Tuijin Jishu/Journal of Propulsion Technology
ISSN: 1001-4055
Vol. 46 No. 1 (2025)

Solutions of Delay Differential Equations Oscillate with Positive Coefficients and Negative Coefficients

P. Sharadha [1], V. Dharmaiah [2]

[1]*Research Scholar, Department of Mathematics, University College of Science, Osmania University Hyderabad, India,*

[2]*Professor (Retd), Department of Mathematics, University College of Science, Osmania University Hyderabad, India,*

***Abstract*:**

In this paper we introduced, the first order linear delay differential equation with positive and negative coefficients

$$r'(t) + A(t)\, r(t - \tau) - B(t)\, r(t - \sigma) = 0, \quad (1.1)$$

where A, B are continuous functions with positive and negative real coefficients and τ, σ are non-negative constants.

The standard form of the above equation is

$$r'(t) + \sum_{i=1}^{k} [\, A_i(t) r(t - \tau_i) - B_i(t) r(t - \sigma_i) \,] = 0 \quad (1.2)$$

where A_i, $B_i \in ([t_1, \infty), R)$ and $\tau_i, \sigma_i \in [0, \infty)$, for i = 1 ,2,3,……,k. We consider a function

$r(t) \in ([t_1 - \delta), R_+)$ for some $t \geq t_1$ and t_1 is hold for conditions (1.1) or (1.2), where r(t) is a continuous function and $\delta = \max\{ \tau_i , \sigma_i \}$ and $1 \leq i \leq k$.

***Keywords*:** Oscillatory criteria, first order delay differential equation, positive coefficients and negative coefficients, eventually positive and eventually negative.

1.Introduction:

In this chapter, we introduced the first-order linear delay differential equation (DDE) of the type

$$r'(t) + A(t)\, r(t - \tau) - B(t)\, r(t - \sigma) = 0, \quad (1.1)$$

Here A and B are continuous real valued functions and τ and σ are non-negative constants. The generalization of equation (1.1) is

$$r'(t) + \sum_{i=1}^{k} [\, A_i(t)\, r(t - \tau_i) - B_i(t) r(t - \sigma_i) \,] = 0 \quad (1.2)$$

where A_i, B_i are real valued continuous functions and τ_i, σ_i are non-negative constants, i =1,2,…,k. By a solution of (1.1) or (1.2), we mean a function $r(t) \in C([t - \delta], R)$, where $\delta = \{ \tau, \sigma\}$ (or $\delta = \max_{1 \leq i \leq k} \{ \tau_i , \sigma_i \}$).

Qian and Ladas [8] have obtained the following well-known oscillatory criterian for equation (1.1)

$$\lim_{t \to \infty} inf \left\{ \int_{t-\delta}^{t} [\, A(s) - B(s + \sigma - \tau)\,]\, ds \right\} > \frac{1}{e}. \quad (1.3)$$

Elabbasy and Saker [3] have found the following oscillation criteria for the generalized equation (1.3)

$$\lim_{t \to \infty} inf \left\{ \int_{t-\delta}^{t} \sum_{i=1}^{k} [A_i(s) - B_i(s + \sigma_i - \tau_i)] \right\} > \frac{1}{e}. \quad (1.4)$$

Many authors have considered the DDE.

$$r'(t) + A(t)\, r(\sigma(t)) = 0. \quad (1.5)$$

Myshkis [12] has established that all the solutions of (1.5) oscillates, if

$$\lim_{t\to\infty} sup\,[t-\sigma(t)] < \infty, \quad \lim_{t\to\infty} inf\,\{A(t)\,[t-\sigma(t)]\} > \frac{1}{e} \tag{1.6}$$

In 1972, Ladas, Laksmikantham, and Papadakis [10] showed the same results.

Let us take that $\tau(t) = T$,

$$\lim_{t\to\infty} sup \int_T^t A(s)\,ds > 1 \tag{1.7}$$

In 1989, Ladas [7] and In 1982, kopadaze and canurija [6] replaced (1.7) by

$$\lim_{t\to\infty} inf \int_T^t A(s)\,ds > \frac{1}{e} \tag{1.8}$$

If the inequation

$$\int_T^t A(s)\,ds \leq \frac{1}{e}$$

(1.9) Is true eventually,

Hence, the solution to equation (1.5) is non-oscillatory.

