

Copmleteness of Fuzzy Gamma-M-Normed Linear Space

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Abstract: This paper introduces the conception of fuzzy gamma-m-normed linear space in accordance with the theory of fuzzy n-normed linear space and it is proved that in a finite dimensional fuzzy gamma m-normed linear space, fuzzy m-normed linear spaces are the same, up to the fuzzy norm equivalence. Also the paper introduces fuzzy gamma-2-normed linear spaces, fuzzy right gamma-m-normed linear spaces and its properties and fuzzy gamma-m-normed linear space which can be analyzed by using the fuzzy n-normed linear space. The paper uses an applications with examples for algebraic operations of fuzzy set theory. The most important concepts fuzzy gamma ring, fuzzy gamma divison ring, fuzzy gamma vector space, fuzzy gamma ring have already been introduced. Using these concepts. Using these concepts it initiated the fuzzy Gamma-m-normed linear space and suggested a theorem for the gamma norm function which is continuous. So far the earlier research has been done in the general t-norm in a fuzzy n-normed linear space and proven that if t-norm is chosen other than “minimum” then the decomposition theorem of a fuzzy norm into a family of crisp norms may not hold. The paper identified completeness of fuzzy gamma m-normed linear space and constructed a norm function and satisfies the axioms of it fuzzy gamma m-normed linear space and additionally provided an example with proof in which a sequence is a Cauchy sequence and converges sequence in fuzzy gamma m-normed linear space if and only if it is Cauchy sequence and Convergence sequence in completeness of fuzzy gamma m-normed linear space. This paper originates the notion of completeness, and produces some results on it in fuzzy Gamma-m-normed linear space. Also a necessary condition and theorem for completeness of a sequence in fuzzy gamma m-normed linear space is suggested.

Keywords: fuzzy gamma ring, fuzzy division ring, fuzzy gamma vector space, fuzzy gamma linear space, fuzzy gamma normed linear space, fuzzy Gamma-2-normed linear space, fuzzy right gamma-m-normed linear space,

1. Introduction:

In 1987 the notion of a Gamma-ring which is more generalization than a ring was introduced by N.Nobusawa[1],and W.E.Barnes was developed the concepts in gamma-ring such as prime and primary ideals, gamma homomorphism's. after that many mathematicians determined the some interesting results on gamma-ring in accordance with concepts of Barnes and Nobusawa.[2] Gahler was introduced the theory of 2-norm,n-norm on a linear space[3] Bag and Samantha [4] introduced the notion of fuzzy norm on a linear space subsequently theory of fuzzy norm on a linear space[5,6,7]. First time A.K.Katsaras[8,9] introduced the concept of fuzzy normed linear space. M.Demirci [10,11] interposed the notion of fuzzy equality and smooth group by using fuzzy binary operations. The Fuzzy n-normed linear space introduced by AL.Narayanan and S.Vijayabalaji [12,13,14,15,16] and Reddy, B. S.[17,18] proposed the concept of Fuzzy anti n-normed linear space and provide some results convergence sequence and Cauchy sequence in fuzzy anti n-normed linear space. The notion of n-normed left gamma-linear space is convection by S Kalaiselvan and S.Shivaramakrishnan [19]. All the authors inspired by Lotfi A. Zadeh [20] in his fuzzy set theory.

Motivated by the previously mentioned theory, we present the concept of fuzzy gamma-m-normed linear spaces, denoting the convergent and cauchy sequence within these spaces. Additionally, we confirmed a few of the findings.

2. PRILIMANARIES:

2.1. Definition: An illustration of the degree to which each input contributes is provided by the characteristic function. Every processed input is assigned a weight, the functional overlap between the inputs is defined, and to ascertain their impact on the fuzzy output sets of the final output conclusion, the rules employ the input membership values as weight factors. A function that meets our needs for ease of use, speed, simplicity, and efficiency can be defined as an arbitrary curve whose shape we can describe. This function is then defuzzed into a crisp output that powers the system. In the continuous unit interval $[0,1]$, where the end point values "0" and "1" denote different degrees of membership, Professor Zadeh expanded the concept of binary membership.

