Tuijin Jishu/Journal of Propulsion Technology
ISSN: 1001-4055
Vol. 46 No. 1 (2024)

On Global Dominator Coloring on Mycielskian Graphs

A. Mohammed Abid[*1], A. Mohammed Yasin[2] and V. Kamal Nasir[3]

[*1, 2, 3]*PG & Research Department of Mathematics, The New College, Chennai, Tamil Nadu 600 014, India*

(Affiliated to University of Madras, Chennai, Tamil Nadu 600 005, India)

Abstract:- A graph G that has proper vertex coloring with dom-color classes and anti-dom-color classes assigned to each vertex is said to have a global dominator coloring. Our goal in this study is to determine the global dominator chromatic number bounds for both Mycielskian graphs and iterated Mycielskian graphs. In addition, we describe the graphs that attain the bounds.

Mathematics Subject Classification: *05C15, 05C69*

Keywords*: Proper Coloring, Dominator Coloring, Global Dominator Coloring.*

1. Introduction

Every graph $G = (V, E)$ that is taken into consideration here is simple. We refer [4] for terms related to graph theory. Two important and extensively researched topics in graph theory are graph coloring and domination. The book by Chartrand and Zhang [1] provides an outstanding discussion of a number of graph coloring problems. We refer Haynes et al.'s work [2] for the fundamental concepts of domination. Numerous advanced domination problems are covered in the book edited by Haynes et al. [3].

When colors are applied to the vertices of a graph G such that neighboring vertices have different colors, this is defined to as proper vertex coloring. The smallest cardinality of all these colors is the chromatic number of a graph G, denoted as $\chi(G)$. Graph colorings and domination problems are frequently related [5]. A newly developed family of graph colorings was recently introduced, leading to domination relations between color classes and vertices. The colorings that fall under this category are strong coloring [12], strict strong coloring [7], dominator coloring [6] and global dominator coloring [10].

A graph G that has proper vertex coloring with dom-color classes and anti-dom-color classes are assigned to each vertex is said to have a global dominator coloring. The smallest cardinality of all such colors is called global dominator chromatic number of a graph G, denoted as $\chi_{gd}(G)$. By Hamid et al. [10], the notation of global dominator coloring was introduced. Throughout this paper the global dominator coloring is referred as gd-coloring. For any graph G, let $\mathcal{C} = \{V_i : 1 \le i \le k\}$ be a gd-coloring in which each V_i represents a color class. The gd-coloring with χ_{gd}-colors is called χ_{gd}-coloring of G. If each $v \in V$ has a dominating color class different from V_i, then the color class is referred to as $Pn(V_i, \mathcal{C}) = \phi$.

2. Mycielskian Graph

To construct a graph where the clique number is small and the chromatic number is high, Mycielski presented the Mycielskian graph $\mu(G)$ [11] in the following manner: "vertex set $V(\mu(G)) = V(G) \cup V' \cup \{u\}$ and edge set $E(\mu(G)) = E(G) \cup \{v_i v_j' : v_i v_j \in E(G)\} \cup \{u v_i' : 1 \le i \le n\}$, where $V = \{v_1, v_2, \dots, v_n\}$ and $V' = \{v_1', v_2', \dots, v_n'\}$ in which v and v' are called twin vertices and u is called a root vertex".

This section describes how we arrived at the global dominator chromatic number bounds for Mycielskian graphs and how the graphs that achieved these bounds were characterized.

Theorem 2.1. *For any graph G,*

$$\chi_{gd}(G) + 1 \le \chi_{gd}(\mu(G)) \le \chi_{gd}(G) + 2.$$

Furthermore, the gd-coloring $\mu(G)$ is equal to $\chi_{gd}(G)+1$ if and only if $Pn(V_i,\mathcal{C})=\phi$, for some i.

Proof.

Let any χ_{gd}-coloring of G be represented by $\mathcal{C}$. Then, $\chi_{gd}(\mu(G)) \le \chi_{gd}(G)+2$, since $\mathcal{C} \cup \{V'\} \cup \{u\}$ is a gd-coloring of $\mu(G)$ utilizing $\chi_{gd}(G)+2$ colors. Let $u \in V_1$ and let $\mathcal{C}=\{V_i : 1 \le i \le \chi_{gd}\}$ be a gd-coloring of $\mu(G)$.

Case 2.1. $V_1=\{u\}$.

Randomly select a vertex $v' \in V_i$ and its twin is recolored with color i for each color class $V_i \subset V'$ but not on V. This coloring is restricted to G, which results in a gd-coloring of G with $\chi_{gd}(G)-1$ colors.

Case 2.2. $V_1 \ne \{u\}$.