In 1995, Elbert and stavrolakis [4] obtained the infinite integral conditions for the oscillations of (1.5) when

$$\int_T^t A(s)\,ds \geq \frac{1}{e} \qquad \lim_{t\to\infty} \int_T^t A(s)\,ds = \frac{1}{e} \tag{1.10}$$

They also proved that if

$$\sum_{i=1}^{\infty} \left\{\int_{t_i}^{t_{i+1}} A(s)\,\mathrm{ds}\right\} - \frac{1}{e} = \infty \tag{1.11}$$

then all solutions of equation (1.5) oscillate.

In 1996 Li [11] proved that $\int_T^t A(s)\,ds > \frac{1}{e}$ and the integral $t_1 \to \infty$ and $t_1 > 0$, when

$$\int_{t_1}^{\infty} A(t)\,[\,\int_T^t A(s)\,ds\,]\,\mathrm{dt} - \frac{1}{e} = \infty \tag{1.12}$$

then all solutions of equation (1.5) oscillate.

Sufficient conditions were established for the oscillation of DDE in 1996 by Domshlak and Stavorolakis [1].

$$r'(t) + A(t)\,r\,(t-\sigma) = 0 \tag{1.13}$$

also in 1999 Domshlak and Stavrolakis [2] and Jaros and Stavrolakis [5] considered the DDE

$$r'(t) + c_1(t)r(t-\sigma) + c_2(t)\,r\,(t-\tau) = 0 \tag{1.14}$$

Our objective in this chapter is to provide infinite-integral conditions for oscillations of equations (1.1) and (1.2) by using generalised characteristic equation and the function of

the form $\frac{r(t)}{r(t-\tau_i)}$.

2. Oscillatory Lemma's:

Lemma –1: Assume that A_j and σ_j for j = 1, 2,…,m be positive real valued continuous functions defined on the interval $[t_1, \infty)$. Then the differential inequation

$$r'(t) + \sum_{j=1}^{m} A_j(t)\,r\,(t - \sigma_j(t)) \leq 0, \tag{2.1}$$

which eventually has positive solutions, then the differential equation is

$$g'(t) + \sum_{j=1}^{m} A_j(t)g(t - \sigma_j(t)) = 0, \tag{2.2}$$

has an eventually positive solution.

Lemma– 2: Let

$$r'(t) + \sum_{j=1}^{m} G_j(t) r(t - \sigma_j(t)) = 0, \quad (2.3)$$

and Let $\lim_{t\to\infty} sup \int_t^{t+\sigma_j} G_j(s)ds > 0$ for j and let r(t) eventually have a positive solution to equation (2.3), for j.

$$\lim_{t\to\infty} inf \frac{r(t-\sigma_j)}{r(t)} < \infty \quad (2.4)$$

Lemma – 3: Let $t_1 \in R$ be a constant real value, and let r(t) be a continuous real-valued function that is defined on the interval $[t_1, \infty)$. If r(t) satisfies the inequality,

$$r(t) \leq \max_{t-\sigma\leq s\leq t} r(s) + b \text{ for all } t_1 \leq t \text{ and } b < 0,\ \sigma > 0$$

Then the function r(t) is can not be a non-negative function.

Lemma – 4: If the solution to (2.3) is eventually positive, then

$$\int_s^{s+\sigma_j} G_j(t)dt < 1, \quad j = 1,2,...,m \quad (2.5)$$

after a long delay, we find an eventually positive solution for (2.5).

3. Oscillatory solutions for (1.1):

We propose infinite integral conditions and show that all solutions of (1.1) are oscillate under the proposed conditions. We require the following lemma in order to prove our proposed theorem:

Lemma - 5: suppose that

(C_1) Let t_1 be a real constant. Let A, B are positive real-valued continuous functions defined on the interval $[t_0, \infty)$. Let τ, σ are non-negative constants and $\tau \geq \sigma$,

(C_2) $A(t) \geq B(t + \sigma - \tau)$, for $t \geq t_1 + \tau - \sigma$.