2.2. Definition: Fuzzy sets theory is an extension of classical set theory and elements have varying degree of membership a logic based on two truth values. If $\mathbb{U}$ is the universe of discourse, a set F is said to be fuzzy set in F if there exists a function $\mu: F \rightarrow [0,1]$ and it is denoted by a set of ordered pairs as $F = \{(u, \mu_F(u)) / u \in \mathbb{U}\}$

2.3. Definition: Suppose $\mathbb{U}$ is a discrete and finite universe of discourse the fuzzy set F is written as

$$F = \mu_F(u_1)/u_1 + \mu_F(u_2)/u_2 + \mu_F(u_3)/u_3 + \dots = \sum_{u \in \mathbb{U}} \mu_F(u) = \{(u_i, \mu_F(u_i)) / u_i \in \mathbb{U}\}$$

2.4 .Definition: Suppose $\mathbb{U}$ is a continuous and finite universe of discourse the fuzzy set F is written as $F = \int_{\mathbb{U}} \mu_F(u) du$, here the summation and integration signs indicate the collection of all elements u in the universe of discourse $\mathbb{U}$ along with their associated membership values $\mu_F(u)$

2.5.Example: $\mathbb{U} = \{H^1 = \text{Finland}, H^2 = \text{Luxembourg}, H^3 = \text{Thailad}, H^4 = \text{India}, H^5 = \text{Turkiya}, H^6 = \text{Ukraine}, H^7 = \text{Zimbabwe}, H^8 = \text{Afghanistan}\}$ is the universe of discourse of all countries then

(i) Let fuzzy set F_1 represents "Ranking in the World Happiness Report" if fuzzy set represented as $F_1 = \{(\text{Finland}, 1.00), (\text{Luxebourg}, 0.94), (\text{Thailad}, 0.60), (\text{India}, 0.14), (\text{Tukiya}, 0.33), (\text{Ukraine}, 0.29), (\text{Zimbabwe}, 0.07), (\text{Afghanistan}, 0.00)\}$ or $F_1 = \{(H^1, 1.00), (H^2, 0.94), (H^3, 0.60), (H^4, 0.14), (H^5, 0.33), (H^6, 0.29), (H^7, 0.07), (H^8, 0.00)\}$ also it can be represented in summation form

$$F_1 = 1.00/\text{Finland} + 0.94/\text{Luxembourg} + 0.60/\text{Thailad} + 0.14/\text{India} + 0.33/\text{Turkiya}$$

$$+ 0.29/\text{Ukraine} + 0.07/\text{Zimbabwe} + 0.00/\text{Afghanistan} \quad \text{Or}$$

$$F_1 = 1.00/H^1 + 0.94/H^2 + 0.60/H^3 + 0.14/H^4 + 0.33/H^5 + 0.29/H^6 + 0.07/H^7 + 0.00/H^8$$

(ii) Let fuzzy set F_2 represents "Ranking in the Human Development Report" then the fuzzy set represented as $F_2 = \{(\text{Finland}, 0.99), (\text{Luxembourg}, 0.89), (\text{Thailad}, 0.82), (\text{India}, 0.30), (\text{Turkiya}, 0.76), (\text{Ukraine}, 0.48), (\text{Zimbabwe}, 0.17), (\text{Afghanistan}, 0.05)\} = \{(H^1, 0.99), (H^2, 0.89), (H^3, 0.82), (H^4, 0.30), (H^5, 0.76),$

$(H^6, 0.48), (H^7, 0.17), (H^8, 0.05)\}$ also it can be represented in summation form $F_2 = 0.99/\text{Finland} + 0.89/\text{Luxembourg} + 0.82/\text{Thailad} + 0.30/\text{India} + 0.76/\text{Turkiya}$

$$+ 0.48/\text{Ukraine} + 0.17/\text{Zimbabwe} + 0.05/\text{Afghanistan}$$

$$= 0.99/H^1 + 0.89/H^2 + 0.82/H^3 + 0.30/H^4 + 0.76/H^5 + 0.48/H^6 + 0.17/H^7 + 0.05/H^8$$

2.6. Definition: If $\mathbb{U}$ is the universe of discourse, a set F_1 and F_2 fuzzy sets with characteristic functions $\mu_{F_1}(u)$ and $\mu_{F_2}(u)$ respectively the fuzzy set operations are defined as given bellow

(1) Union: the union of two fuzzy sets F_1 and F_2 is defined as

$$\mu_{F_1 \cup F_2}(u) = \mu_{F_1}(u) \vee \mu_{F_2}(u) = \max\{\mu_{F_1}(u), \mu_{F_2}(u)\}, \text{ for all } u \in \mathbb{U}$$

(2) Intersection: the intersection of two fuzzy sets F_1 and F_2 is defined as for all $u \in \mathbb{U}$,

$$\mu_{F_1 \cap F_2}(u) = \mu_{F_1}(u) \wedge \mu_{F_2}(u) = \min\{\mu_{F_1}(u), \mu_{F_2}(u)\}$$