Clearly $V_1 \cap V(G) \ne \phi$. Let $S=V_1 \cap V(G)$. It is evident that every $v \in V(G)$ fails to dominate the color class V_1. Select a vertex $v' \in V_i$ at random and apply color i to its twin vertex, for each color class $V_i \subset V'$. For every $v \in S$, apply the twin vertex v' color. Let C_1 be the restricted coloring to G. Now, we claim that C_1 is a proper coloring of G. Assume $uv \in E(G)$. If $u,v \in V-S$, then u and v recieve distinct colors. In case, $u \in S$ and $v \notin S$, then also they receive different colors. Considering that $u's$ new color is the same as u', its twin. Hence, different colors are given to u and v. Consider a vertex $v \in V$ which dominates the color class V_i in $\mathcal{C}_1$ and anti-dominates some color class $V_j,\ j \ne i$ in $\mathcal{C}_1$. The vertex v continues to dominate the color class $V_i \in \mathcal{C}_1$ and anti-dominate the color class $V_j,\ j \ne i \in C_1$, since $\{i,j\} \ne 1$. Consequently, inequality holds since $\chi_{gd}(G) \le \chi_{gd}(\mu(G))-1$.

Let $\mathcal{C}=\{V_i : 1 \le i \le \chi_{gd}\}$ be a gd-coloring of G using χ_{gd}-colors such that, for some i, $Pn(V_i,\mathcal{C})=\phi$. Let $\mathcal{C}_1=(\mathcal{C}-V_i) \cup \{V'\} \cup \{V_i \cup u\}$ be a coloring of $\mu(G)$ in which color i is assigned to the root vertex u and twin vertices receive a single color. It is evident from $\mathcal{C}_1$ of $\mu(G)$ that every $v \in V(G)$ both dominates and anti-dominates some color class. The root vertex u dominate the color classes V' and anti-dominate some color class $V_i \in \mathcal{C}$ of G, while the twin vertices $v' \in V'$ dominate and anti-dominate those color classes as its twin vertex $v \in V(G)$. Since $\chi_{gd}(\mu(G)) \le \chi_{gd}(G)+1$, equality holds.

On the other hand, assume that $\chi_{gd}(\mu(G))=\chi_{gd}(G)+1$. Let $u \in V_1$ and let $\mathcal{C}=\{V_i : 1 \le i \le k\}$ be a χ_{gd}-coloring of $\mu(G)$, where $k=\chi_{gd}(G)+1$.

Case 2.3. $V_1=\{u\}$.

Assume $V_k=V'$. The coloring $\mathcal{C}_1=\mathcal{C}-(\{V_k\} \cup \{u\})$ of G is a gd-coloring utilizing at most $\chi_{gd}(\mu(G))-2$ colors. Which is not possible. So, let us assume that $V_k \subset V'$. Selecting a representation $v' \in V_i$ for any $V_i \subset V'$ and its twin v being recolored with color i; the other vertices are colored according to $\mathcal{C}$ of $\mu(G)$ yields a new coloring $\mathcal{C}_1$ that is obtained from $\mathcal{C}$. It is evident that $\mathcal{C}_1$ is a gd-coloring of G. We now claim that a color class V_i exists in $\mathcal{C}_1$ such that $Pn(V_i,\mathcal{C}_1)=\phi$.

Assume the case where each $v \in V$ fails to dominate the color class V_k. Then in the restricted coloring $\mathcal{C}_1, Pn(V_k,\mathcal{C}_1)=\phi$. Select a vertex $v_j' \in V_k$, if a vertex $v \in V(G)$ dominate the color class $V_k \subset V'$. Its twin vertex $v_j \in V_i$ is evident for some $i \ne \{1,k\}$. If the color class V_i is not dominated by any vertex, then $Pn(V_i,\mathcal{C}_1)=\phi$. Let us assume that the color class V_i is dominated by a vertex $v \in V(G)$. Thus, in this instance, both the color classes V_i and V_k are still dominated by the vertex $v \in V(G)$. Thus, $Pn(V_i,\mathcal{C}_1)=\phi$.

Case 2.4. $V_1 \ne \{u\}$.

It is evident no vertex $v \in V(G)$ dominate the color class $V_1 \in \mathcal{C}$. In this case, the root vertex u dominates some color class $V_k \subseteq V'$. Let $V_k=V'$. Then the restricted coloring $\mathcal{C}_1$ to G implies that $Pn(V_1,\mathcal{C}_1)=\phi$. For $V_k \subset V'$. Let us assume that the color class V_k is dominated by a vertex $v \in V(G)$. Consequently, the twin vertex v' cannot dominate the color class V_k. In this instance, v' dominate the color class V_1. We now take into consideration a restricted coloring $\mathcal{C}_1$ to G in the following manner: we randomly choose a vertex $v' \in V_i$ for each color class

$V_i \subset V'$, $i \neq k$, and its twin v is recolored with color i, $i \neq k$; the other vertices have the same color as $\mathcal{C}$ of $\mu(G)$. The restricted coloring is a gd-coloring of G with $\chi_{gd}(\mu(G)) - 1$ colors, where $Pn(V_1, \mathcal{C}_1) = \phi$.