(C_3) $\int_{t-\tau}^{t-\sigma} B(s)ds \leq 1$ for $t_1 + \tau \leq t$

Assume that r(t) has an eventually positive solution to equation (1.1), and set

$$h(t) = r(t) - \int_{t-\tau}^{t-\sigma} (B(s+\sigma)r(s))\, ds, \quad \text{for } t_1 + \tau - \sigma \leq t \quad (3.1)$$

For contradiction, h(t) holds, is a non-increasing positive function, and satisfies the equation.

$$h'(t) + [A(t) - B(t + \sigma - \tau)]\, h(t - \tau) \leq 0. \quad (3.2)$$

we found the proof of this theorem in [9].

Theorem – 6: Assume that conditions (C_1), (C_2) and (C_3) of Lemma (1.5) hold.

Let $G(t) = A(t) - B(t + \sigma - \tau)$ then,

(C_4) $\int_t^{t+\tau} G(s)ds > 0$

(C_5) $\int_{t_0}^{\infty} G(t) \text{ In} \left[e \int_t^{t+\sigma} G(s)ds\right] dt = \infty.$

Then all solutions of (1.1) oscillate.

Proof: However, let's say that (1.1) has a final solution, where the function r(t) is nonnegative. From the previous theorem, we can conclude that the function h(t) is non-negative and holds for (3.2). The above theorem has determined the DDE. i..e,

$$g'(t) + [A(t) - B(t + \sigma - \tau)]\, g(t - \tau) = 0 \quad (3.3)$$

Finally, we get a non-negative solution. Let $\mu(s) = -g'(s)/g(s)$.

Then $\mu(s)$ is a positive and continuous function, then $\exists\, s_1 \geq s_0$ such that $g(s_1) > 0$

and

$$g(s) = g(s_1) e^{\left(-\int_{s_1}^{s} \mu(t)\, dt\right)} \quad (3.4)$$

Moreover, The generalised characteristic equation holds if and only if $\mu(s)$ is

$$\mu(s) = G(s)\, e^{\left(-\int_{s-\tau}^{s} \mu(t)\, dt\right)} \quad (3.5)$$

we can prove that

$$r + \frac{In(eu)}{u} \leq e^{ur} \quad \text{for } u > 0. \quad (3.6)$$

we determine that $P(s) = \int_{s}^{s+\tau} G(t)dt$ By using (3.6) we get

$$\mu(s) = G(s)\, e^{\left(P(s) \frac{1}{P(s)} \int_{s-\tau}^{s} \mu(t)\, dt\right)}$$

$$\geq G(s) \left[\frac{1}{P(s)} \int_{s-\tau}^{s} \mu(t)\, dt + \frac{In(e\, P(s))}{P(s)}\right] \quad (3.7)$$

or

$$\left(\int_{s}^{s+\tau} G(t)dt\right) \mu(s) - G(s) \int_{s-\tau}^{s} \mu(t)\, dt \geq G(s)\left(Ine \int_{s}^{s+\tau} G(t)dt\right) \quad (3.8)$$

Then $T_2 > T_1$,

$$\int_{T_1}^{T_2} \mu(s) \left(\int_{s}^{s+\tau} G(t)\, dt\right) ds - \int_{T_1}^{T_2} G(s) \left(\int_{s-\tau}^{s} \mu(t) dt\right) ds$$

$$\geq \int_{T_1}^{T_2} G(s) \left(Ine \int_{s}^{s+\tau} G(t)dt\right) ds \quad (3.9)$$

By changing the order of integrations, we get,

$$\int_{T_1}^{T_2} G(s) \left(\int_{s-\tau}^{s} \mu(t)\, dt\right) ds \geq \int_{T_1}^{T_2-\tau} \left(\int_{t}^{t+\tau} G(s)\, \mu(t) ds\right) dt. \quad (3.10)$$

Thus,

$$\int_{T_1}^{T_2} G(s) \left(\int_{s-\tau}^{s} \mu(t)\, dt\right) ds \geq \int_{T_1}^{T_2-\tau} \mu(t) \left(\int_{t}^{t+\tau} G(s) ds\right) dt. \quad (3.11)$$

There fore,

$$\int_{T_1}^{T_2} G(s) \left(\int_{s-\tau}^{s} \mu(t)\, dt\right) ds \geq \int_{T_1}^{T_2-\tau} \mu(s) \left(\int_{s}^{s+\tau} G(t) dt\right) ds. \quad (3.12)$$

