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# Copmpleteness of Fuzzy Gamma-M-Normed Linear Space

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***Abstract:*** This paper introduces the conception of fuzzy gamma-m-normed linear space in accordance with the theory of fuzzy n-normed linear space and it is proved that in a finite dimensional fuzzy gamma m-normed linear space, fuzzy m-normed linear spaces are the same, up to the fuzzy norm equivalence. Also the paper introduces fuzzy gamma-2-normed linear spaces, fuzzy right gamma-m-normed linear spaces and its properties and fuzzy gamma-m-normed linear space which can be analyzed by using the fuzzy n-normed linear space. The paper uses an applications with examples for algebraic operations of fuzzy set theory. The most important concepts fuzzy gamma ring, fuzzy gamma divison ring, fuzzy gamma vector space, fuzzy gamma ring have already been introduced. Using these concepts. Using these concepts it initiated the fuzzy Gamma-m-normed linear space and suggested a theorem for the gamma norm function which is continuous. So far the earlier research has been done in the general t-norm in a fuzzy n-normed linear space and proven that if t-norm is chosen other than "minimum" then the decomposition theorem of a fuzzy norm into a family of crisp norms may not hold. The paper identified completeness of fuzzy gamma m-normed linear space and constructed a norm function and satisfies the axioms of it fuzzy gamma m-normed linear space and additionally provided an example with proof in which a sequence is a Cauchy sequence and converges sequence in fuzzy gamma m-normed linear space if and only if it is Cauchy sequence and Convergence sequence in completeness of fuzzy gamma m-normed linear space. This paper originates the notion of completeness, and produces some results on it in fuzzy Gamma-m-normed linear space. Also a necessary condition and theorem for completeness of a sequence in fuzzy gamma m-normed linear space is suggested.

***Keywords****:* fuzzy gamma ring, fuzzy division ring, fuzzy gamma vector space, fuzzy gamma linear space, fuzzy gamma normed linear space, fuzzy Gamma-2-normed linear space, fuzzy right gamma-m-normed linear space,

## 1. Introduction:

In 1987 the notion of a Gamma-ring which is more generalization than a ring was introduced by N.Nobusawa[1],and W.E.Barnes was developed the concepts in gamma-ring such as prime and primary ideals, gamma homomorphism's. after that many mathematicians determined the some interesting results on gamma-ring in accordance with concepts of Barnes and Nobusawa.[2] Gahler was introduced the theory of 2-norm,n-norm on a linear space[3] Bag and Samantha [4] introduced the notion of fuzzy norm on a linear space subsequently theory of fuzzy norm on a linear space[5,6,7]. First time A.K.Katsaras[8,9] introduced the concept of fuzzy normed linear space. M.Demirci [10,11] interposed the notion of fuzzy equality and smooth group by using fuzzy binary operations. The Fuzzy n-normed linear space introduced by AL.Narayanan and S.Vijayabalaji [12,13,14,15,16] and Reddy, B. S.[17,18] proposed the concept of Fuzzy anti n-normed linear space and provide some results convergence sequence and Cauchy sequence in fuzzy anti n-normed linear space. The notion of n-normed left gamma-linear space is convection by S Kalaiselvan and S.Shivaramakrishnan [19]. All the authors inspired by Lotfi A. Zadeh [20] in his fuzzy set theory.

Motivated by the previously mentioned theory, we present the concept of fuzzy gamma-m-normed linear spaces, denoting the convergent and cauchy sequence within these spaces. Additionally, we confirmed a few of the findings.

**2. PRILIMANARIES:**

**2.1. Definition:** An illustration of the degree to which each input contributes is provided by the characteristic function. Every processed input is assigned a weight, the functional overlap between the inputs is defined, and to ascertain their impact on the fuzzy output sets of the final output conclusion, the rules employ the input membership values as weight factors. A function that meets our needs for ease of use, speed, simplicity, and efficiency can be defined as an arbitrary curve whose shape we can describe. This function is then defuzzed into a crisp output that powers the system. In the continuous unit interval [0,1], where the end point values "0" and "1" denote different degrees of membership, Professor Zadeh expanded the concept of binary membership.

**2.2. Definition:** Fuzzy sets theory is an extension of classical set theory and elements have varying degree of membership a logic based on two truth values. If $\mathbb{U}$ is the universe of discourse, a set F is said to be fuzzy set in F if there exists a function $\mu: F \longrightarrow [0,1]$ and it is denoted by a set of ordered pairs as $F=\{(u, \mu_F(u)) / u \in \mathbb{U}\}$

**2.3. Definition:** Suppose $\mathbb{U}$ is a discrete and finite universe of discourse the fuzzy set F is written as

$F=\mu_F(u_1)/u_1+\mu_F(u_1)/u_1+\mu_F(u_1)/u_1+\ldots\ldots=\sum_{\mathbb{U}} \mu_F(u_1)=\{(u_\ell, \mu_F(u_\ell)) / u_\ell \in \mathbb{U}\}$

**2.4 .Definition:** Suppose $\mathbb{U}$ is a continuous and finite universe of discourse the fuzzy set F is written as$F=\int_{\mathbb{U}} \mu_F(u_\ell)$ ,here the summation and integration signs indicate the collection of all elements $u$ in the universe of discourse $\mathbb{U}$ along with their associated membership values $\mu_F(u_\ell)$

**2.5.Example:**$\mathbb{U}=\{H^1$=Finland,$H^2$=Luxembourg,$H^3$=Thailad,$H^4$=India,$H^5$=Turkiya,$H^6$=Ukraine,$H^7$=Zimbabwe,$H^8$=Afghanistan $\}$ is the universe of discourse of all countries then

(i) Let fuzzy set $F_1$ represents "Ranking in the World Happiness Report" if fuzzy set represented as $F_1$ ={(Finland,1.00), (Luxebourg,0.94),(Thailad,0.60),(India,0.14),(Tukiya,0.33),(Ukraine,0.29),(Zimbabwe,0.07), (Afghanistan , 0.00) } or $F_1=\{(H^1, 1.00), (H^2, 0.94), (H^3, 0.60), (H^4, 0.14), (H^5, 0.33), (H^6, 0.29), (H^7, 0.07), (H^8, 0.00)\}$ also it can be represented in summation form

$F_1$=1.00/Finland+ 0.94/Luxembourg + 0.60Thailad + 0.14/India + 0.33Turkiya

+ 0..29/Ukraine+0.07/Zimbabwe+0.00/Afghanistan Or

$F_1 = 1.00/H^1+0.94/H^2 + 0.60/H^3+ 0.14/H^4 + 0.33/H^5 + 0.29/H^6+ 0.07/H^7 +0.00/H^8$

(ii) Let fuzzy set $F_2$ represents "Ranking in the Human Development Report " then the fuzzy set represented as $F_2$={(Finland, 0.99), (Luxembourg, 0.89),( Thailad, 0.82),(India, 0.30), (Turkiya, 0.76), (Ukraine, 0.48), (Zimbabwe, 0.17), (Afghanistan , 0.05) } $==\{(H^1, 0.99), (H^2, 0.89), (H^3, 0.82), (H^4, 0.30), (H^5, 0.76),$

$(H^6, 0.48), (H^7, 0.17), (H^8,0.05)\}$ also it can be represented in summation form $F_2$=0.99/Finland+0.89/Luxembourg + 0.82/ Thailad + 0.30/India + 0.76/ Turkiya

+ 0.48/Ukraine+0.17/Zimbabwe +0.05/Afghanistan

$=0.99/H^1+0.89/H^2 + 0.82/H^3+ 0.30/H^4 + 0.76/H^5 + 0.48/H^6+ 0.17/H^7 +0.05/H^8$

**2.6. Definition:** If $\mathbb{U}$ is the universe of discourse, a set $F_1$and $F_2$ fuzzy sets with characteristic functions $\mu_{F1}(u)$ and $\mu_{F2}(u)$ respectively the fuzzy set operations are defined as given bellow

(1) Union: the union of two fuzzy sets $F_1$and $F_2$ is defined as

$\mu_{F1\cup F2}(u)=\mu_{F1}(u) \vee \mu_{F2}(u)=\max\{\mu_{F1}(u), \mu_{F2}(u)\}$, for all $u \in \mathbb{U}$

(2) Intersection: the intersection of two fuzzy sets $F_1$and $F_2$ is defined as for all $u \in$ ,

$\mu_{F1\cap F2}(u)=\mu_{F1}(u) \wedge \mu_{F2}(u)=\min\{\mu_{F1}(u), \mu_{F2}(u)\}$

(3) Compliment: the compliment of F is denoted by $F^C$ and it is defined as $\mu_{F^c}(u)=1-\mu_F(u)$

(4) Algebraic Sum: the sum of the two fuzzy sets $\mu_{F1}(u)$ , $\mu_{F2}(u)$ is $\mu_{F1}(u)+\mu_{F2}(u)$ it is defined as

$\mu_{F1+F2}(u) = \mu_{F1}(u) + \mu_{F2}(u) - \mu_{F1}(u) .\mu_{F2}(u)$

(5)Algebraic Product: The product of the two fuzzy sets is $\mu_{F1}(u).\mu_{F2}(u)$ it is defined as

$\mu_{F1.F2}(u) = \mu_{F1}(u) .\mu_{F2}(u)$

(6) Bounded Sum : the bounded sum of the two fuzzy sets $\mu_{F1}(u)$ , $\mu_{F2}(u)$ is $\mu_{F1}(u)\oplus\mu_{F2}(u)$ it is defined as $\mu_{F1\oplus F2}(u) = \min\{1, \mu_{F1}(u) + \mu_{F2}(u)\}$

(7) Bounded difference: the bounded difference $\ominus$ of the two fuzzy sets $\mu_{F1}(u)$ , $\mu_{F2}(u)$ is $\mu_{F1}(u) \ominus \mu_{F2}(u)$ it is defined as $\mu_{F1\ominus F2}(u) = \max\{0, \mu_{F1}(u) - \mu_{F2}(u)\}$

**Table -1. Outcomes of the operations on fuzzy sets.**

| | $H^1$ | $H^2$ | $H^3$ | $H^4$ | $H^5$ | $H^6$ | $H^7$ | $H^8$ |
|---|---|---|---|---|---|---|---|---|
| $\mu_{F1\cup F2}$ | 1.00 | 0.94 | 0.82 | 0.30 | 0.76 | 0.48 | 0.17 | 0.05 |
| $\mu_{F1\cap F2}$ | 0.99 | 0.89 | 0.60 | 0.14 | 0.33 | 0.29 | 0.07 | 0.00 |
| $\mu_{F1}{}^c$ | 0.00 | 0.06 | 0.18 | 0.70 | 0.24 | 0.52 | 0.83 | 0.95 |
| $\mu_{F1+F2}$ | 1.00 | 0.99 | 0.928 | 0.39 | 0.83 | 0.63 | 0.22 | 0.05 |
| $\mu_{F1.F2}$ | 0.99 | 0.83 | 0.49 | 0.04 | 0.25 | 0.13 | 0.01 | 0.00 |
| $\mu_{F1\oplus F2}$ | 1.00 | 1.00 | 1.00 | 0.44 | 1.00 | 0.77 | 0.24 | 0.05 |
| $\mu_{F1\ominus F2}$ | 0.01 | 0.05 | 0.22 | 0.16 | 0.43 | 0.18 | 0.10 | 0.05 |

**2.7.Definition:** Let $F_G$ be any group and mapping $\mu_{FG} : F_G \longrightarrow [0,1]$ is called fuzzy subgroup

if for all $f_{g1}, f_{g2} \in F_G$

(1) $\mu_{FG}(f_{g1} . f_{g2}) \geq \min \{ \mu_{FG}(f_{g1}) ,\mu_{FG}(f_{g2}) \}$

(2) $\mu_{FG}((f_{g1})^{-1}) = \mu_{FG}(f_{g1})$

(3) Let $f_{ge}$ be the identity element of fuzzy group $F_G$ such that $\mu_{FG}((f_g) \leq \mu_{FG}(f_{ge})$,for all $f_g \in F_G$

**2.8.Definition:** Let $F_R$ is fuzzy additive abelian group, and $F_\Gamma$ be any additive group the mapping

$\mathcal{Z}_{\mathcal{R}}: F_R \times F_\Gamma \times F_R \rightarrow F_R$ and it is defined as $\mathcal{Z}_{\mathcal{R}}(f_{r1}, f_\gamma, f_{r2}) = f_{r1}.f_\gamma.f_{r2}$ such that $F_R$ is called as fuzzy Gamma ring if it satisfies the following properties let for any $f_{r1}, f_{r2}$ $f_{r3}$, $\in F_R$ and $f_{\gamma1}, f_{\gamma2}$ and $f_\gamma \in F_\Gamma$

(1) $\mathcal{Z}_{\mathcal{R}}(f_{r1}+f_{r2},f_\gamma,f_{r3}) = \mathcal{Z}_{\mathcal{R}}(f_{r1}, f_\gamma,f_{r2})+ \mathcal{Z}_{\mathcal{R}}(f_{r2}, f_\gamma ,f_{r3})$

(2) $\mathcal{Z}_{\mathcal{R}}(f_{r1},f_{\gamma1}+f_{\gamma2},f_{r2})=\mathcal{Z}_{\mathcal{R}}(f_{r1},f_{\gamma1},f_{r2})+ \mathcal{Z}_{\mathcal{R}}(f_{r1}, f_{\gamma2},f_{r2})$