3. Iterated Mycielskian Graph

In this section we give a bound for the global dominator chromatic number of a iterated Mycielsian graph $\mu^k(G)$ in terms of global dominator chromatic number of a graph G. The graphs that achieve the bounds are then characterized. Applying the Mycielskian operation k-times recursively for a graph G, we obtain the k-iterated Mycielskian graph $\mu^k(G)$, $k \geq 1$.

Theorem 3.1. *Let G be a graph, then*

$$\chi_{gd}(G) + k \leq \chi_{gd}(\mu^k(G)) \leq \chi_{gd}(G) + k + 1$$

where $k \geq 1$. Furthermore, the gd-coloring of $\mu^k(G)$ is equal to $\chi_{gd}(G) + k$ if and only if $Pn(V_i, \mathcal{C}) = \phi$, for some i.

Proof.

Suppose some χ_{gd}-coloring $\mathcal{C}$ of G has $Pn(V_i, \mathcal{C}) = \phi$, for some i. Then by applying Theorem 2.1 k-times, we have $\chi_{gd}(\mu^k(G)) = \chi_{gd}(G) + k$. If not, we have $\chi_{gd}(\mu(G)) = \chi_{gd}(G) + 2$ by Theorem 2.1. We now claim that the $\mu(G)$ of any graph G has a χ_{gd}-coloring such that $Pn(V_i, \mathcal{C}) = \phi$.

Suppose $\mathcal{C} = \{V_i : 1 \leq i \leq k\}$, is a χ_{gd}-coloring of G where $Pn(V_1, \mathcal{C}) = \phi$. Then the coloring $\mathcal{C}_1 = (\mathcal{C} - \{V_1\}) \cup \{V_1 \cup \{u\}\}$ is a gd-coloring of $\mu(G)$ utilizing χ_{gd} colors where $Pn(V_1, \mathcal{C}) = \phi$. Suppose $\mathcal{C}$ has no coloring in which $Pn(V_i, \mathcal{C}) = \phi$. Then, we have $\chi_{gd}(\mu(G)) = \chi_{gd}(G) + 2$, by Theorem 2.1. In this case, $\mathcal{C}_1 = \{V_i : 1 \leq i \leq k\} \cup \{V', \{u\}\}$ is a gd-coloring of $\mu(G)$ using χ_{gd} colors in which $Pn(V_1, \mathcal{C}) = \phi$. Then, we have $\chi_{gd}(\mu^k(G)) = \chi_{gd}(G) + k + 1$, applying Theorem 2.1 $k - 1$ times.

Next, we prove "if and only if" condition. Let $\chi_{gd}(\mu^k(G)) = \chi_{gd}(G) + k$. In case no χ_{gd}-coloring of G contains a color class V_i such that $Pn(V_i, \mathcal{C}) = \phi$. Then, by above argument $\chi_{gd}(\mu^k(G)) = \chi_{gd}(G) + k + 1$, which is a contradiction.

On the other hand, consider a case where a χ_{gd}-coloring of G has a color class V_i such that $Pn(V_i, \mathcal{C}) = \phi$. Thus, using the prior argumentation, $\chi_{gd}(\mu^k(G)) = \chi_{gd}(G) + k$.

Refrences

[1] G. Chartrand and Zhang P, *Chromatic graph theory, Chapman and Hall*, CRC, 2009.
[2] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of domination in graphs, Marcel Dekker Inc.*, (1998).
[3] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Domination in graphs - Advanced topics, Marcel Dekker Inc.*, (1998).
[4] G. Chartrand and Lesniak, *Graphs and Digraphs, Fourth Edition* (CRC Press, Boca Raton, 2005).
[5] M. Chellali and L. Volkmann, *Relations between the lower domination parameters and the chromatic number of a graph, Discrete Math. 274*, (2004) 1-8.
[6] R. Gera, *On Dominator coloring in graphs, Graph Theory Notes N. Y. 52* (2007) 25-30.
[7] M. Haddad and H. Kheddouci, *A strict strong coloring of trees, Inf. Process. Lett. 109* (2009) 1047-1054.
[8] A. Mohammed Abid and T. R. Ramesh Rao, *Dominator coloring of Mycielskian graph, Australasian Journal of Combinatorics*, Volume 73(2)(2019), pp. 274-279.
[9] A. Mohammed Abid and T. R. Ramesh Rao, *On strict strong coloring of graphs, Discrete Mathematics Algorithms and Applications*, (2021) 2150040.
[10] I. S. Hamid and M. Rajeswari, *Global dominator coloring of graphs, Discussion Mathematicae Graph Theory 39* (2019) 325-339.

[11] J. Mycielski, Sur le coloriage des graphes, *Colloq. Math.* **3** (1955), 161-162.
[12] I.E. Zverovich, *A new kind of graph coloring*, J. Algorithms **58** (2) (2006), 118-133.