Hence from equations (3.11) and (3.12), we get

$$\int_{T_1}^{T_2} \mu(s) \left(\int_{s}^{s+\tau} G(t) dt\right) ds - \int_{T_1}^{T_2-\tau} \mu(t) \left(\int_{t}^{t+\tau} G(s) ds\right) dt$$

$$\geq \int_{T_1}^{T_2} \mu(s) \left(\int_{s}^{s+\tau} G(t) dt\right) ds - \int_{T_1}^{T_2} G(s) \left(\int_{s-\tau}^{s} \mu(t)\, dt\right) ds. \quad (3.13)$$

From (3.9) and (3.13), follows that

$$\int_{T_2-\tau}^{T_2} \mu(s) \left(\int_{s}^{s+\tau} G(t) dt\right) ds \geq \int_{T_1}^{T_2} G(s) \left(Ine \int_{s}^{s+\tau} G(t) dt\right) ds. \quad (3.14)$$

However, by Lemma- 4, we have

$$\int_s^{s+\tau} G(t)dt < 1 \tag{3.15}$$

Then, by conditions (3.14) and (3.15), we get

$$\int_{T_2-\tau}^{T_2} \mu(s)ds \geq \int_{T_1}^{T_2} G(s)\,(\text{Ine}\int_s^{s+\tau} G(t)dt)\,ds. \tag{3.16}$$

Or

$$\text{In}\,\frac{g(T_2-\tau)}{g(T_2)} \geq \int_{T_1}^{T_2} G(s)\,(\text{Ine}\int_s^{s+\tau} G(t)dt)\,ds. \tag{3.17}$$

According to (C_5) and finally by lemma-2, we have

$$\lim_{t\to\infty} \frac{g(t-\tau)}{g(t)} \leq \infty. \tag{3.18}$$

It completes the proof and this is the contradiction for (1.1).

Hence, all solutions are oscillates for (1.1).

4. Oscillatory solutions for (1.2):

Our aim in this section is to propose infinite integral conditions and prove that all solutions of equation (1.2) oscillate. We require the following theorem.

Theorem - 7: Let

(D_1) A_i, B_i are continuous functions with real values on the interval $[t_1, \infty)$ and τ_i, σ_i are non-negative constants for i = 1, 2,…,k

(D_2) A partition of the set {1, 2,…,k} into q disjoint subsets $L_1, L_2,\dots,L_q$, such that $l \in L_i$ implies that $\sigma_i \leq \tau_i$,

(D_3) $A_i(t) \geq \sum_{l\in Li} B_l(t+\sigma_l-\tau_i)$ for $t \geq t_1 + \tau_i - \sigma_l$, and i = 1, 2,…,q.

(D_4) $\sum_{i=1}^{q}\sum_{l\in L_i}\int_{t-\tau_i}^{t-\sigma_l} B_l(s)\,ds \leq 1$ for $t \geq t_1 + \tau_i$.

Let r(t) be an eventually positive solution of (1.2), Define,

$$h(t) = r(t) - \sum_{i=1}^{q}\sum_{l\in L_i}\int_{t-\tau_i}^{t-\sigma_l} B_l(s+\sigma_l)\,r(s)ds, \quad t_1+\tau_i-\sigma_l \leq t \tag{4.1}$$

Consequently, h(t) is a positive and non-increasing function.

Proof: Let $t_2 \geq t_1 + \delta$, and r(t) is not a negative function for $t \geq t_2 - \delta$, where $\delta = \max_{1\leq i\leq k}\{\tau_i\}$.

From (3.1) we have

$$h'(t) = r'(t) - \sum_{i=1}^{q}\sum_{l\in L_i} B_l(t)\,r(t-\sigma_l) + \sum_{i=1}^{q}\sum_{l\in L_i} B_l(t+\sigma_l-\tau_i)r(t-\tau_i).$$