(3) $\mathcal{Z}_{\mathcal{R}}(f_{r1},f_\gamma,f_{r2}+f_{r3}) = \mathcal{Z}_{\mathcal{R}}(f_{r1}, f_\gamma ,f_{r2})+ \mathcal{Z}_{\mathcal{R}}(f_{r1}, f_\gamma ,f_{r3})$

(4) $\mathcal{Z}_{\mathcal{R}}((f_{r1}, f_{\gamma1},f_{r2}), f_{\gamma2},f_{r3}) = \mathcal{Z}_{\mathcal{R}}(f_{r1}, f_{\gamma1},(f_{r2}, f_\gamma ,f_{r3}))$

**2.9.Definition:** Let $\mu_{\mathcal{LI}}$ be any non-empty fuzzy sub set of fuzzy Gamma ring $F_R$ is said to be fuzzy left ideal of $F_R$ if it satisfies the following properties as fuzzy Gamma if for all

$f_{r1}, f_{r2}$ $f_{r3}$, $\in F_R$ and $f_\gamma \in F_\Gamma$ such that

(i) $\mu_{\mathcal{LI}}(f_{r1} - f_{r2}) \geq \min\{\mu_{\mathcal{LI}}(f_{r1}), \mu_{\mathcal{LI}}(f_{r2})\}$

(ii) $\mu_{\mathcal{LI}}(f_{r2} + f_{r1} - f_{r2}) \geq \mu_{\mathcal{LI}}(f_{r1})$

(iii)$\mu_{\mathscr{G}\mathscr{J}}(f_{r2}f_\gamma(f_{r1}+f_{r3})-f_{r2}\,f_\gamma f_{r3}) \geq \mu_{\mathscr{G}\mathscr{J}}(f_{r1})$

**2.10. Definition:** Let $(F\nu,+,\times)$ is a vector space over a field $\mathcal{K}\nu$ and mapping $\mu\nu$: $F\nu \longrightarrow [0,1]$ is a fuzzy set of $F\nu$ is said to be fuzzy vector subspace of $\mathcal{K}\nu$ if it satisfies the following properties for all $f_{\nu1}, f_{\nu2} \in F\nu$ and $k_\nu \in \mathcal{K}\nu$

(1) $\mu(f_{\nu1}, f_{\nu2}) \geq \min\{\mu\nu(f_{\nu1}), \mu\nu(f_{\nu2})\}$
(2) $\mu((f_{\nu1})^{-1}) \geq \mu\nu(f_{\nu1})$
(3) $\mu(k_\nu\, f_{\nu1}) \geq \mu\nu(f_{\nu1})$

**2.11.Definition:** If $\mu\nu$: $F\nu \longrightarrow [0,1]$ is a fuzzy vector subspace of $F\nu$ over a field $\mathcal{K}\nu$ if and only if $\mu\nu(k_{\nu1}f_{\nu1}+k_{\nu2}f_{\nu2}) \geq \min\{\mu\nu(f_{\nu1}), \mu\nu(f_{\nu2})\}$,for all $f_{\nu1}, f_{\nu2} \in F\nu$ and for all $k_{\nu1}, k_{\nu2} \in \mathcal{K}\nu$

**2.12. Definition:** If the fuzzy Gamma-ring and $F_D$ have an identity element and just one non-zero ideal, then they are referred to as division fuzzy Gamma-rings.

**2.13. Definition:** Let $F\nu$ is fuzzy vector space if $\widetilde{f_{v1}}(m)$=V for all $m \in \mathscr{M}$ then a mapping $\|.,.\|$: $F\nu \longrightarrow [0,1]$ is said to be a fuzzy norm on the soft vector space $F\nu$ if $\|.,.\|$ is satisfies the following properties

(1) $\|\widetilde{f_{v1}}\| \geq \tilde{0}$ , for all $\widetilde{f_{v1}} \in F\nu$

(2) $\|\widetilde{f_{v1}}\| = \tilde{0} \Leftrightarrow \widetilde{f_{v1}} = \tilde{0}$

(3) $\|\,.\,\widetilde{f_{v1}}\| = |k_\nu|\,\|\widetilde{f_{v1}}\|$ , for all $\widetilde{f_{v1}} \in F\nu$ and for every soft scalar $k_\nu \in \mathcal{K}\nu$

(4)$\|\widetilde{f_{v1}} + \widetilde{f_{v2}}\| \leq \|\widetilde{f_{v1}}\| + \|\widetilde{f_{v2}}\|$, for all $\widetilde{f_{v1}}, \widetilde{f_{v2}} \in F\nu$

The fuzzy vector space F with fuzzy norm $\|.,.\|$ on $F\nu$ is said to be a fuzzy normed linear space and is denoted by $(F\nu\ \|.,.\|, \mathscr{M})$. (1),(2),(3) and (4) are called to be fuzzy norm axiom .

**2.14. Definition:** Let $F_D$ be a fuzzy division Gamma-ring possessing identification 1and let $(F\nu,+)$ be a fuzzy abelian group. and the function $\mathcal{Z}\nu$ :$F_D$x $F_\Gamma$x $F\nu \longrightarrow F\nu$ and it is defined as $\mathcal{Z}\nu(f_v, f_\gamma, f_d) = f_v.\, f_\gamma.f_d$ then $F\nu$ is called as a right fuzzy Gamma-vector space over $F_D$ if the following properties holds for every $f_{v1}, f_{v2} \in F\nu$, $f_{d1}$, $f_{d2} \in F_D$ and $f_{\gamma1}$, $f_{\gamma2}$ and $f_\gamma \in F_\Gamma$

$(F_\gamma\text{-}V^1)$ $(f_{v1}+f_{v2}, f_\gamma, f_{d1}) = \mathcal{Z}(f_{v1}, f_\gamma, f_{d1}) + \mathcal{Z}\nu(f_{v2}, f_\gamma, f_{d1})$

$(F_\gamma\text{-}V^2)$ $(f_{v1}, f_\gamma, f_{d1}, f_{d2}) = \mathcal{Z}(f_{v1}, f_\gamma, f_{d1}) + \mathcal{Z}\nu(f_{v1}, f_\gamma\, f_{d2})$

$(F_\gamma\text{-}V^3)$ $(f_{v1}, f_{\gamma1}, f_{d1}, f_{\gamma2}, f_{d2})) = \mathcal{Z}((f_{v1}, f_{\gamma1}, f_{d1},), f_{\gamma2}, f_{d2})$

$(F_\gamma\text{-}V^4)$ $(f_{v1}, f\gamma, 1) = f_{v1}$, for some $f_\gamma \in F_\Gamma$, the elements $f_{v1}, f_{v2}$ are called fuzzy vectors in $F\nu$, $f_{d1}, f_{d2}$ are called fuzzy scalars in $F_D$.

**2.15.Definition:** The unit closed interval [0, 1] and $I_f$ be any closed sub-interval of [0,1] and is defined by

$I_f = [I_f^- \, I_f^+]$ where $0 \leq I_f^- \leq I_f^+ \leq 1$ suppose $\mathcal{C}[0,1]$ be the set of all closed sub interval of [0,1]

that is $\mathcal{C}[0,1] = \{ I_f / I_f = [I_f^- \, I_f^+] ,\ I_f^-, \leq I_f^+$ and $I_f^-, I_f^+ \in [0,1]\}$

**2.16.Definition:** Consider F be any set and a mapping $F^I$:$F \rightarrow \mathcal{C}[0,1]$ and the set is represented as

$\{F^I(\mathfrak{f}) = [F^{I-}(\mathfrak{f}), F^{I+}(\mathfrak{f})] / F^{I-}, F^{I+}$ are fuzzy subsets of F $, F^{I-}(\mathfrak{f}) \leq F^{I+}(\mathfrak{f})$, for all $\mathfrak{f} \in F\}$ and this set is also called as interval-valued fuzzy subset of F.

**2.17. Definition** Let $(F\nu,+,X)$ is a vector space over a field $\mathcal{K}\nu$ with dimension m and $\mu$ is a fuzzy subset of $F\nu$ such that $\mu\nu(k_{\nu1}\, f_{\nu1}+k_{\nu2}\, f_{\nu2}) \geq \min\{\mu\nu(f_{\nu1}), \mu\nu(f_{\nu2})\}$ , for all $f_{\nu1}, f_{\nu2} \in F\nu$ and for all $k_{\nu1}, k_{\nu2} \in \mathcal{K}\nu$

**2.18. Definition:** Let $\Delta^F$ be a binary operation and is mapping from [0,1]x[0,1] to [0,1]

that is $\Delta^F$ :[0,1]x[0,1] $\rightarrow$[0,1] is said to be continuous triangular-norm or t-norm if it satisfies the following axioms

(i)$\Delta^F$ is associative and commutative

That is for every $f^1, f^2, f^3 \in [0,1]$ such that $\Delta^F((f^1, f^2), f^3) = \Delta^F(f^1, (f^2, f^3))$ and

$\Delta^F(f^1, f^2) = \Delta^F(f^2, f^1)$

(ii) $\Delta^F$ is continuous

(iii) $\Delta^F(f^1, 1) = f^1$, for all $f^1 \in [0,1]$

(iv) $\Delta^F(f^1, f^2) \leq \Delta^F(f^3, f^4)$ whenever $f^1 \leq f^3$ and $f^2) \leq f^4$, for all $f^1, f^2, f^3, f^4 \in [0,1]$

**2.19. Example:** We have two examples of continuous t-norm $\Delta^F(f^1, f^2) = f^1 . f^2$ $\Delta^F(f^1, f^2) = \min\{f^1, f^2\}$.

**2.20.Remark:** For any $f^1, f^2 \in (0,1)$ with $f^1 > f^2$ there exists $f^3, f^4 \in (0,1)$ such that $\Delta^F(f^1, f^3) \geq f^2$ and

for any $f^4 \in (0,1)$ there exists $f^5, f^6 \in (0,1)$ such that $\Delta^F(f^5, f^6) \geq f^4$, $\Delta^F(f^1, f^3) \geq f^2$

**2.21. Definition:** Let $F_L$ linear space over a fuzzy field F a real valued function $\|,...,\|: F_L \times F_L \rightarrow [0,1]$ and it satisfies the following properties

(1) $\|v_{r1}, v_{r2}\| = 0$ if and only $v_{r1}, v_{r2}$ are linearly independent over F.

(2) $\|v_{r1}, v_{r2}\| = \|v_{r2}, v_{r1}\|$

(3) $\|v_{r1}, kv_{r2}\| = k\|v_{r1}, v_{r2}\|$

(4) $\|v_{r1}, v_{r2} + v_{r3}\| \leq \|v_{r1}, v_{r2}\| + \|v_{r1}, v_{r3}\|$

Is called the soft m-norm on $F_L$ and the pair $(F_L, \|..,.\|)$ is called the fuzzy 2-normed linear space.

**2.22. Definition:** Let $F_{\mathbb{R}}$ be a real vector space over with dimension m over a field F a real valued

function $\|.,\ldots,.\|: F_{\mathbb{R}} \rightarrow [0,1]$ and it satisfies the following properties

(1) $\|v_{r1}, v_{r2}, v_{r3}, \ldots, v_{rm-1}, v_{rm}\| = 0$, if and only if $v_{r1}, v_{r2}, v_{r3}, \ldots, v_{rm-1}, v_{rm}$ are linearly independent over F.

(2) $\|v_{r1}, v_{r2}, v_{r3}, \ldots, v_{rm-1}, v_{rm}\|$ is invariant under any permutation.

(3) $\|v_{r1}, v_{r2}, v_{r3}, \ldots, v_{rm-1}, kv_{rm}\| = k\|v_{r1}, v_{r2}, v_{r3}, \ldots, v_{rm-1}, v_{rm}\|$

(4) $\|v_{r1}, v_{r2}, v_{r3}, \ldots, v_{rm-1}, v_{rm} + v'_r\| \leq \|v_{r1}, v_{r2}, v_{r3}, \ldots, v_{rm-1}, v_{rm}\| + \|v_{r1}, v_{r2}, v_{r3}, \ldots, v_{rm-1}, v'_{rm}\|$ is called the m-norm on $F_{\mathbb{R}}$ and the pair $(F_{\mathbb{R}}, \|.,\ldots,.\|)$ is called the m-normed linear space.

**2.23.Definition :** Let $F_{\mathcal{V}}$ be a linear space in a field F ,Fuzzy m-normed linear space is defined as a fuzzy subset $F_{\mathcal{N}}$ of $F_{\mathcal{V}}$ x $F_{\mathcal{V}}$ x…x $F_{\mathcal{V}}$ x $F_{\mathcal{V}}$ (m-times)x$(-\infty,\infty)$ and the pair $(F_{\mathcal{V}}, F_{\mathcal{N}})$.

$F_{\mathcal{V}}$ x $F_{\mathcal{V}}$ x…x $F_{\mathcal{V}}$ x $F_{\mathcal{V}}$ (m-times)x$(-\infty,\infty)$ is referred to as a fuzzy m-norm on $F_{\mathcal{V}}$ if and only if.

(1) $F_{\mathcal{N}}(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vm-1}, f_{vm}, f_t) \geq 0$, for all $f_t \in (-\infty,\infty)$.