(3) Compliment: the compliment of F is denoted by F^C and it is defined as $\mu_{F^C}(u) = 1 - \mu_F(u)$

(4) Algebraic Sum: the sum of the two fuzzy sets $\mu_{F_1}(u)$, $\mu_{F_2}(u)$ is $\mu_{F_1}(u) + \mu_{F_2}(u)$ it is defined as

$$\mu_{F_1+F_2}(u) = \mu_{F_1}(u) + \mu_{F_2}(u) - \mu_{F_1}(u) \cdot \mu_{F_2}(u)$$

(5) Algebraic Product: The product of the two fuzzy sets is $\mu_{F_1}(u) \cdot \mu_{F_2}(u)$ it is defined as

$$\mu_{F_1 \cdot F_2}(u) = \mu_{F_1}(u) \cdot \mu_{F_2}(u)$$

(6) Bounded Sum : the bounded sum of the two fuzzy sets $\mu_{F_1}(u)$, $\mu_{F_2}(u)$ is $\mu_{F_1}(u) \oplus \mu_{F_2}(u)$ it is defined as

$$\mu_{F_1 \oplus F_2}(u) = \min \{1, \mu_{F_1}(u) + \mu_{F_2}(u)\}$$

(7) Bounded difference: the bounded difference $\ominus$ of the two fuzzy sets $\mu_{F_1}(u)$, $\mu_{F_2}(u)$ is $\mu_{F_1}(u) \ominus \mu_{F_2}(u)$ it is

$$\text{defined as } \mu_{F_1 \ominus F_2}(u) = \max\{0, \mu_{F_1}(u) - \mu_{F_2}(u)\}$$

Table -1. Outcomes of the operations on fuzzy sets.

	H ¹	H ²	H ³	H ⁴	H ⁵	H ⁶	H ⁷	H ⁸
$\mu_{F_1 \cup F_2}$	1.00	0.94	0.82	0.30	0.76	0.48	0.17	0.05
$\mu_{F_1 \cap F_2}$	0.99	0.89	0.60	0.14	0.33	0.29	0.07	0.00
$\mu_{F_1^c}$	0.00	0.06	0.18	0.70	0.24	0.52	0.83	0.95
$\mu_{F_1+F_2}$	1.00	0.99	0.928	0.39	0.83	0.63	0.22	0.05
$\mu_{F_1 \cdot F_2}$	0.99	0.83	0.49	0.04	0.25	0.13	0.01	0.00
$\mu_{F_1 \oplus F_2}$	1.00	1.00	1.00	0.44	1.00	0.77	0.24	0.05
$\mu_{F_1 \ominus F_2}$	0.01	0.05	0.22	0.16	0.43	0.18	0.10	0.05

2.7.Definition: Let F_G be any group and mapping $\mu_{F_G} : F_G \rightarrow [0,1]$ is called fuzzy subgroup

if for all $f_{g1}, f_{g2} \in F_G$

$$(1) \mu_{F_G}(f_{g1} \cdot f_{g2}) \geq \min \{ \mu_{F_G}(f_{g1}), \mu_{F_G}(f_{g2}) \}$$

$$(2) \mu_{F_G}((f_{g1})^{-1}) = \mu_{F_G}(f_{g1})$$

$$(3) \text{ Let } f_{ge} \text{ be the identity element of fuzzy group } F_G \text{ such that } \mu_{F_G}(f_g) \leq \mu_{F_G}(f_{ge}), \text{ for all } f_g \in F_G$$

2.8.Definition: Let F_R is fuzzy additive abelian group, and F_Γ be any additive group the mapping

$Z_{\mathcal{A}} : F_R \times F_\Gamma \times F_R \rightarrow F_R$ and it is defined as $Z_{\mathcal{A}}(f_{r1}, f_\gamma, f_{r2}) = f_{r1} \cdot f_\gamma \cdot f_{r2}$ such that F_R is called as fuzzy Gamma ring if it satisfies the following properties let for any $f_{r1}, f_{r2}, f_{r3} \in F_R$ and $f_{\gamma1}, f_{\gamma2}$ and $f_\gamma \in F_\Gamma$

$$(1) Z_{\mathcal{A}}(f_{r1}+f_{r2}, f_\gamma, f_{r3}) = Z_{\mathcal{A}}(f_{r1}, f_\gamma, f_{r2}) + Z_{\mathcal{A}}(f_{r2}, f_\gamma, f_{r3})$$

