\lambda_i|^2 + n(n-1)\left(\prod_{i=1}^{n}\lambda_i\right)^{\frac{2}{n}}$$

$$\left(E_{DH}(G)\right)^2 \geq 2m + d(G) + n(n-1)\Delta^{\frac{2}{n}}$$

$$\therefore E_{DH}(G) \geq \sqrt{2m + d(G) + n(n-1)\Delta^{\frac{2}{n}}}$$

Theorem 7.5. Let G be a graph with a minimum dominating hub set $D(G)$. If the minimum dominating hub energy $E_{DH}(G)$ of G is rational number, then $E_{DH}(G) \equiv |D|(mod\ 2)$

Proof: Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be minimum hub eigenvalues of a graph G of which $\lambda_1, \lambda_2, \ldots, \lambda_p$ are positive and the remaining are negative then

$$\sum_{i=1}^{n}|\lambda_i| = \left(\lambda_1 + \lambda_2 + \cdots + \lambda_p\right) - \left(\lambda_{p+1} + \lambda_{p+2} + \cdots + \lambda_n\right)$$

$$= 2\left(\lambda_1 + \lambda_2 + \cdots + \lambda_p\right) - \left(\lambda_1 + \lambda_2 + \cdots + \lambda_n\right)$$

$$E_{DH}(G) = 2\left(\lambda_1 + \lambda_2 + \cdots + \lambda_p\right) - |D|$$

The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_p$ are algebraic integers implies that their sum is also an algebraic integer. Therefore $2\left(\lambda_1 + \lambda_2 + \cdots + \lambda_p\right)$ must be an integer if $E_{DH}(G)$ is rational. Hence the result.

8. Conclusion

The theory of domination and energy of graphs are both central topics in modern graph theory, with a wide range of applications in various fields such as network theory, computer science, chemistry, and even physics. In recent years many scholars are working in this area and also, they are introducing new domination parameters. In this paper we have initiated the domination parameter for hub energy. We have significant progress in our research by calculating the exact values of the minimum dominating hub energy for standard family graphs and establishing bounds for this parameter in terms of other graph properties such as the degree and order of the graph.

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