(2) $F_{\mathcal{N}}(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vm-1}, f_{vm}, f_t) = 0$ if and only if $f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vm-1}, f_{vm}$, are linearly dependent if for all $f_t \geq 0$, and $f_t \in (-\infty,\infty)$

(3) $F_{\mathcal{N}}(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vm-1}, f_{vm}, f_t)$ is invariant under any permutation of $f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vm-1}, f_{vm}, f_t$.

(4) $F_{\mathcal{N}}(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vm-1}, f_{\Delta} f_{vm}, f_t) = F_{\mathcal{N}}(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vm-1}, f_{vm}, f_t, \frac{f_t}{|f_{\Delta}|})$, where $f_{\Delta} \in F$

(5) $F_{\mathcal{N}}(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vm-1}, f_{vm1} + f_{vm2}, f_{t1} + f_{t2}) \geq \min\{F_{\mathcal{N}}(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vm-1}, f_{vm1}, f_{t1}), F_{\mathcal{N}}(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vm-1}, f_{vm2}, f_{t2})\}$

(6) $F(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vm-1}, f_{vm}, f_t)$ is a left continuous and non-decreasing function of $f_t \in (-\infty,\infty)$ such that

$F(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vm-1}, f_{vm}, f_t) = 0$.

**2.24.Definition:** Let $(F\nu,\|.,\ldots,.\|)$ be a fuzzy m-normed linear space and a sequence $\{f_{vr}\}_{r=1}^{\infty}$ in $(F\nu,\|.,\ldots,.\|)$ is said to convergence to $f_v \in F\nu$ if for every $\in>0$ there exists appositive number M such that $\lim_{r\to\infty} \|(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vr-1}, f_{vr} - f_v)\| = 0$

That is $\lim_{r\to\infty} \|(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vr-1}, f_{vr})\| = f_v$

**2.25. Definition:** In a fuzzy m-normed linear space $(F\nu,\|.,\ldots,.\|)$ a sequence $\{f_{vr}\}_{r=1}^{\infty}$is said to be Cauchy sequenc if if for every $\varepsilon>0$ there exists appositive number M such that $\|(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vr-1}, f_{vr} - f_{v\ell})\| < \varepsilon$ ,whenever $r, \ell \geq M$. And it is represented by $\lim_{r,\ell\to\infty} \|(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vr-1}, f_{vr} - f_{v\ell})\| = 0$.

**2.26. Definition:** The fuzzy m-normed linear space $(F\nu,\|.,\ldots,.\|)$ is said to be complete if every Cauchy sequence is convergent in it.

**2.27.Definition:** Let $E_V$ be subset of fuzzy m-normed linear space $(F\nu,\|.,\ldots,.\|)$ and is said to be bounded if there exists a positive real constant η such that

$\|(e_{v1}, e_{v2}, e_{v3}, \ldots, e_{vr-1}, e_{vr})\| \leq \eta$ for all $e_{v1}, e_{v2}, e_{v3}, \ldots, e_{vr-1}, e_{vr} \in E_V$.

**2.28.Definition:** Let $(F\nu,\|.,\ldots,.\|)$ be a fuzzy m- normed linear space for any $(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vm-1}, f_{vm}) \in F\nu$, then set $\{(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vm-1}, f_{vm}) \in F\nu / \|(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vm} - f_{v0})\| < \rho\}$ is called an open ball center at $f_{v0}$ with radius ρ and it is represented as $B_\rho(f_{v0}) = \{(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vm-1}, f_{vm}) \in F\nu / \|(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vm} - f_{v0})\| < \rho\}$

Similarly we can define a closed ball

$B_\rho(f_{v0}) = \{(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vm-1}, f_{vm})\ F\nu / \|(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vm} - f_{v0})\| \leq \rho\}$ .

**2.29.Definition:** In the fuzzy m-normed linear space $(F\nu,\|.,\ldots,.\|)$ a sequence $\{f_{vr}\}_{r=1}^{\infty}$is called bounded or norm bounded if there is a constant η such that that $\|(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vr-1}, f_{vr})\| \leq \eta$, for all r.

**2.30. Remark:** Every convergent sequence is Cauchy and norm bounded in the fuzzy m-normed linear space $(F\nu,\|.,\ldots,.\|)$.

**2.31.Theorem:** Let $(F\nu,\|.,\ldots,.\|)$ be a fuzzy m-normed linear space and let $\{f_{vr}\}_{r=1}^{\infty}$ and $\{f'_{vr}\}_{r=1}^{\infty}$ be two Cauchy sequences in $(F\nu,\|.,\ldots,.\|)$ such that $\|(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vr-1}, f_{vr})\| \longrightarrow f_v$ ,

$\|(f'_{v1}, f'_{v2}, f'_{v3}, \ldots, f'_{vr-1}, f'_{vr})\| \longrightarrow f'_v$, as $r\to\infty$ .Let $\{k_{vr}\}_{r=1}^{\infty}$ be a sequence in $\mathcal{K}\nu$ where $\mathcal{K}\nu$ is being the field of scalars be such that $k_{vr} \to k_{v0}$ as $r\to\infty$ then the following axioms holds:

(F-1) $\|(f_{v1} \pm f'_{v1}, f_{v2} \pm f'_{v2}, f_{v3} \pm f'_{v3}, \ldots, f_{vr} \pm f'_{vr})\| \longrightarrow f_v \pm f'_v$ , as $r\to\infty$

(F-2) $\|(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vr-1}, k_{vr} f_{vr})\| \longrightarrow k_v . f_v$ , as $r\to\infty$.

(F-3) Let $\{f_{vr}\}_{r=1}^{\infty}$ and $\{f'_{vr}\}_{r=1}^{\infty}$ be two Cauchy sequences in $(F\nu,\|.,\ldots,.\|)$ and $\{k_{vr}\}_{r=1}^{\infty}$ be a Cauchy sequence in then $\{f_{vr} + f'_{vr}\}_{r=1}^{\infty}$ and $\{k_{vr} f_{vr}\}_{r=1}^{\infty}$ are also Cauchy sequence in $(F\nu,\|.,\ldots,.\|)$.

**Proof:** (1) We have

$\|(f_{v1} \pm f'_{v1}, f_{v2} \pm f'_{v2}, f_{v3} \pm f'_{v3}, \ldots, f_{vr} \pm f'_{vr}V - (f_v \pm f'_v)\| \leq (\|(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vr} - f_v\| +$

$\|(f'_{v1}, f'_{v2}, f'_{v3}, \ldots, f'_{vr} - f'_v)\| \longrightarrow 0$,as $r\to\infty$,

Since $\lim_{r\to\infty} \|(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vr-1}, f_{vr} - f_v)\| = 0$ and

$\lim_{r\to\infty} \|(f'_{v1}, f'_{v2}, f'_{v3}, \ldots, f'_{vr-1}, f'_{vr} - f'_v)\| = 0$

Hence $\|(f_{v1} \pm f'_{v1}, f_{v2} \pm f'_{v2}, f_{v3} \pm f'_{v3}, \ldots, f_{vr} \pm f'_{vr}\| \longrightarrow f_v \pm f'_v$ , as $r\to\infty$.

(2) Given that $k_{vr} \to k_{v0}$ as $r\to\infty \Rightarrow |k_{vr} - k_{v0}| \to 0$, as $r\to\infty$ then there exists a constant λ such that $|k_{vr}| \leq \lambda$ for all r.

Now we consider

$\|(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vr-1}, k_{vr} f_{vr} - k_{v0} f_{v0})\| = \|(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vr-1}, k_{vr} f_{vr} - k_{vr} f_{v0} + k_{vr} f_{v0} - k_{v0} f_{v0})\| = \|(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vr-1}, k_{vr}(f_{vr} - f_{v0}) + (k_{vr} - k_{v0}) f_{v0})\|$

$\leq \|(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vr-1}, k_{vr}(f_{vr} - f_{v0}))\| + \|(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vr-1}, (k_{vr} - k_{v0}) f_{v0})\|$

$\leq |k_{vr}| \, \|(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vr-1}, f_{vr} - f_{v0}))\| + |k_{vr} - k_{v0}| \, \|(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vr-1}, f_{v0})\|$

$\leq |\lambda| \, \|(f_{v1}, f_{v2}, f_{v3}, \ldots f_{vr-1}, f_{vr} - f_{v0}))\| + |k_{vr} - k_{v0}| \, \|(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vr-1}, f_{v0})\| \to 0$ as $r \to \infty$.

So $\|(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vr-1}, k_{vr} f_{vr})\| \to k_{v0} . f_{v0}$, as $r \to \infty$.

(3) We have $\|(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vr-1}, f_{vr} - f_{v\ell})\| \to 0$, as $r, \ell \to \infty$ And

$\|(f'_{v1}, f'_{v2}, f'_{v3}, \ldots, f'_{vr-1}, f'_{vr} - f'_{v\ell})\| \to 0$, as $r, \ell \to \infty$

Therefore $\|(f_{v1} + f'_{v1}, f_{v2} + f'_{v2}, f_{v3} + f'_{v3}, \ldots f_{vr-1} + f_{vr-1}, f_{vr} + f'_{vr} - (f_{v\ell} + f'_{v\ell})\|$

$\leq [\, \|(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vr-1}, f_{vr} - f_{v\ell})\| + \|(f'_{v1}, f'_{v2}, f'_{v3}, \ldots, f'_{vr-1}, f'_{vr} - f'_{v\ell})\|] \to 0$, as $r, \ell \to \infty$

This implies that $\{f_{vr} + f'_{vr}\}_{r=1}^{\infty}$ is Cauchy sequence in $(F\nu, \|.,\ldots,.\|)$

Since $\{k_{vr}\}_{r=1}^{\infty}$ be a Cauchy sequence of scalars in K, the scalar field is complete and $\{k_{vr}\}_{r=1}^{\infty}$ is convergent sequence and hence $\{k_{vr}\}_{r=1}^{\infty}$ is bounded.

Also we consider

$\|(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vr-1}, k_{vr} f_{vr} - k_{v\ell} f_{v\ell})\| = \|(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vr-1}, k_{vr} f_{vr} - k_{v\ell} f_{v\ell} + k_{vr} f_{v\ell} - k_{vr} f_{v\ell})\| = \|(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vr-1}, k_{vr}(f_{vr} - f_{v\ell}))\| + \|(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vr-1}, (k_{vr} - k_{v\ell}) f_{v\ell})\| \leq [|k_v|\,\| + |k_{vr} - k_{v\ell}| \, \|(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vr-1}, f_{v\ell})\|] \to 0$, as $r, \ell \to \infty$

Since $\{k_{vr}\}_{r=1}^{\infty}$ is bounded and $\{f_{vr}\}_{r=1}^{\infty}$ is norm bounded

Hence $\{k_{vr} f_{vr}\}_{r=1}^{\infty}$ is a Cauchy sequence in $(F\nu, \|.,\ldots,.\|)$.

**2.32. Remark:** The norm function is continuous as it follows by **2.31.Theorem**.

**2.33 .Example:** Let $(F\nu, \|.,\ldots,.\|)$ be a fuzzy m- normed linear space. We define continuous t-norm

$\Delta^F(f^1, f^2) = f^1 . f^2$ , $\Delta^F(f^1, f^2) = \min\{f^1, f^2\}$, for all $f^1, f^2 \in [0,1]$.

$F\nu(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vm-1}, f_{vm}, f_t) = \dfrac{\|(f_{v1}, f_{v2}, f, \ldots, f_{vm-1}, f_{vm})\|}{f_t + \|(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vm-1}, f_{vm})\|}$ then the set

$S = \{(F\nu, F\nu(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vm-1}, f_{vm}, f_t)) / (f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vm-1}, f_{vm}, f_t) \in F\nu^M\}$ is a fuzzy m-normed linear space.

**Proof:** (1) Obviously for every $f_t \in (-\infty, \infty)$ such that $F\nu(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vm-1}, f_{vm}, f_t) \geq 0$.

(2) $F\nu(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vm-1}, f_{vm}, f_t) = 0 \Leftrightarrow \dfrac{\| f_{v1}, f_{v2}, \ldots, f_{vm-1}, f_{vm})\|}{f_t + \|(f_{v1}, f_{v2}, \ldots, f_{vm-1}, f_{vm})\|} = 0$

$\Leftrightarrow \|(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vm-1}, f_{vm})\| = 0$

$\Leftrightarrow f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vm-1}, f_{vm}$ are linearly dependent if for all $f_t \geq 0$, and $f_t \in (-\infty, \infty)$

(3) $F\nu(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vm-1}, f_{vm}, f_t) = \dfrac{\|(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vm-1}, f_{vm})\|}{f_t + \|(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vm-1}, f_{vm})\|} = \dfrac{\|(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vm-1}, f_{vm})\|}{f_t + \|(f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vm}, f_{vm-1})\|} =$ ..so on

Hence it is invariant under any permutation of $f_{v1}, f_{v2}, f_{v3}, \ldots, f_{vm-1}, f_{vm}, f_t$.