$$(2) Z_{\mathcal{A}}(f_{r1}, f_{\gamma1}+f_{\gamma2}, f_{r2}) = Z_{\mathcal{A}}(f_{r1}, f_{\gamma1}, f_{r2}) + Z_{\mathcal{A}}(f_{r1}, f_{\gamma2}, f_{r2})$$

$$(3) Z_{\mathcal{A}}(f_{r1}, f_\gamma, f_{r2}+f_{r3}) = Z_{\mathcal{A}}(f_{r1}, f_\gamma, f_{r2}) + Z_{\mathcal{A}}(f_{r1}, f_\gamma, f_{r3})$$

$$(4) Z_{\mathcal{A}}((f_{r1}, f_{\gamma1}, f_{r2}), f_{\gamma2}, f_{r3}) = Z_{\mathcal{A}}(f_{r1}, f_{\gamma1}, (f_{r2}, f_\gamma, f_{r3}))$$

2.9.Definition: Let $\mu_{\mathcal{A}}$ be any non-empty fuzzy sub set of fuzzy Gamma ring F_R is said to be fuzzy left ideal of F_R if it satisfies the following properties as fuzzy Gamma if for all

$f_{r1}, f_{r2}, f_{r3} \in F_R$ and $f_\gamma \in F_\Gamma$ such that

$$(i) \mu_{\mathcal{A}}(f_{r1} - f_{r2}) \geq \min \{ \mu_{\mathcal{A}}(f_{r1}), \mu_{\mathcal{A}}(f_{r2}) \}$$

$$(ii) \mu_{\mathcal{A}}(f_{r2} + f_{r1} - f_{r2}) \geq \mu_{\mathcal{A}}(f_{r1})$$

$$(iii) \mu_{\mathcal{G}}(f_{r2}f_{\gamma}(f_{r1}+f_{r3})-f_{r2}f_{\gamma}f_{r3}) \geq \mu_{\mathcal{G}}(f_{r1})$$

2.10. Definition: Let $(F_V, +, \times)$ is a vector space over a field $\mathcal{K}_V$ and mapping $\mu_V: F_V \rightarrow [0,1]$ is a fuzzy set of F_V is said to be fuzzy vector subspace of $\mathcal{K}_V$ if it satisfies the following properties for all $f_{v1}, f_{v2} \in F_V$ and $k_v \in \mathcal{K}_V$

$$(1) \mu(f_{v1}, f_{v2}) \geq \min \{ \mu_V(f_{v1}), \mu_V(f_{v2}) \}$$

$$(2) \mu((f_{v1})^{-1}) \geq \mu_V(f_{v1})$$

$$(3) \mu(k_v f_{v1}) \geq \mu_V(f_{v1})$$

2.11. Definition: If $\mu_V: F_V \rightarrow [0,1]$ is a fuzzy vector subspace of F_V over a field $\mathcal{K}_V$ if and only if

$$\mu_V(k_{v1}f_{v1} + k_{v2}f_{v2}) \geq \min \{ \mu_V(f_{v1}), \mu_V(f_{v2}) \}, \text{ for all } f_{v1}, f_{v2} \in F_V \text{ and for all } k_{v1}, k_{v2} \in \mathcal{K}_V$$

2.12. Definition: If the fuzzy Gamma-ring and F_D have an identity element and just one non-zero ideal, then they are referred to as division fuzzy Gamma-rings.

2.13. Definition: Let F_V is fuzzy vector space if $\widetilde{f_{v1}}(m) = V$ for all $m \in \mathcal{M}$ then a mapping $\|\cdot\|: F_V \rightarrow [0,1]$ is said to be a fuzzy norm on the soft vector space F_V if $\|\cdot\|$ is satisfies the following properties

$$(1) \|\widetilde{f_{v1}}\| \geq \widetilde{0}, \text{ for all } \widetilde{f_{v1}} \in F_V$$

$$(2) \|\widetilde{f_{v1}}\| = \widetilde{0} \Leftrightarrow \widetilde{f_{v1}} = \widetilde{0}$$

$$(3) \|\cdot \widetilde{f_{v1}}\| = \|k_v\| \|\widetilde{f_{v1}}\|, \text{ for all } \widetilde{f_{v1}} \in F_V \text{ and for every soft scalar } k_v \in \mathcal{K}_V$$

$$(4) \|\widetilde{f_{v1}} + \widetilde{f_{v2}}\| \leq \|\widetilde{f_{v1}}\| + \|\widetilde{f_{v2}}\|, \text{ for all } \widetilde{f_{v1}}, \widetilde{f_{v2}} \in F_V$$

The fuzzy vector space F with fuzzy norm $\|\cdot\|$ on F_V is said to be a fuzzy normed linear space and is denoted by $(F_V, \|\cdot\|, \mathcal{M})$. (1),(2),(3) and (4) are called to be fuzzy norm axiom.