Hence

$$h'(t) = r'(t) - \sum_{i=1}^{k} B_i(t)r(t-\tau_i) + \sum_{i=1}^{q}\sum_{l\in L_i} B_l(t+\sigma_l-\tau_i)r(t-\tau_i).$$

from equation (1.2), we have

$$h'(t) = \sum_{i=1}^{q}\sum_{l\in L_i} B_l(t+\sigma_l-\tau_i)r(t-\tau_i) - \sum_{i=1}^{q}(A_i(t)r(t-\tau_i)) - \sum_{i=q+1}^{k}(A_i(t)r(t-\tau_i)). \tag{4.2}$$

we know that

$$\sum_{i=q+1}^{k}(A_i(t)r(t-\tau_i)) > 0 \tag{4.3}$$

Then

$$h'(t) \le -\left[\sum_{i=1}^{q} A_i(t) - \left(\sum_{l\in L_i} B_l(t+\sigma_l - \tau_i)x(t-\tau_i)\right)\right] \quad (4.4)$$

By using (D_3), we have $h'(t) \le 0$, for $t_2 + \delta \le t$. It follows that h(t) is a decreasing function. It is necessary to show that h(t) has no negative function.

If not, there is $t_3 \ge t_2$ such that $h(t_3) \le 0$. There is $t_4 > t_3$ such that $h(t) < h(t_4)$ for all $t \ge t_4$, since $h'(t) \le 0$ for $t \ge t_2 + \delta$ and $h'(t) \ne 0$ on $[t_2 + \delta, \infty)$. Accordingly, for $t \ge t_4$, it follows from (3.1).

$$r(t) = h(t) + \sum_{i=1}^{q}\sum_{l\in L_i}\int_{t-\tau_i}^{t-\sigma_l} B_l(s+\sigma_{nl})r(s)\,ds$$

$$\le h(t_4) + \sum_{i=1}^{q}\sum_{l\in L_i}\int_{t-\tau_i}^{t-\sigma_l} B_l(s+\sigma_l)r(s)\,ds$$

$$\le h(t_4) + \sum_{i=1}^{q}\sum_{i\in L_i}\int_{t-\tau_i}^{t-\sigma_l} B_l(s+\sigma_l)ds\left(\max_{t-\delta\le s\le t} r(s)\right). \quad (4.5)$$

Thus,

$$r(t) \le h(t_4) + \sum_{i=1}^{q}\sum_{l\in L_i}\int_{t-\tau_i}^{t-\sigma_l} B_l(s+\sigma_l)ds\left(\max_{t-\delta\le s\le t} r(s)\right). \quad (4.6)$$

Hypothesis (D_4) yields

$$r(t) \le h(t_4) + \max_{t-\delta\le s\le t} r(s) \quad \text{for all } t \ge t_4, \quad (4.7)$$ here $h(t_4) \le 0$.

According to Lemma-3, r(t) can't be a positive function on $[t_4, \infty)$. Thus, the contradicting $r(t) > 0$.

Therefore, h(t) is a decreasing and non-negative function.

Theorem – 8: Let (D_1), (D_2), (D_3) and (D_4) are true, $\lambda_q = \max\{\lambda_1, \lambda_2, \ldots, \lambda_q\}$,

$\sum_{i=1}^{q}\int_t^{t+\tau_i} G_i(s)ds > 0$ for $t \ge t_1$ for $t_1 > 0$. And

Let

(D_5) $\limsup_{t\to\infty} \int_t^{t+\lambda_q} G_q(s)ds > 0$

(D_6) $\int_{t_1}^{\infty}\left(\sum_{i=1}^{q} G_i(t)\right) In\left[e\sum_{i=1}^{q}\int_t^{t+\tau_i} G_i(s)ds\right]dt = \infty$

where $G_i(t) = A_i(t) - \sum_{l\in L_i} B_l(t+\sigma_l - \tau_i)$.

Hence, all the solutions of (1.2) oscillates.

Proof: On the contrary, by the theorem-5, we have the continuous function h(t) has a nonnegative integer, and it is defined by (4.1), and also equation (1.2) has non-negative solutions of r(t). Thus, we have

$$h'(t) + \sum_{i=1}^{q}[A_i(t) - \sum_{l\in L_i} B_l(t+\sigma_l - \tau_i)]\, r(t-\tau_i) \le 0. \quad (4.8)$$

Here $0 < h(t) \le r(t)$, where h(t) is a non-negative function with a constant coefficient.

$$h'(t) + \sum_{i=1}^{q}[A_i(t) - \sum_{l\in L_i} B_l(t+\sigma_l - \tau_i)]\, h(t-\tau_i) \le 0. \quad (4.9)$$

Then, by Lemma -1, we know that the DDE is

$$g'(t) + \sum_{i=1}^{q}[A_i(t) - \sum_{l\in L_i} B_l(t+\sigma_l - \tau_i)]\, g(t-\tau_i) \le 0. \quad (4.10)$$

finally yields a non-negative solution.