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(4) $F\nu\ (f_{v1}, f_{v2}, f_{v3,\ldots,}f_{vm-1}, f_{\Delta}f_{vm}, f_t) = \dfrac{\|(f_{v1},f_{v2},f_{v3},\ldots,f_{vm-1},f_{\Delta}f_{vm})\|}{f_t + \|(f_{v1},f_{v2},f_{v3},\ldots,f_{vm-1},f_{\Delta}f_{vm})\|}$

$$= \frac{f_{\Delta}\|(f_{v1},f_{v2},f_{v3},\ldots,f_{vm-1},f_{vm})\|}{f_t + f_{\Delta}\|(f_{v1},f_{v2},f_{v3},\ldots,f_{vm-1},f_{vm})\|} = \frac{\|(f_{v1},f_{v2},f_{v3},\ldots,f_{vm-1},f_{vm})\|}{\dfrac{f_t}{f_{\Delta}} + \|(f_{v1},f_{v2},f_{v3},\ldots,f_{vm-1},f_{vm})\|}$$

$= F\nu\ (f_{v1}, f_{v2}, f_{v3,\ldots,}f_{vm-1}, f_{vm}, f_{t,}\ \frac{f_t}{|f_{\Delta}|})$, where $f_{\Delta} \in F$

(5) Without loss of generality we consider that

$F_V(f_{v1}, f_{v2}, f_{v3,\ldots,}f_{vm-1}, f_{vm1}, f_{t1}) \le F_V(f_{v1}, f_{v2}, f_{v3,\ldots,}f_{vm-1}, f_{vm2}, f_{t2})$

$$\Rightarrow \frac{\|(f_{v1},f_{v2},f_{v3},\ldots,f_{vm-1},f_{vm1})\|}{f_{t1} + \|(f_{v1},f_{v2},f_{v3},\ldots,f_{vm-1},f_{vm1})\|} \le \frac{\|(f_{v1},f_{v2},f_{v3},\ldots,f_{vm-1},f_{vm2})\|}{f_{t2} + \|(f_{v1},f_{v2},f_{v3},\ldots,f_{vm-1},f_{vm2})\|}$$

$$\Rightarrow f_{t2}\|(f_{v1},f_{v2},f_{v3},\ldots,f_{vm-1},f_{vm1})\| + [\|(f_{v1},f_{v2},f_{v3},\ldots,f_{vm-1},f_{vm1})\| . \|(f_{v1},f_{v2},f_{v3},\ldots,f_{vm-1},f_{vm2})\|]$$
$$\le f_{t1}\|(f_{v1},f_{v2},f_{v3},\ldots,f_{vm-1},f_{vm2})\| + [\|(f_{v1},f_{v2},f_{v3},\ldots,f_{vm-1},f_{vm1})\| . \|(f_{v1},f_{v2},f_{v3},\ldots,f_{vm-1},f_{vm2})\|]$$

$$\Rightarrow f_{t2}\|(f_{v1},f_{v2},f_{v3},\ldots,f_{vm-1},f_{vm1})\| \le f_{t1}\|(f_{v1},f_{v2},f_{v3},\ldots,f_{vm-1},f_{vm2})\|$$

$$\Rightarrow f_{t1}\|(f_{v1},f_{v2},f_{v3},\ldots,f_{vm-1},f_{vm2})\| - f_{t2}\|(f_{v1},f_{v2},f_{v3},\ldots,f_{vm-1},f_{vm1})\| \ge 0 \quad \ldots\ldots.(1)$$

Now we consider,

$F_V(f_{v1}, f_{v2}, f_{v3},\ldots, f_{vm-1}, f_{vm1} + f_{vm2}, f_{t1} + f_{t2}) - F_V(f_{v1}, f_{v2}, f_{v3},\ldots, f_{vm-1}, f_{vm1}, f_{t1})$

$$\Leftrightarrow \frac{\|(f_{v1},f_{v2},f_{v3},\ldots,f_{vm-1},f_{vm1} + f_{vm2})\|}{f_{t1} + f_{t2} + \|(f_{v1},f_{v2},f_{v3},\ldots,f_{vm-1},f_{vm1} + f_{vm2})\|} - \frac{\|(f_{v1},f_{v2},f_{v3},\ldots,f_{vm-1},f_{vm1})\|}{f_{t1} + \|(f_{v1},f_{v2},f_{v3},\ldots,f_{vm-1},f_{vm1})\|}$$

$$\Leftrightarrow [\|(f_{v1},f_{v2},f_{v3},..,f_{vm-1},f_{vm1} + f_{vm2})\|][f_{t1} + \|(f_{v1},f_{v2},f_{v3},\ldots,f_{vm-1},f_{vm1})\|] -$$
$$[\|(f_{v1},f_{v2},f_{v3},\ldots,f_{vm-1},f_{vm1})\|][f_{t1} + f_{t2} + \|(f_{v1},f_{v2},f_{v3},\ldots,f_{vm-1},f_{vm1} + f_{vm2})\|]$$

$$\Leftrightarrow f_{t1}\Box(f_{v1},f_{v2},f_{v3},\ldots,f_{vm-1},f_{vm2})\Box - f_{t2}\Box(f_{v1},f_{v2},f_{v3},\ldots,f_{vm-1},f_{vm1})\Box\ldots..(2)$$

From (1) and (2) we obtain

$F_V(f_{v1}, f_{v2}, f_{v3},\ldots, f_{vm-1}, f_{vm1} + f_{vm2}, f_{t1} + f_{t2}) - F_V(f_{v1}, f_{v2}, f_{v3},\ldots, f_{vm-1}, f_{vm1}, f_{t1}) \ge 0$

Hence $F_V(f_{v1}, f_{v2}, f_{v3},\ldots, f_{vm-1}, f_{vm1} + f_{vm2}, f_{t1} + f_{t2}) \ge \min\{F_V(f_{v1}, f_{v2}, f_{v3},\ldots, f_{vm-1}, f_{vm1}, f_{t1}), F_V(f_{v1}, f_{v2}, f_{v3},\ldots, f_{vm-1}, f_{vm2}, f_{t2})\}$

(6) Clearly $F(f_{v1}, f_{v2}, f_{v3},\ldots, f_{vm-1}, f_{vm}, f_t)$ is a left continuous and non-decreasing function of

$f_t \in (-\infty, \infty)$ such that

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$$\lim_{m\to\infty} F_V(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm}, f_t) = \lim_{m\to\infty} \frac{\| (f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm}) \|}{f_t + \| (f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm}) \|} = 0 .$$

**2.34. Definition:** A sequence $\{f_{vr}\}_{r=1}^{\infty}$ in fuzzy m-normed linear space S is said to be convergent to $f_v$ if given that $\delta > 0$, $\delta \in (0,1)$ and $f_t > 0$ there exists appositive number M such that

$$F_N(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm} - f_v, f_t) < \delta \text{ for all } m \geq M.$$

$$\textit{that is } \lim_{m\to\infty} F_N(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm} - f_v, f_t) < \delta \text{ for all } m \geq M.$$

**2.35. Definition:** A sequence $\{f_{vr}\}_{r=1}^{\infty}$ in fuzzy m-normed linear space S is said to be Cauchy sequence if given that $\delta > 0$,and $\delta$ in (0,1) and $f_t > 0$ there exists appositive number M such that

$F_V$ $(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm} - f_{v\ell}, f_t) < \delta$ for all m, $\ell \geq M$.

$$F_N(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm} - f_{v\ell}, f_t) < \delta \text{ for all } m, l \geq M.$$

$$\textit{that is } \lim_{m,l\to\infty} F_N(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm} - f_{v\ell}, f_t) < \delta \text{ for all } m, l \geq M.$$

**2.36. Theorem:** In a fuzzy m-normed linear space S, every convergent sequence is Cauchy sequence.

**2.37. Definition:** A fuzzy m-normed linear space S is said to be complete if every Cauchy sequence in S is convergent.

**3. MAIN RESULTS AND DISCUSSIONS:**

**3.1. Definition:** Let $F_V$ be a fuzzy right Gamma linear space over $F_D$ a real valued function

$\|,\dots,\|: F_V \times F_V \to [0,\infty)$ is called fuzzy Gamma-2-normed linear space over $F_D$, It is denoted by $(F_V, \|.,\dots,.\|)$. if it satisfies the following properties for every $f_{v1}, f_{v2} \in F_V$, $f_{d1}, f_{d2}, f_{d3}$ $F_D$ and $f_\gamma \in F_\Gamma$

(1) $\| f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2} \| = 0 \Leftrightarrow f_{v1}, f_{v2}$ linearly independent over $F_D$

(2) $\|f_{v1} f_\gamma f_{d1}, f_\sigma (f_{v2} f_\gamma f_{d2})\| = f_\sigma \| f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2} \|$, for any $f_\sigma \in F_\Gamma$

(3) $\| f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2} + f_{v3} f_\gamma f_{d3}\| \leq \| f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2} \| + \| f_{v1} f_\gamma f_{d1}, f_{v3} f_\gamma f_{d3} \|$

**3.2.Definition:** Let $F_V$ be a fuzzy right Gamma linear space over $F_D$ a real valued function

$\|,\dots,\|: F_V \times F_V \times F_V \dots \times F_V$ (m-times)$\times[0,\infty)$ is called fuzzy right Gamma-m-normed linear space over $F_D$ if it satisfies the following properties , for any $f_{v1}, f_{v2}, \dots, f_{vm-1} f_{vm} \in F_V$, $f_{d1}, f_{d2}, f_{d3}, \dots f_{dm-1}, f_{dm} \in F_D$ and $f_\gamma \in F_\Gamma$

(1) $\| f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2}, \dots, f_{vm-1} f_\gamma f_{dm-1}, f_{vm} f_\gamma f_{dm} \| = 0 \Leftrightarrow f_{v1}, f_{v2}, \dots, f_{vm-1} f_{vm}$ are linearly independent over $F_D$.

(2) $\| f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2}, \dots, f_{vm-1} f_\gamma f_{dm-1}, f_{vm} f_\gamma f_{dm} \| = 0$ , is invariant under any permutation of $f_{v1}, f_{v2}, \dots, f_{vm-1} f_{vm}$ .

(3) $\| f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2}, \dots, f_{vm-1} f_\gamma f_{dm-1}, f_\sigma (f_{vm} f_\gamma f_{dm}) \| = f_\sigma \| f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2}, \dots, f_{vm-1} f_\gamma f_{dm-1}, f_{vm} f_\gamma f_{dm} \|$ ,

for any $f_\sigma \in F_\Gamma$

(4) $\| f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2}, \dots, f_{vm-1} f_\gamma f_{dm-1}, f_{vm1} f_\gamma f_{dm1} + f_{vm2} f_\gamma f_{dm2} \| \leq \| f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2}, \dots, f_{vm-1} f_\gamma f_{dm-1}, f_{vm1} f_\gamma f_{dm1} \| + \| f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2}, \dots, f_{vm-1} f_\gamma f_{dm-1}, f_{vm2} f_\gamma f_{dm2} \|$

It is represented by $(F_V, \|.,\dots,.\|)$ likewise fuzzy left Gamma-m-normed linear space over $F_D$ can be defined a similar manner.

**3.3.Definition:** Over $F_D$ let $F_V$ be a fuzzy Gamma linear space a real valued function

‖,....,‖: $F_V\times F_V \times F_V\ldots\ldots\times F_V$ (m-times)$\times[0,\infty)$ is called fuzzy Gamma m-normed linear space over $F_D$ if it is either fuzzy right Gamma-m-normed linear space over $F_D$ or fuzzy left Gamma-m-normed linear space over $F_D$

**3.4. Definition:** Let ($F_V$, ‖.,….,. ‖) be a fuzzy Gamma-m-normed linear space and a sequence $\{ f_{vr}\, f_\gamma\, f_{dr}\}_{r=1}^{\infty}$ in ($F_V$, ‖.,….,. ‖) is said to convergence to $f_v\, f_\gamma\, f_d \in F_V$ if for every $\varepsilon>0$ there exists appositive number M such that $\|f_{v1}\, f_\gamma\, f_{d1}, f_{v2}\, f_\gamma f_{d2},\ldots,f_{vr-1}\, f_\gamma\, f_{dr-1},f_{vr}f_\gamma f_{dr} - f_v\, f_\gamma\, f_d\ \| < \varepsilon$

That is $\lim_{r\to 0} \| (f_{v1}f_\gamma f_{d1}, f_{v2}f_\gamma f_{d2}, f_{v3}f_\gamma f_{d3},\ldots, f_{vr-1}f_\gamma f_{dr-1}, f_{vr}f_\gamma f_{dr} - f_v f_\gamma f_d) \| = 0$

Which implies that $\lim_{r\to 0} \| (f_{v1}f_\gamma f_{d1}, f_{v2}f_\gamma f_{d2}, f_{v3}f_\gamma f_{d3},\ldots, f_{vr-1}f_\gamma f_{dr-1}, f_{vr}f_\gamma f_{dr}) \| = f_v f_\gamma f_d$

Where $f_{v1}, f_{v2}\, f_{v3},\ldots,f_{vr-1}, f_{vr}$ and $f_v \in F_V$ ,$f_{d1}, f_{d2}, f_{d3},\ldots,f_{dr-1}, f_{dr}$, and $f_d \in F_D$ , $f_\gamma \in F_\Gamma$

**3.5. Definition:** In a fuzzy Gamma-m-normed linear space ($F_V$, ‖.,….,. ‖) a

Sequence $\{ f_{vr}\, f_\gamma\, f_{dr}\}_{=1}^{\infty}$ is said to be Cauchy sequence if for every $\varepsilon>0$ there exists appositive number M such that $\| (f_{v1}f_\gamma f_{d1}, f_{v2}f_\gamma f_{d2}, f_{v3}f_\gamma f_{d3},\ldots, f_{vr-1}f_\gamma f_{dr-1}, f_{vr}f_\gamma f_{dr} - f_{v\ell}f_\gamma f_{d\ell}) \| < \varepsilon$

Whenever $r,\ell \geq M$. and it is represented by

$$\lim_{r,l\to\infty} \| (f_{v1}f_\gamma f_{d1}, f_{v2}f_\gamma f_{d2}, f_{v3}f_\gamma f_{d3},\ldots, f_{vr-1}f_\gamma f_{dr-1}, f_{vr}f_\gamma f_{dr} - f_{v\ell}f_\gamma f_{d\ell}) \| = 0,$$

Where $f_{vr}$ , $f_{v\ell}$ ,$f_v \in F_V$ , $f_{dr}$ , $f_{d\ell}$ , $f_d \in F_D$ for r=1,2,3…… and $f_\gamma \in F_\Gamma$

**3.6. Definition:** The fuzzy Gamma-m-normed linear space ($F_V$, ‖.,….,. ‖) is said to be complete if every Cauchy sequence is convergent in it.

**3.7. Definition:** Let $E_V$ be subset of fuzzy Gamma-m-normed linear space ($F_V$, ‖.,….,. ‖) and is said to be bounded if there exists a positive real constant $\eta_0$ such that

$\|(e_{v1}\, f_\gamma\, f_{d1},e_{v2}\, f_\gamma\, f_{d2},e_{v3}\, f_\gamma\, f_{d3},\ldots, e_{vr-1}\, f_\gamma\, f_{dr-1},e_{vr}\, f_\gamma\, f_{dr}) \| \leq \eta_0$ ,

for all $e_{v1}$ ,$e_{v2}$ ,$e_{v3},\ldots, e_{vr-1}$ ,$e_{vr}\in E_V$, $f_{d1}, f_{d2}, f_{d3},\ldots,f_{dr-1}, f_{dr}$, and $f_d \in F_D$ and $f_\gamma\in F_\Gamma$

**3.8.Definition:** Let ($F_V$, ‖.,….,. ‖) be a fuzzy Gamma-m- normed linear space for any $(f_{v1}\, f_\gamma\, f_{d1}, f_{v2}\, f_\gamma f_{d2},\ldots,f_{vm-1}\, f_\gamma\, f_{dm-1}, f_{vm}\, f_\gamma f_{dm}) \in F_V$, then set $\{( f_{v1}\, f_\gamma\, f_{d1}, f_{v2}\, f_\gamma f_{d2}, \ldots,f_{vm-1}\, f_\gamma\, f_{dm-1}, f_{vm}\, f_\gamma f_{dm}) \in F_V /\|( f_{v1}\, f_\gamma\, f_{d1}, f_{v2}\, f_\gamma f_{d2},\ldots,f_{vm-1}\, f_\gamma\, f_{dm-1}, f_{vm}\, f_\gamma f_{dm} - f_{v0}\, f_\gamma f_{d0}) \|<\rho_0\}$ is called an open ball center at $f_{v0}\, f_\gamma f_{d0}$ with radius $\rho_0$ and it is represented as

$B_{\rho0}(f_{v0}\, f_\gamma f_{d0})= \{( f_{v1}\, f_\gamma\, f_{d1}, f_{v2}\, f_\gamma f_{d2},\ldots,f_{vm-1}\, f_\gamma\, f_{dm-1}, f_{vm}\, f_\gamma f_{dm}) \in F_V /\|( f_{v1}\, f_\gamma\, f_{d1}, f_{v2}\, f_\gamma f_{d2},\ldots,f_{vm-1}\, f_\gamma\, f_{dm-1}, f_{vm}\, f_\gamma f_{dm} - f_{v0}\, f_\gamma f_{d0}) \|<\rho_0\}$

Similarly we can define a closed ball $B_{\rho0}(f_{v0}\, f_\gamma f_{d0})= \{( f_{v1}\, f_\gamma\, f_{d1}, f_{v2}\, f_\gamma f_{d2},\ldots,f_{vm-1}\, f_\gamma\, f_{dm-1}, f_{vm}\, f_\gamma f_{dm}) \in F_V /\|( f_{v1}\, f_\gamma\, f_{d1}, f_{v2}\, f_\gamma f_{d2}, \ldots,f_{vm-1}\, f_\gamma\, f_{dm-1}, f_{vm}\, f_\gamma f_{dm} - f_{v0}\, f_\gamma f_{d0}) \|\leq\rho_0\}$ Where $f_{v1}, f_{v2}\, f_{v3},\ldots, f_{vm-1}, f_{vm}$ and $f_{v0} \in F_V$ ,$f_{d1}, f_{d2}, f_{d3},\ldots,f_{dm-1}, f_{dm}$, and $f_{d0} \in F_D$ and $f_\gamma \in F_\Gamma$

**3.9. Definition:** In the fuzzy Gamma-m-normed linear space ($F_V$, ‖.,….,. ‖) a sequence $\{f_{vr}f_\gamma f_{dr}\}_{r=1}^{\infty}$ is said to be bounded or norm bounded if there is a constant $\eta_0$ such that that $\|( f_{v1}\, f_\gamma\, f_{d1}, f_{v2}\, f_\gamma f_{d2},\ldots,f_{vr-1}\, f_\gamma\, f_{dr-1}, f_{vr}\, f_\gamma f_{dr}) \|\leq \eta_0$, for all $f_{vi}$ , $f_{vj}$ ,$f_v \in F_V$ , $f_{di}$ , $f_{dj}$ , $f_d \in F_D$ for i,j=1,2,..r, and $f_\gamma\in F_\Gamma$

**3.10. Remark:** Every convergent sequence is Cauchy and norm bounded in the fuzzy Gamma-m-normed linear space ($F_V$, ‖.,….,. ‖).

**3.11.Theorem:** Let ($F_V$, ‖.,….,. ‖) be a fuzzy Gamma-m-normed linear space and let $\{f_{vr}f_\gamma f_{dr}\}_{r=1}^{\infty}$ and $\{f'_{vr}f_\gamma f'_{dr}\}_{r=1}^{\infty}$ be two Cauchy sequences in ($F_V$, ‖.,….,. ‖) such that

$\|(f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2},\ldots,f_{vr-1} f_\gamma f_{dr-1}, f_{vr} f_\gamma f_{dr})\| \to f_v f_\gamma f_d$ ,

$\|(f'_{v1} f_\gamma f'_{d1}, f'_{v2} f_\gamma f'_{d2}, \ldots,f'_{vr-1} f_\gamma f'_{dr-1}, f'_{vr} f_\gamma f'_{dr})\| \to f'_v f_\gamma f'_d$ , as $r\to\infty$ .

Let $\{k_{vr}\}_{r=1}^{\infty}$ be a sequence in $\mathcal{v}$ where $\mathcal{K}v$ is being the field of scalars be such that $k_{vr}\to k_v$ as $r\to\infty$ then the following axioms holds:

(1) $\|(f_{v1} f_\gamma f_{d1}\pm f'_{v1} f_\gamma f'_{d1} , f_{v2} f_\gamma f_{d2} \pm f'_{v2} f_\gamma f'_{d2} , f_{v3} f_\gamma f_{d3} \pm f'_{v3} f_\gamma f'_{d3} ,\ldots,f_{vr} f_\gamma f_{dr}\pm f'_{vr} f_\gamma f'_{dr})\| \to f_v f_\gamma f_d \pm f'_v f_\gamma f'_d$ , as $r\to\infty$

(2) $\|(f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2},\ldots,f_{vr-1} f_\gamma f_{dr-1}, k_{vr}( f_{vr} f_\gamma f_{dr} ))\| \to k_v(f_v f_\gamma f_d)$ , as $r\to\infty$.

(3) Let $\{f_{vr}f_\gamma f_{dr}\}_{r=1}^{\infty}$ and $\{f'_{vr}f_\gamma f'_{dr}\}_{r=1}^{\infty}$ be two Cauchy sequences in $(F_V, \|.,\ldots,.\|)$ and $\{k_{vr}\}_{r=1}^{\infty}$ be a Cauchy sequence in $F_V$ then $\{f_{vr}f_\gamma f_{dr} + f'_{vr}f_\gamma f'_{dr}\}_{r=1}^{\infty}$ and $\{k_{vr}f_{vr}f_\gamma f_{dr}\}_{r=1}^{\infty}$ are also Cauchy sequence in $(F_V, \|.,\ldots,.\|)$.

**Proof:** (1) We have $\lim_{r\to\infty}\| (f_{v1}f_\gamma f_{d1}, f_{v2}f_\gamma f_{d2}, f_{v3}f_\gamma f_{d3},\ldots, f_{vr-1}f_\gamma f_{dr-1}, f_{vr}f_\gamma f_{dr} - f_v f_\gamma f_d)\| = 0$ and

$$\lim_{r\to\infty}\| (f'_{v1}f_\gamma f'_{d1}, f'_{v2}f_\gamma f'_{d2}, f'_{v3}f_\gamma f'_{d3},\ldots, f'_{vr-1}f_\gamma f'_{dr-1}, f'_{vr}f_\gamma f'_{dr} - f'_v f_\gamma f'_d\| = 0$$

Then which implies that $\| (f_{v1}f_\gamma f_{d1} \pm f'_{v1}f_\gamma f'_{d1}, f_{v2}f_\gamma f_{d2} \pm f'_{v2}f_\gamma f'_{d2}, f_{v3}f_\gamma f_{d3} \pm f'_{v3}f_\gamma f'_{d3},\ldots, f_{vr}f_\gamma f_{dr} \pm f'_{vr}f_\gamma f'_{dr} - ( f_v f_\gamma f_d \pm f'_v f_\gamma f'_d))\|$

$$\leq [\| (f_{v1}f_\gamma f_{d1}, f_{v2}f_\gamma f_{d2}, f_{v3}f_\gamma f_{d3},\ldots, f_{vr-1}f_\gamma f_{dr-1}, f_{vr}f_\gamma f_{dr} - f_v f_\gamma f_{d_r})\| + \| (f'_{v1}f'_\gamma f'_{d1}, f'_{v2}f'_\gamma f'_{d2}, f'_{v3}f'_\gamma f'_{d3}, .f'_{vr-1}f'_\gamma f'_{dr-1}, f'_{vr}f'_\gamma f'_{dr} - f'_v f'_\gamma f'_d)\|] \to 0$$

as $r \to \infty$

Hence $\| (f_{v1}f_\gamma f_{d1} \pm f'_{v1}f_\gamma f'_{d1}, f_{v2}f_\gamma f_{d2} \pm f'_{v2}f_\gamma f'_{d2}, f_{v3}f_\gamma f_{d3} \pm f'_{v3}f_\gamma f'_{d3},\ldots, f_{vr}f_\gamma f_{dr} \pm f'_{vr}f_\gamma f'_{dr})\| \to f_v f_\gamma f_d \pm f'_v f_\gamma f'_d$

as $r\to\infty$.

(2) Given that $k_{vr}\to k_{v0}$ as $r\to\infty \Rightarrow |k_{vr}-k_{v0}|\to 0$, as $r\to\infty$ then there exists a constant $\lambda$ such that $|k_{vr}| \leq \lambda$ for all r.

Now we consider $\|(f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2},\ldots,f_{vr-1} f_\gamma f_{dr-1}, k_{vr}( f_{vr} f_\gamma f_{dr} )- k_{v0}(f_{v0}f_{\gamma 0}f_{d0}))\|$

$= \|(f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2},\ldots,f_{vr-1} f_\gamma f_{dr-1}, k_{vr}( f_{vr} f_\gamma f_{dr})- k_{vr}( f_{v0} f_{\gamma 0}f_{d0} )+ k_{vr}(f_{v0} f_{\gamma 0}f_{d0})- k_{v0}(f_{v0}f_{\gamma 0}f_{d0} ))\|$

$\leq \|(f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2},\ldots,f_{vr-1} f_\gamma f_{dr-1}, k_{vr}( f_{vr} f_\gamma f_{dr}- f_{v0} f_{\gamma 0}f_{d0}))\| + \|(f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2},\ldots,f_{vr-1} f_\gamma f_{dr-1}, (k_{vr} - k_{v0})f_{v0}f_{\gamma 0}f_{d0} )\| \leq |k_{vr}|\|(f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2},\ldots,f_{vr-1} f_\gamma f_{dr-1}, f_{vr} f_\gamma f_{dr}- f_{v0} f_{\gamma 0}f_{d0})\| + | k_{vr} - k_{v0}| \|(f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2},\ldots,f_{vr-1} f_\gamma f_{dr-1}, f_{v0}f_{\gamma 0}f_{d0})\| \leq \lambda\| (f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2},\ldots,f_{vr-1} f_\gamma f_{dr-1}, f_{vr} f_\gamma f_{dr}- f_{v0} f_{\gamma 0}f_{d0})\| +$

$| k_{vr} - k_{v0}|\| (f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2},\ldots,f_{vr-1} f_\gamma f_{dr-1}, f_{v0}f_{\gamma 0}f_{d0} )\| \to 0$ , as $r\to\infty$.

So $\|(f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2},\ldots,f_{vr-1} f_\gamma f_{dr-1}, k_{vr}( f_{vr} f_\gamma f_{dr} ))\| \to k_{v0}(f_{v0}f_{\gamma 0}f_{d0})$, as $r\to\infty$.