Let $\mu(s) = \frac{-g'(s)}{g(s)}$. Then $\mu(s)$ is a positive and continuous function, and there exists $s_1 \ge s_0$ with

$g(s_1) > 0$. such that

$$g(s) = g(s_1)\, e^{\left(-\int_{s_1}^{s} \mu(t)\, dt\right)}$$

Moreover, μ(s) holds for the generalized characteristic equation

$$\mu(s) = \sum_{i=1}^{q} G_i(s)\, e^{\left(\int_{s-\tau_i}^{s} \mu(t)\, dt\right)}$$

Here,

$$G_i(s) = A_i(s) - \sum_{l \in L_i}^{q} B_l(t + \sigma_l - \tau_i)$$

Let

$$Q(s) = \sum_{i=1}^{q} \int_{s}^{s+\tau_i} G_i(t)dt$$

From (3.6) we get

$$\mu(s) = \sum_{i=1}^{q} G_i(s)\, e^{\left(B(s)\frac{1}{B(s)}\int_{s-\tau_i}^{s} \mu(t)dt\right)}$$

$$\geq \sum_{i=1}^{q} G_i(s) \left[\frac{1}{B(s)} \int_{s-\tau_i}^{s} \mu(t)dt\right] + \sum_{i=1}^{k} G_i(s) \frac{In(eB(s))}{B(s)} \quad (4.11)$$

or

$$\mu(s) \sum_{i=1}^{q} \int_{s}^{s+\tau_i} G_i(t)dt - \sum_{i=1}^{q} G_i(s) \int_{s-\tau_i}^{s} \mu(t)dt$$

$$\geq \sum_{i=1}^{q} G_i(s)\left(\text{In e} \int_{s}^{s+\tau_i} G_i(t)dt\right) \quad (4.12)$$

Then for $T_2 > T_1$,

$$\int_{T_1}^{T_2} \mu(s)\left(\sum_{i=1}^{q} \int_{s}^{s+\tau_i} G_i(t)dt\right)ds - \int_{T_1}^{T_2} \sum_{i=1}^{q} G_i(s) \left(\int_{s-\tau_i}^{s} \mu(t)dt\right) ds$$

$$\geq \int_{T_1}^{T_2} \sum_{i=1}^{q} G(s)\left(In\ e \int_{s}^{s+\tau_i} G_i(t)dt\right)ds \quad (4.13)$$

By changing the order of integration, we get

$$\int_{T_1}^{T_2} \sum_{i=1}^{q} G_i(s) \left(\int_{s-\tau_i}^{s} \mu(t)dt\right) ds \geq \int_{T_1}^{T_2-\tau_i}\left(\int_{s}^{s+\tau_i} \sum_{i=1}^{q} G_i(t)\mu(t)dt\right)ds.$$

$$\geq \int_{T_1}^{T_2-\tau_i} \mu(t) \left(\int_{t}^{t+\tau_i} \sum_{i=1}^{q} G_i(s)ds\right)dt.$$

$$\geq \sum_{i=1}^{q} \int_{T_1}^{T_2-\tau_i} \mu(s) \left(\int_{s}^{s+\tau_i} G_i(t)dt\right) ds. \quad (4.14)$$

From (4.13) and (4.14), it follows that

$$\int_{T_1}^{T_2} \mu(s) \left(\sum_{i=1}^{q} \int_{s}^{s+\tau_i} G_i(t)\, dt\right) ds - \int_{T_1}^{T_2-\tau_i} \mu(s) \left(\int_{s}^{s+\tau_i} \sum_{i=1}^{q} G_i(t)\, dt\right)ds$$

$$\geq \int_{T_1}^{T_2} \sum_{i=1}^{k} G_i(s) In\left(e \sum_{i=1}^{q} \int_{s}^{s+\tau_i} G_i(t)\, dt \right)ds \quad (4.15)$$