(3) We have $\{f_{vr}f_\gamma f_{dr}\}_{r=1}^{\infty}$ and $\{f'_{vr}f_\gamma f'_{dr}\}_{r=1}^{\infty}$ be two Cauchy sequences in $(F_V, \|.,\ldots,.\|)$ then $\|( f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2},\ldots,f_{vr-1} f_\gamma f_{dr-1}, f_{vr} f_\gamma f_{dr} - f_{v\ell} f_\gamma f_{d\ell})\| \to 0$, as $r, \ell \to \infty$ and

$\|( f'_{v1} f_\gamma f'_{d1}, f'_{v2} f_\gamma f'_{d2},\ldots,f'_{vr-1} f_\gamma f'_{dr-1}, f'_{vr} f_\gamma f'_{dr} - f'_{v\ell} f_\gamma f'_{d\ell})\| \to 0$ ,as $r, \ell \to \infty$

Therefore $\|( f_{v1} f_\gamma f_{d1}+f'_{v1} f_\gamma f'_{d1} ,f_{v2} f_\gamma f_{d2} +f'_{v2} f_\gamma f'_{d2} ,f_{v3} f_\gamma f_{d3} +f'_{v3} f_\gamma f'_{d3},\ldots,f_{vr} f_\gamma f_{dr}+f'_{vr} f_\gamma f'_{dr} -(f_{v\ell} f_\gamma f_{d\ell} \pm f'_{v\ell} f_\gamma f'_{d\ell}))\|$

$\leq \|(f_{v1} f_\gamma f_{d1},f_{v2} f_\gamma f_{d2},\ldots,f_{vr-1} f_\gamma f_{dr-1},f_{vr} f_\gamma f_{dr} - f_{v\ell} f_\gamma f_{d\ell})\| + \|(f'_{v1} f_\gamma f'_{d1},f'_{v2} f_\gamma f'_{d2},\ldots,f'_{vr-1} f_\gamma f'_{dr-1},f'_{vr} f_\gamma f'_{dr} - f'_{v\ell} f_\gamma f'_{d\ell})\| \to 0$ ,as $r, \ell \to \infty$

This implies that $\{f_{vr}f_\gamma f_{dr} + f'_{vr}f_\gamma f'_{dr}\}_{r=1}^{\infty}$ is Cauchy sequence in $(F_V, \|.,\ldots,.\|)$.

Since $\{k_{vr}\}_{r=1}^{\infty}$ be a Cauchy sequence of scalars in $F_V$ the scalar field is complete and $\{k_{vr}\}_{r=1}^{\infty}$ is convergent sequence and hence $\{k_{vr}\}_{r=1}^{\infty}$ is bounded.

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Also we consider $\| ( f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2}, \ldots, f_{vr-1} f_\gamma f_{dr-1}, k_{vr} (f_{vr} f_\gamma f_{dr}) - k_{v\ell} (f_{v\ell} f_\gamma f_{d\ell})) \|$

$= \|( f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2}, \ldots, f_{vr-1} f_\gamma f_{dr-1}, k_{vr}(f_{vr} f_\gamma f_{dr}) - k_{vr} (f_{v\ell} f_\gamma f_{d\ell}) + k_{vr} (f_{v\ell} f_\gamma f_{d\ell}) - k_{v\ell} (f_{v\ell} f_\gamma f_{d\ell})) \|$

$\leq \| ( f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2}, \ldots, f_{vr-1} f_\gamma f_{dr-1}, k_{vr}(f_{vr} f_\gamma f_{dr} - f_{v\ell} f_\gamma f_{d\ell})) \| +$

$\|( f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2}, \ldots, f_{vr-1} f_\gamma f_{dr-1}, ( k_{vr} - k_{v\ell} )(f_{v\ell} f_\gamma f_{d\ell})) \|$

$\leq |k_{vr}| \, \|(f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2}, \ldots, f_{vr-1} f_\gamma f_{dr-1}, f_{vr} f_\gamma f_{dr} - f_{v\ell} f_\gamma f_{d\ell}) \| + |k_{vr} - k_{v\ell}| \, \|( f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2}, \ldots, f_{vr-1} f_\gamma f_{dr-1}, , f_{v\ell} f_\gamma f_{d\ell}) \| \to 0$, as $r, \ell \to \infty$

Since $\{k_{vr}\}_{r=1}^{\infty}$ is bounded and $\{f_{vr} f_\gamma f_{dr}\}_{r=1}^{\infty}$ is norm bounded .

Hence $\{k_{vr} f_{vr} f_\gamma f_{dr}\}_{r=1}^{\infty}$ is a Cauchy sequence in $(F_V, \|.,\ldots,.\|)$.

**3.12. Remark:** The norm function is continuous as it follows by **3.11.Theorem**.

**3.13. Definition:** let $F_V$ Gamma linear space over a field F. A fuzzy subset $F_\Gamma$ of

$F_V \times F_V \times \ldots\ldots \times F_V$ $F_V$ (m-times) $\times (-\infty,\infty)$ and the pair $(F_V, F_\Gamma)$ is called fuzzy Gamma-m-normed linear space and $F_V \times F_V \times \ldots\ldots\ldots \times F_V \times F_V$ (m-times) $\times (-\infty,\infty)$ is called as a fuzzy m-norm on $F_V$ if and only if

(1) $F_\Gamma(f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2}, \ldots, f_{vm-1} f_\gamma f_{dm-1}, f_{vm} f_\gamma f_{dm}, f_t) \geq 0$ , for all $f_t \in (-\infty,\infty)$.

(2) $F_\Gamma(f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2}, \ldots, f_{vm-1} f_\gamma f_{dm-1}, f_{vm} f_\gamma f_{dm}, f_t)=0$ if and only if $f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2}, \ldots, f_{vm-1} f_\gamma f_{dm-1}, f_{vm} f_\gamma f_{dm}$ are linearly dependent if for all $f_t \geq 0$, and $f_t \in (-\infty,\infty)$ .

(3) $F_\Gamma(f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2}, \ldots, f_{vm-1} f_\gamma f_{dm-1}, f_{vm} f_\gamma f_{dm}, f_t)$ is invariant under any permutation of

$f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2}, \ldots, f_{vm-1} f_\gamma f_{dm-1}, f_{vm} f_\gamma f_{dm}, f_t$.

(4) $F_\Gamma(f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2}, \ldots, f_{vm-1} f_\gamma f_{dm-1}, f_\Delta (f_{vm} f_\gamma f_{dm}), f_t) = F_\Gamma(f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2}, \ldots, f_{vm-1} f_\gamma f_{dm-1}, f_{vm} f_\gamma f_{dm}, \frac{f_t}{|f_\Delta|})$ where $f_\Delta \in F$.

(5) $F_\Gamma(f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2}, \ldots, f_{vm-1} f_\gamma f_{dm-1}, f_{vm1} f_\gamma f_{dm1} + f_{vm2} f_\gamma f_{dm2}, f_{t1} + f_{t2}) \geq \min\{ F_\Gamma(f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2}, \ldots, f_{vm-1} f_\gamma f_{dm-1}, f_{vm1} f_\gamma f_{dm1}, f_{t1}), F_\Gamma(f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2}, \ldots, f_{vm-1} f_\gamma f_{dm-1}, f_{vm2} f_\gamma f_{dm2}, f_{t2})$

(6) $F_\Gamma(f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2}, \ldots f_{vm-1} f_\gamma f_{dm-1}, f_{vm} f_\gamma f_{dm}, f_t)$ is a left continuous and non-decreasing function of $f_t \in (-\infty,\infty)$ such that $\lim_{r \to 0} F_\Gamma(f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2}, f_{v3} f_\gamma f_{d3}, \ldots, f_{vm-1} f_\gamma f_{dm-1}, f_{vm} f_\gamma f_{dm}, f_t) = 0$

**3.14. Example:** Let $(F_V, \|.,\ldots,.\|)$ be a fuzzy Gamma-m- normed linear space. We define continuous t-norm $\Delta^F (f^1, f^2) = f^1 . f^2$ , $\Delta^F (f^1, f^2) = \min \{f^1, f^2\}$, for all $f^1, f^2 \in [0,1]$.

$$F_\Gamma(f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2}, \ldots f_{vm-1} f_\gamma f_{dm-1}, f_{vm} f_\gamma f_{dm}, f_t) = \frac{\| (f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2}, \ldots, f_{vm-1} f_\gamma f_{dm-1}, f_{vm} f_\gamma f_{dm}) \|}{f_t + \| (f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2}, \ldots, f_{vm-1} f_\gamma f_{dm-1}, f_{vm} f_\gamma f_{dm}) \|}$$

then the set $S = \{( F_V , F_\Gamma(f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2}, \ldots f_{vm-1} f_\gamma f_{dm-1}, f_{vm} f_\gamma f_{dm}, f_t) / (f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2}, \ldots, f_{vm-1} f_\gamma f_{dm-1}, f_{vm} f_\gamma f_{dm}, f_t) \in F_V{}^M \}$ or the set is $(F_V, F_\Gamma)$ a fuzzy Gamma-m-normed linear space.

**Proof:** (1) Obviously for every $f_t \in (0,\infty)$

$$\| (f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2}, \ldots, f_{vm-1} f_\gamma f_{dm-1}, f_{vm} f_\gamma f_{dm}) \| \geq 0 \Leftrightarrow \left[ \frac{\| (f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2}, \ldots, f_{vm-1} f_\gamma f_{dm-1}, f_{vm} f_\gamma f_{dm}) \|}{f_t + \| (f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2}, \ldots, f_{vm-1} f_\gamma f_{dm-1}, f_{vm} f_\gamma f_{dm}) \|} \right] \geq 0$$

$$\Leftrightarrow F_\Gamma(f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2}, \ldots f_{vm-1} f_\gamma f_{dm-1}, f_{vm} f_\gamma f_{dm}, f_t) \geq 0.$$

(2) We have

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$$F_\Gamma(f_{v1}f_\gamma f_{d1},\ f_{v2}f_\gamma f_{d2},\ \ldots f_{vm-1}f_\gamma f_{dm-1},\ f_{vm}f_\gamma f_{dm},\ f_t)=0 \Leftrightarrow \frac{\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm}f_\gamma f_{dm})\|}{f_t+\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm}f_\gamma f_{dm})\|}=0$$

$$\Leftrightarrow \|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm}f_\gamma f_{dm})\|=0$$

$\Leftrightarrow f_{v1}f_\gamma f_{d1},\ f_{v2}f_\gamma f_{d2},\ \ldots f_{vm-1}f_\gamma f_{dm-1},\ f_{vm}f_\gamma f_{dm}$ are linearly dependent if for all $f_t \geq 0$, and $f_t$ in $(-\infty,\infty)$ .

(3) $$F_\Gamma(f_{v1}f_\gamma f_{d1},\ f_{v2}f_\gamma f_{d2},\ \ldots f_{vm-1}f_\gamma f_{dm-1},\ f_{vm}f_\gamma f_{dm},\ f_t)=\frac{\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm}f_\gamma f_{dm})\|}{f_t+\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm}f_\gamma f_{dm})\|}$$

$$=\frac{\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm}f_\gamma f_{dm},f_{vm-1}f_\gamma f_{dm-1})\|}{f_t+\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots f_{vm}f_\gamma f_{dm},f_{vm-1}f_\gamma f_{dm-1})\|}=F_\Gamma(f_{v1}f_\gamma f_{d1},\ f_{v2}f_\gamma f_{d2},\ldots,\ f_{vm}f_\gamma f_{dm},f_{vm-1}f_\gamma f_{dm-1},\ f_t)=\ldots\ldots\text{so on.}$$

This proves that $F_\Gamma(f_{v1}f_\gamma f_{d1},\ f_{v2}f_\gamma f_{d2},\ \ldots f_{vm-1}f_\gamma f_{dm-1},\ f_{vm}f_\gamma f_{dm},\ f_t)$ is invariant under any permutation of $f_{v1}f_\gamma f_{d1},\ f_{v2}f_\gamma f_{d2},\ \ldots f_{vm-1}f_\gamma f_{dm-1},\ f_{vm}f_\gamma f_{dm}$ .

(4) Now we assume that LHS

$$F_\Gamma(f_{v1}f_\gamma f_{d1},\ f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},k_\Delta(f_{vm}f_\gamma f_{dm}),f_t)=\frac{\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},k_\Delta(f_{vm}f_\gamma f_{dm}))\|}{f_t+\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},k_\Delta(f_{vm}f_\gamma f_{dm}))\|}$$

$$=\frac{k_\Delta\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},(f_{vm}f_\gamma f_{dm})\|}{f_t+k_\Delta\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},(f_{vm}f_\gamma f_{dm})\|}=\frac{\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},(f_{vm}f_\gamma f_{dm})\|}{\frac{f_t}{|k_\Delta|}+\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},(f_{vm}f_\gamma f_{dm})\|}$$

$=F_\Gamma(f_{v1}\ f_\gamma\ f_{d1},\ f_{v2}\ f_\gamma f_{d2},\ \ldots\ldots\ldots\ f_{vm-1}\ f_\gamma\ f_{dm-1},\ f_{vm}\ f_\gamma f_{dm}\ ,\frac{f_t}{|k_\Delta|})$ , where $k_\Delta$ in =RHS

(5) Without loss of generality we consider that

$F_\Gamma(f_{v1}\ f_\gamma\ f_{d1},f_{v2}\ f_\gamma f_{d2},\ldots\ldots,f_{vm-1}\ f_\gamma\ f_{dm-1},\ f_{vm1}\ f_\gamma f_{dm1}\ ,\ f_s)\leq F_\Gamma(f_{v1}\ f_\gamma\ f_{d1},\ f_{v2}\ f_\gamma f_{d2},\ldots,f_{vm-1}\ f_\gamma\ f_{dm-1},\ f_{vm2}\ f_\gamma f_{dm2}\ ,\ f_t)$

$$\Leftrightarrow \frac{\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm1}f_\gamma f_{dm1})\|}{f_s+\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm1}f_\gamma f_{dm1})\|}\leq\frac{\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm2}f_\gamma f_{dm2})\|}{f_t+\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm2}f_\gamma f_{dm2})\|}$$

$$\Leftrightarrow\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm1}f_\gamma f_{dm1})\|(f_t+\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm2}f_\gamma f_{dm2})\|)$$
$$\leq\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm2}f_\gamma f_{dm2})\|(f_s+\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm1}f_\gamma f_{dm1})\|)$$

$$\Leftrightarrow f_t\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm1}f_\gamma f_{dm1})\|\leq f_s\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm2}f_\gamma f_{dm2})\|$$

$$\Leftrightarrow f_t\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm1}f_\gamma f_{dm1})\|-f_s\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm2}f_\gamma f_{dm2})\|\leq 0$$

$$\Leftrightarrow f_s\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm2}f_\gamma f_{dm2})\|\text{-}f_t\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm1}f_\gamma f_{dm1})\|\geq 0.....(1)$$

Consider again without loss of generality $F_\Gamma(f_{v1}\ f_\gamma\ f_{d1},f_{v2}\ f_\gamma f_{d2},\ldots,f_{vm-1}\ f_\gamma\ f_{dm-1},\ f_{vm1}\ f_\gamma f_{dm1}+\ f_{vm2}\ f_\gamma f_{dm2},f_s+f_t)$-$F_\Gamma(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1},f_\gamma f_{dm-1},f_{vm1}f_\gamma f_{dm1},f_s)$=

$$\frac{\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm1}f_\gamma f_{dm1}+f_{vm2}f_\gamma f_{dm2})\|}{f_s+f_t+\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm1}f_\gamma f_{dm1}+f_{vm2}f_\gamma f_{dm2})\|}-\frac{\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm1}f_\gamma f_{dm1})\|}{f_s+\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm1}f_\gamma f_{dm1})\|}$$

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$$= \frac{\|(f_{v1}f_\gamma f_{d1},\ f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm1}f_\gamma f_{dm1}+f_{vm2}f_\gamma f_{dm2})\|\,(f_s+\|(f_{v1}f_\gamma f_{d1},\ f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm1}f_\gamma f_{dm1})\|)-\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm1}f_\gamma f_{dm1})\|\,((f_s+f_t)+\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm1}f_\gamma f_{dm1}+f_{vm2}f_\gamma f_{dm2})\|)}{(f_s+\|(f_{v1}f_\gamma f_{d1},\ f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm1}f_\gamma f_{dm1})\|)[(f_s+f_t)+\|(f_{v1}f_\gamma f_{d1},\ f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm1}f_\gamma f_{dm1}+f_{vm2}f_\gamma f_{dm2})\|]}$$

$$\Leftrightarrow \frac{f_s\,\|(f_{v1}f_\gamma f_{d1},\ f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm2}f_\gamma f_{dm2})\| - f_t\,\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm1}f_\gamma f_{dm1})\|}{(f_s+\|(f_{v1}f_\gamma f_{d1},\ f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm1}f_\gamma f_{dm1})\|)[(f_s+f_t)+\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm1}f_\gamma f_{dm1}+f_{vm2}f_\gamma f_{dm2})\|]} \ge 0$$

$$\frac{\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm1}f_\gamma f_{dm1}+f_{vm2}f_\gamma f_{dm2})\|}{f_s+f_t+\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm1}f_\gamma f_{dm1}+f_{vm2}f_\gamma f_{dm2})\|} - \frac{\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm1}f_\gamma f_{dm1})\|}{f_s+\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm1}f_\gamma f_{dm1})\|} \ge 0$$

$$\Leftrightarrow \frac{\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm1}f_\gamma f_{dm1}+f_{vm2}f_\gamma f_{dm2})\|}{f_s+f_t+\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm1}f_\gamma f_{dm1}+f_{vm2}f_\gamma f_{dm2})\|} \ge \frac{\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm1}f_\gamma f_{dm1})\|}{f_s+\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm1}f_\gamma f_{dm1})\|}$$

$\Leftrightarrow F_\Gamma(f_{v1}\ f_\gamma\ f_{d1}, f_{v2}\ f_\gamma f_{d2},..,f_{vm-1}\ f_\gamma\ f_{dm-1},\ f_{vm1}\ f_\gamma f_{dm1}+\ f_{vm2}\ f_\gamma f_{dm2}\ ,\ f_s+\ f_t) \ge F_\Gamma(f_{v1}\ f_\gamma\ f_{d1},\ f_{v2}\ f_\gamma f_{d2},\ldots,f_{vm-1}\ f_\gamma\ f_{dm-1},\ f_{vm1}\ f_\gamma f_{dm1},\ f_s)$ ……. (2)

From (1) and (2) we obtain

$F_\Gamma(f_{v1}\ f_\gamma\ f_{d1},f_{v2}\ f_\gamma f_{d2},\ldots,f_{vm-1}\ f_\gamma\ f_{dm-1},\ f_{vm1}\ f_\gamma f_{dm1}+\ f_{vm2}\ f_\gamma f_{dm2}\ ,\ f_s+\ f_t) \ge \min\{\ F_\Gamma(f_{v1}\ f_\gamma\ f_{d1},\ f_{v2}\ f_\gamma f_{d2},\ldots,f_{vm-1}\ f_\gamma\ f_{dm-1},f_{vm1}\ f_\gamma f_{dm1},\ f_s),\ F_\Gamma(f_{v1}\ f_\gamma\ f_{d1},\ f_{v2}\ f_\gamma f_{d2},\ldots,f_{vm-1}\ f_\gamma\ f_{dm-1},f_{vm2}\ f_\gamma f_{dm2}\ ,\ f_t)\ \}$

(6) We have clearly $F_\Gamma(f_{v1}f_\gamma f_{d1},\ f_{v2}f_\gamma f_{d2},\ldots\ldots f_{vm-1}f_\gamma f_{dm-1},\ f_{vm}f_\gamma f_{dm},\ f_t)$,

is a left continuous and non-decreasing function of $f_t$ in $(-\infty,\infty)$ such that

$$\lim_{m\to\infty} F_\Gamma(f_{v1}f_\gamma f_{d1},\ f_{v2}f_\gamma f_{d2},\ldots\ldots f_{vm-1}f_\gamma f_{dm-1},\ f_{vm}f_\gamma f_{dm},\ f_t)$$
$$= \lim_{m\to\infty} \frac{\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm}f_\gamma f_{dm})\|}{f_t+\|(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vm-1}f_\gamma f_{dm-1},f_{vm}f_\gamma f_{dm})\|} = 0$$

**3.15.Definition:** Let $(F_V,F_\Gamma)$be the fuzzy Gamma-m-normed linear space , a sequence $\{f_{vr}f_\gamma f_{dr}\}_{r=1}^{\infty}$ in $(F_V,F_\Gamma)$ is convergent to $f_v\ f_\gamma f_d$ if for every δ in (0,1) and $f_t$ in $(0,\infty)$ there exists appositive number M such that

$$F_\Gamma(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vr-1}f_\gamma f_{dr-1},f_{vr}f_\gamma f_{dr} - f_v f_\gamma f_d,f_t) < \delta,\ \text{for all } r \ge M$$
$$\Rightarrow \lim_{r\to\infty} F_\Gamma(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vr-1}f_\gamma f_{dr-1},f_{vr}f_\gamma f_{dr} - f_v f_\gamma f_d,f_t) = 0$$

**3.16.Definition:** Let $(F_V,F_\Gamma)$ be the fuzzy Gamma-m-normed linear space , a sequence $\{f_{vr}f_\gamma f_{dr}\}_{r=1}^{\infty}$ in $((F_V,F_\Gamma)$is said to be Cauchy sequence if for every δ is in (0,1) and $f_t\in(0,\infty)$ there exists appositive number M such that

$F_\Gamma(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vr-1}f_\gamma f_{dr-1},f_{vr}\ f_\gamma f_{dr}-\ f_{v\ell}\ f_\gamma f_{d\ell}\ ,\ f_t)<\delta$ ,for all r ,ℓ≥M ,

That is $\lim_{r,l\to\infty} F_\Gamma(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vr-1}f_\gamma f_{dr-1},f_{vr}f_\gamma f_{dr} - f_{v\ell}f_\gamma f_{d\ell},\ f_t) = 0$

**3.17. Definition:** A fuzzy Gamma-m-normed linear space $(F_V,F_\Gamma)$is said to be complete if every Cauchy sequence in $(F_V,F_\Gamma)$is convergence sequence .

**3.18. Theorem:** Let $(F_V,F_\Gamma)$ be the fuzzy Gamma-m-normed linear space and let $\{f_{vr}f_\gamma f_{dr}\}_{r=1}^{\infty}$ be a sequence in it then

(1) A sequence $\{f_{vr}f_\gamma f_{dr}\}_{r=1}^{\infty}$ convergence to $f_v\ f_\gamma f_d$ if and only if $F_\Gamma(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vr-1}f_\gamma f_{dr-1},f_{vr}f_\gamma f_{dr}-f_v\ f_\gamma f_d,f_t)\to 0$, as $r\to\infty$.

(2) Every convergence sequence is a Cauchy sequence in $(F_V,F_\Gamma)$.

**Proof:** Suppose a sequence $\{f_{vr}f_\gamma f_{dr}\}_{r=1}^{\infty}$ convergence to $f_v f_\gamma f_d$ if fix $f_t > 0$, then for any given δ in (0,1) there is a positive integer r ≥ M, such that

$$F_\Gamma(f_{v1}f_\gamma f_{d1}, f_{v2}f_\gamma f_{d2}, \ldots, f_{vr-1}f_\gamma f_{dr-1}, f_{vr}f_\gamma f_{dr} - f_v f_\gamma f_d, f_t) < \delta \text{ , for all } r \geq M$$

$$\Rightarrow F_\Gamma(f_{v1}f_\gamma f_{d1}, f_{v2}f_\gamma f_{d2}, \ldots, f_{vr-1}f_\gamma f_{dr-1}, f_{vr}f_\gamma f_{dr} - f_v f_\gamma f_d, f_t) \to 0, \text{ as } r \to \infty.$$

$$\textit{that is } \lim_{r\to\infty} F_\Gamma(f_{v1}f_\gamma f_{d1}, f_{v2}f_\gamma f_{d2}, \ldots, f_{vr-1}f_\gamma f_{dr-1}, f_{vr}f_\gamma f_{dr}, f_t) = f_v f_\gamma f_d$$

Conversely if for each $f_t \in (0,\infty)$ $F_\Gamma(f_{v1}f_\gamma f_{d1}, f_{v2}f_\gamma f_{d2}, \ldots, f_{vr-1}f_\gamma f_{dr-1}, f_{vr} f_\gamma f_{dr} - f_v f_\gamma f_d, f_t) \to 0$, as $r \to \infty$ then for every δ in (0,1)there exists positive integer r≥M, such that $F_\Gamma(f_{v1}f_\gamma f_{d1}, f_{v2}f_\gamma f_{d2}, \ldots, f_{vr-1}f_\gamma f_{dr-1}, f_{vr} f_\gamma f_{dr} - f_v f_\gamma f_d, f_t) < \delta$ , for all r ≥M,

Hence a sequence $\{f_{vr}f_\gamma f_{dr}\}_{r=1}^{\infty}$ is convergence to $f_v f_\gamma f_d$ in $(F_V, F_\Gamma)$

(2) Suppose a sequence $\{f_{vr}f_\gamma f_{dr}\}_{r=1}^{\infty}$ is convergence in $(F_V, F_\Gamma)$ and if it is convergence to $f_v f_\gamma f_d$. Let $f_t$ in (0,∞) and $0 < \varepsilon < 1$ then we choose δ is in (0,1)such that $\delta \, \Delta^F \, \delta < \varepsilon$ , since $\{f_{vr}f_\gamma f_{dr}\}_{r=1}^{\infty}$ is convergence to $f_v f_\gamma f_d$

Which implies that there exists positive integer r ≥ M, such that

$$F_\Gamma(f_{v1}f_\gamma f_{d1}, f_{v2}f_\gamma f_{d2}, \ldots, f_{vr-1}f_\gamma f_{dr-1}, f_{vr}f_\gamma f_{dr} - f_v f_\gamma f_d, \frac{f_t}{2}) < \delta, \text{ for all } r \geq M,$$

We consider that

$$F_\Gamma(f_{v1}f_\gamma f_{d1}, f_{v2}f_\gamma f_{d2}, \ldots, f_{vr-1}f_\gamma f_{dr-1}, f_{vr}f_\gamma f_{dr} - f_{v\ell}f_\gamma f_{d\ell}, f_t)$$

$$= F_\Gamma(f_{v1}f_\gamma f_{d1}, f_{v2}f_\gamma f_{d2}, \ldots, f_{vr-1}f_\gamma f_{dr-1}, f_{vr}f_\gamma f_{dr} - f_v f_\gamma f_d + f_v f_\gamma f_d - f_{v\ell}f_\gamma f_{d\ell}, \frac{f_t + f_t}{2})$$

$$\leq \{F_\Gamma(f_{v1}f_\gamma f_{d1}, f_{v2}f_\gamma f_{d2}, \ldots, f_{vr-1}f_\gamma f_{dr-1}, f_{vr}f_\gamma f_{dr} - f_v f_\gamma f_d, \frac{f_t}{2}) \Delta^F F_\Gamma(f_{v1}f_\gamma f_{d1}, f_{v2}f_\gamma f_{d2}, \ldots, f_{vr-1}f_\gamma f_{dr-1}, f_{v\ell}f_\gamma f_{d\ell} - f_v f_\gamma f_d, \frac{f_t}{2})\}$$

$$< \delta \, \Delta^F \delta$$

$$< \varepsilon, \text{for all } r, \ell \geq M$$

Therefore $\{f_{vr}f_\gamma f_{dr}\}_{r=1}^{\infty}$ is a cauchy sequence in $(F_V, F_\Gamma)$.