Hence

$$\sum_{i=1}^{q} \int_{T_1-\tau_i}^{T_2} \mu(s) \left(\int_{t}^{t+\sigma_i} G_i(t)\, dt\right) ds$$

$$\geq \int_{T_1}^{T_2} \sum_{i=1}^{q} G_i(s) In\left(e \int_{s}^{s+\tau_i} \sum_{i=1}^{q} G_i(t)\, dt \right) ds. \quad (4.16)$$

By Lemma- 4, we know that

$$\int_{s}^{s+\tau_i} G_i(t)\, dt < 1, \quad (4.17)$$

Finally, by (4.16) and (4.17), we get

$$\sum_{i=1}^{q}\int_{T_1-\tau_i}^{T_2}\mu(s)\,ds \geq \int_{T_1}^{T_2}\left(\sum_{i=1}^{q}G_i(s)\right) In\left(e\int_{s}^{s+\tau_i}\sum_{i=1}^{q}G_i(t)dt\right) ds$$

Or

$$\sum_{i=1}^{q} In\,\frac{g(T_2-\tau_i)}{g(T_2)} \geq \int_{T_1}^{T_2}\left(\sum_{i=1}^{q}G_i(t)\right) In\left(e\int_{s}^{s+\tau_i}\sum_{i=1}^{q}G_i(t)\,dt\right)ds \qquad (4.18)$$

By (D_6) and lemma-2, we know that

$$\lim_{t\to\infty}\sum_{i=1}^{q}\frac{g(t-\tau_i)}{g(t)} = \infty \qquad (4.19)$$

That is,

$$\lim_{t\to\infty}\frac{g(t-\tau_q)}{g(t)} \leq \infty \qquad (4.20)$$

It completes the proof and this is the contradiction for (1.2).

Hence, all solutions of (1.2) is oscillate.

Conclusion:

In this chapter, in order to identify new oscillatory criteria, we give infinite integral conditions of first-order ordinary delay differential equations with positive and negative coefficients.

Reference:

[1] Domshlak. Y. and I. P. Stavrolakis, "Oscillation of first order delay differential equations in a critical state, Applicabl Analysis., Vol. 61(1996) 359-371.

[2] Domshlak.Y. and I. P. Stavrolakis. "Oscillation tests for delay equations", Rocky Mountain Journal of Mathematics, Vol. 29, no. 4 (1999), 1–11.

[3] Elabbasy. El. M., S. H. Saker, "Oscillation of solutions of delay differential equations" M.Sc. thesis, Mansoura University, Egypt, (1997).

[4] Elbert. A. and I. P. Stavrolakis, "Oscillation and non-oscillation criteria for delay differential equations", Proc. Amer. Math. Sco., Vol. 124, no. 5 (1995) 1503–1511.

[5] Jaros. J and I. P. Stavroulakis, "Oscillation tests for delay equations," The Rocky Mountain Journal of Mathematics, vol. 29, no. 1, pp. 197–207, 1999.

[6] Kopllatadze. R. G and T. A. Canturija, "On oscillatory and monotonic solution of first Order differential equations with deviating arguments", Differentialnye Uravenenija 18 (1982), 1463-1465, in Russian.

[7] Ladas. G and S. W. Schults, "On oscillations of neutral equations with mixed arguments," Hiroshima Mathematical Journal, vol. 19, no. 2, pp. 409–429, 1989.

[8] Ladas. G and C. Qian, "Oscillation in differential equations with positive and negative coefficients", Canad. Math. Bull. 33 (1990), 442–451.

[9] Ladas. G, Qain. C and J. Yan, "A comparison result for oscillation of delay differential Equations", Proc. Amer. Math. Soc. 114 (1992), 939-962.

[10] Ladas. G. S, Lakshmikantham. V, and L.S. Papadakis, "Oscillation of higher-order

retarded differential equations generated by the retarded arguments, in delay and functional differential equations and their applications", Academic Press, New Your, 1972.

[11] Li. B., "Oscillation of First Order Delay Differential Equations", Proc. Amer. Math. Sco., Vol. 124, (1996), 3729–3737.

[12] Myshkis. A. R., "Linear homogeneous differential equations of first order with Deviating Arguments", Uspehi Mat. Nauk 5 (1950), 160–162, in Russian.