**3.19. Remark:** The following examples **3.21.Example and 3.22.Example** shows that there may exist cauchy sequence in the fuzzy Gamma-m-normed linear space $(F_V, F_\Gamma)$ which is not convergent.

**3.20. Remark**: For the following examples **3.21.Example and 3.22.Example** consider a fuzzy Gamma-m-normed linear space $(F_V, F_\Gamma)$as in previous **3.14. Example.**

**3.21.Example:** Let $(F_V, F_\Gamma)$be the fuzzy Gamma-m-normed linear space , a sequence $\{f_{vr}f_\gamma f_{dr}\}_{r=1}^{\infty}$ in $(F_V, F_\Gamma)$ then a sequence $\{f_{vr}f_\gamma f_{dr}\}_{r=1}^{\infty}$ is a convergence in$(F_V, F_\Gamma)$ a sequence $\{f_{vr}f_\gamma f_{dr}\}_{r=1}^{\infty}$is a convergence in$(F_V, \|.,..,.\|)$

**Proof:** We have a sequence $\{f_{vr}f_\gamma f_{dr}\}_{r=1}^{\infty}$is also a convergence in $(F_V, F_\Gamma)$

$$\lim_{r\to\infty} F_\Gamma(f_{v1}f_\gamma f_{d1}, f_{v2}f_\gamma f_{d2}, f_{v3}f_\gamma f_{d3}, \ldots, f_{vr-1}f_\gamma f_{dr-1}, f_{vr}f_\gamma f_{dr} - f_v f_\gamma f_d, f_t) = 0$$

$$\Leftrightarrow \lim_{r\to\infty} \frac{\| (f_{v1}f_\gamma f_{d1}, f_{v2}f_\gamma f_{d2}, \ldots, f_{vr-1}f_\gamma f_{dr-1}, f_{vr}f_\gamma f_{dr} - f_v f_\gamma f_d) \|}{f_t + \| (f_{v1}f_\gamma f_{d1}, f_{v2}f_\gamma f_{d2}, \ldots, f_{vm-1}f_\gamma f_{dr-1}, f_{vr}f_\gamma f_{dr} - f_v f_\gamma f_d) \|} = 0$$

$$\Leftrightarrow \lim_{r\to\infty} \| (f_{v1}f_\gamma f_{d1}, f_{v2}f_\gamma f_{d2}, \ldots, f_{vr-1}f_\gamma f_{dr-1}, f_{vr}f_\gamma f_{dr} - f_v f_\gamma f_d) \| = 0$$

Hence a sequence $\{f_{vr}f_\gamma f_{dr}\}_{r=1}^{\infty}$is a convergence in $(F_V, \|.,\ldots,.\|)$.

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Conversely we have a sequence $\{f_{vr}f_\gamma f_{dr}\}_{r=1}^{\infty}$ is a convergence in (Fv, ‖.,….,. ‖)

$$\Leftrightarrow \lim_{r\to\infty} \| (f_{v1}f_\gamma f_{d1}, f_{v2}f_\gamma f_{d2}, \ldots, f_{vr-1}f_\gamma f_{dr-1}, f_{vr}f_\gamma f_{dr} - f_v f_\gamma f_d) \| = 0$$

$$\Leftrightarrow \lim_{r\to\infty} \frac{\| (f_{v1}f_\gamma f_{d1}, f_{v2}f_\gamma f_{d2}, \ldots, f_{vr-1}f_\gamma f_{dr-1}, f_{vr}f_\gamma f_{dr} - f_v f_\gamma f_d) \|}{f_t + \| (f_{v1}f_\gamma f_{d1}, f_{v2}f_\gamma f_{d2}, \ldots, f_{vm-1}f_\gamma f_{dr-1}, f_{vr}f_\gamma f_{dr} - f_v f_\gamma f_d) \|} = 0$$

$$\lim_{r\to\infty} F_\Gamma(f_{v1}f_\gamma f_{d1}, f_{v2}f_\gamma f_{d2}, f_{v3}f_\gamma f_{d3}, \ldots, f_{vr-1}f_\gamma f_{dr-1}, f_{vr}f_\gamma f_{dr} - f_v f_\gamma f_d, f_t) = 0$$

Hence a sequence $\{f_{vr}f_\gamma f_{dr}\}_{r=1}^{\infty}$ is also a convergence in (Fv, FΓ).

**3.22.Example**: Let (Fv,FΓ)be the fuzzy Gamma-m-normed linear space , a sequence $\{f_{vr}f_\gamma f_{dr}\}_{r=1}^{\infty}$ in (Fv,FΓ)then a sequence $\{f_{vr}f_\gamma f_{dr}\}_{r=1}^{\infty}$ is a Cauchy sequence in(Fv,FΓ) a sequence $\{f_{vr}f_\gamma f_{dr}\}_{r=1}^{\infty}$ is a Cauchy sequence in (Fv, ‖.,….,. ‖).

**Proof:** Suppose a sequence $\{f_{vr}f_\gamma f_{dr}\}_{r=1}^{\infty}$ is a Cauchy sequencein (Fv,FΓ)

$$\lim_{r,l\to\infty} F_\Gamma(f_{v1}f_\gamma f_{d1}, f_{v2}f_\gamma f_{d2}, \ldots, f_{vr-1}f_\gamma f_{dr-1}, f_{vr}f_\gamma f_{dr} - f_{v\ell}f_\gamma f_{d\ell}, f_t) = 0$$

$$\Leftrightarrow \lim_{r,\ell\to\infty} \frac{\| (f_{v1}f_\gamma f_{d1}, f_{v2}f_\gamma f_{d2}, \ldots, f_{vr-1}f_\gamma f_{dr-1}, f_{vr}f_\gamma f_{dr} - f_{v\ell}f_\gamma f_{d\ell}) \|}{f_t + \| (f_{v1}f_\gamma f_{d1}, f_{v2}f_\gamma f_{d2}, \ldots, f_{vm-1}f_\gamma f_{dr-1}, f_{vr}f_\gamma f_{dr} - f_{v\ell}f_\gamma f_{d\ell}) \|} = 0$$

$$\lim_{r,\ell\to\infty} \| (f_{v1}f_\gamma f_{d1}, f_{v2}f_\gamma f_{d2}, \ldots, f_{vr-1}f_\gamma f_{dr-1}, f_{vr}f_\gamma f_{dr} - f_{v\ell}f_\gamma f_{d\ell}) \| = 0$$

Hence a sequence $\{f_{vr}f_\gamma f_{dr}\}_{r=1}^{\infty}$ is a Cauchy sequence in (Fv, ‖.,….,. ‖).

Conversely we prove that a sequence $\{f_{vr}f_\gamma f_{dr}\}_{r=1}^{\infty}$ is a Cauchy sequence in (Fv,FΓ) when it is a Cauchy sequence in (Fv, ‖.,….,. ‖) if for every $\varepsilon$ >0 there exists appositive number M suchthat

$$\Leftrightarrow \| (f_{v1}f_\gamma f_{d1}, f_{v2}f_\gamma f_{d2}, \ldots, f_{vr-1}f_\gamma f_{dr-1}, f_{vr}f_\gamma f_{dr} - f_{v\ell}f_\gamma f_{d\ell}) \| < \varepsilon, \text{ whenever } r, \ell \geq M$$

$$\Leftrightarrow \left| \frac{\| (f_{v1}f_\gamma f_{d1}, f_{v2}f_\gamma f_{d2}, \ldots, f_{vr-1}f_\gamma f_{dr-1}, f_{vr}f_\gamma f_{dr} - f_{v\ell}f_\gamma f_{d\ell}) \|}{f_t + \| (f_{v1}f_\gamma f_{d1}, f_{v2}f_\gamma f_{d2}, \ldots, f_{vm-1}f_\gamma f_{dr-1}, f_{vr}f_\gamma f_{dr} - f_{v\ell}f_\gamma f_{d\ell}) \|} \right| < \varepsilon, \text{ whenever } r, \ell \geq M$$

$$\Leftrightarrow \left| F_\Gamma(f_{v1}f_\gamma f_{d1}, f_{v2}f_\gamma f_{d2}, \ldots, f_{vr-1}f_\gamma f_{dr-1}, f_{vr}f_\gamma f_{dr} - f_{v\ell}f_\gamma f_{d\ell}, f_t) \right| < \varepsilon, \text{ whenever } r, \ell \geq M$$

Hence a sequence $\{f_{vr}f_\gamma f_{dr}\}_{r=1}^{\infty}$ is a Cauchy sequence in (Fv,FΓ).

**3.23.Theorem:** In a fuzzy Gamma-m-normed linear space (Fv,FΓ),every Cauchy sequence has a convergent sub-sequence is complete.

**Proof:** Let (Fv,FΓ) be the fuzzy Gamma-m-normed linear space and let a sequence $\{f_{vr}f_\gamma f_{dr}\}_{r=1}^{\infty}$ is a Cauchy sequence in (Fv,FΓ).

Let $\{f_{vr\ell}f_\gamma f_{dr\ell}\}_{r=1}^{\infty}$ be a subsequence of $\{f_{vr}f_\gamma f_{dr}\}_{r=1}^{\infty}$ and it is convergence to $f_v\ f_\gamma\ f_d$ .now we need to prove that the sequence $\{f_{vr}f_\gamma f_{dr}\}_{r=1}^{\infty}$ is convergence to $f_v\ f_\gamma\ f_d$ , for this let $\varepsilon$ in (0,1) and $f_t \in (0,\infty)$, choose δ is in (0,1)such that $\delta\ \Delta^F\ \delta < \varepsilon$ .

Given that a sequence $\{f_{vr}f_\gamma f_{dr}\}_{r=1}^{\infty}$ is a Cauchy sequence in (Fv,FΓ)that is if for every ϵ in (0,1) and $f_t \in (0,\infty)$ there exists appositive number M such that $F_\Gamma(f_{v1}f_\gamma f_{d1}, f_{v2}f_\gamma f_{d2}, \ldots, f_{vr-1}f_\gamma f_{dr-1}, f_{vr}\ f_\gamma f_{dr} - f_{v\ell}\ f_\gamma f_{d\ell}\ , f_t) < \delta$ , for all r ,ℓ ≥M.

We have the sub-sequence $\{f_{vr\ell}f_\gamma f_{dr\ell}\}_{r=1}^{\infty}$ is also convergent to $f_v$ $f_\gamma$ $f_d$, there exists a r,ℓ >M such that $F_\Gamma(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vr-1}f_\gamma f_{dr-1}, f_{v\ell}\, f_\gamma f_{d\ell} - f_v\, f_\gamma\, f_d, \frac{1}{2}f_t)<\delta$ for all r ,ℓ≥M.

Now $F_\Gamma(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vr-1}f_\gamma f_{dr-1}, f_{v\ell}\, f_\gamma f_{d\ell} - f_v\, f_\gamma\, f_d, f_t) = F_\Gamma(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vr-1}f_\gamma f_{dr-1}, f_{v\ell}\, f_\gamma f_{d\ell} - f_v\, f_\gamma\, f_d,\frac{1}{2}f_t+\frac{1}{2}f_t) \leq F_\Gamma(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vr-1}f_\gamma f_{dr-1},f_{v\ell}f_\gamma f_{d\ell}-f_v\, f_\gamma\, f_d,\frac{1}{2}f_t)\Delta^F F_\Gamma(f_{v1}f_\gamma f_{d1},f_{v2}f_\gamma f_{d2},\ldots,f_{vr-1}f_\gamma f_{dr-1}, f_{v\ell}\, f_\gamma f_{d\ell} - f_v\, f_\gamma\, f_d, \frac{1}{2}f_t)< \delta\, \Delta^F\, \delta< \varepsilon$

Therefore a sequence $\{f_{vr}f_\gamma f_{dr}\}_{r=1}^{\infty}$ is convergence to $f_v$ $f_\gamma$ $f_d$ in $(F_V,F_\Gamma)$

Hence it is complete.

## 4. Discussion and Conclusion

In this research paper, we convicted the concept of fuzzy gamma ring, fuzzy gamma vector space and using this also introduced the notion of fuzzy Gamma m-normed linear space and produced a detailed axioms with theory of fuzzy n-normed linear space. In fuzzy Gamma m-normed linear space obtained some results on cauchy and convergence sequence. Also provided theorems of completeness sequence and Cauchy sequence in fuzzy Gamma m-normed linear space. This work can be extended to Banach fuzzy gamma linear space by introducing the concepts of completeness in fuzzy Gamma m-normed linear space.

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