2.14. Definition: Let F_D be a fuzzy division Gamma-ring possessing identification 1 and let $(F_V, +)$ be a fuzzy abelian group. and the function $\mathcal{Z}_V: F_D \times F_V \times F_V \rightarrow F_V$ and it is defined as $\mathcal{Z}_V(f_v, f_{\gamma}, f_d) = f_v \cdot f_{\gamma} \cdot f_d$ then F_V is called as a right fuzzy Gamma-vector space over F_D if the following properties holds for every $f_{v1}, f_{v2} \in F_V$, $f_{d1}, f_{d2} \in F_D$ and $f_{\gamma1}, f_{\gamma2}$ and $f_{\gamma} \in F_{\Gamma}$

$$(F_V - V^1) (f_{v1} + f_{v2}, f_{\gamma}, f_{d1}) = \mathcal{Z}(f_{v1}, f_{\gamma}, f_{d1}) + \mathcal{Z}_V(f_{v2}, f_{\gamma}, f_{d1})$$

$$(F_V - V^2) (f_{v1}, f_{\gamma}, f_{d1}, f_{d2}) = \mathcal{Z}(f_{v1}, f_{\gamma}, f_{d1}) + \mathcal{Z}_V(f_{v1}, f_{\gamma}, f_{d2})$$

$$(F_V - V^3) (f_{v1}, f_{\gamma1}, f_{d1}, f_{\gamma2}, f_{d2}) = \mathcal{Z}((f_{v1}, f_{\gamma1}, f_{d1}), f_{\gamma2}, f_{d2})$$

$(F_V - V^4) (f_{v1}, f_{\gamma}, 1) = f_{v1}$, for some $f_{\gamma} \in F_{\Gamma}$, the elements f_{v1}, f_{v2} are called fuzzy vectors in F_V , f_{d1}, f_{d2} are called fuzzy scalars in F_D .

2.15. Definition: The unit closed interval $[0, 1]$ and I_f be any closed sub-interval of $[0,1]$ and is defined by

$$I_f = [I_f^-, I_f^+] \text{ where } 0 \leq I_f^- \leq I_f^+ \leq 1 \text{ suppose } \mathcal{C}[0,1] \text{ be the set of all closed sub interval of } [0,1]$$

that is $\mathcal{C}[0,1] = \{ I_f / I_f = [I_f^-, I_f^+], I_f^- \leq I_f^+ \text{ and } I_f^-, I_f^+ \in [0,1] \}$

2.16. Definition: Consider F be any set and a mapping $F^l: F \rightarrow \mathcal{C}[0,1]$ and the set is represented as

$\{F^l(f) = [F^l(f), F^{l+}(f)] / F^l, F^{l+}$ are fuzzy subsets of F , $F^l(f) \leq F^{l+}(f)$, for all $f \in F$ } and this set is also called as interval-valued fuzzy subset of F .

2.17. Definition Let $(F_V, +, X)$ is a vector space over a field $\mathcal{K}_V$ with dimension m and μ is a fuzzy subset of F_V such that $\mu_V(k_{v1}f_{v1} + k_{v2}f_{v2}) \geq \min \{ \mu_V(f_{v1}), \mu_V(f_{v2}) \}$, for all $f_{v1}, f_{v2} \in F_V$ and for all $k_{v1}, k_{v2} \in \mathcal{K}_V$

2.18. Definition: Let Δ^F be a binary operation and is mapping from $[0,1] \times [0,1]$ to $[0,1]$

that is $\Delta^F: [0,1] \times [0,1] \rightarrow [0,1]$ is said to be continuous triangular-norm or t-norm if it satisfies the following axioms

(i) Δ^F is associative and commutative

That is for every $f^1, f^2, f^3 \in [0,1]$ such that $\Delta^F((f^1, f^2), f^3) = \Delta^F(f^1, (f^2, f^3))$ and

$$\Delta^F(f^1, f^2) = \Delta^F(f^2, f^1)$$

(ii) Δ^F is continuous

(iii) $\Delta^F(f^1, 1) = f^1$, for all $f^1 \in [0,1]$

(iv) $\Delta^F(f^1, f^2) \leq \Delta^F(f^3, f^4)$ whenever $f^1 \leq f^3$ and $f^2 \leq f^4$, for all $f^1, f^2, f^3, f^4 \in [0,1]$

2.19. Example: We have two examples of continuous t-norm $\Delta^F(f^1, f^2) = f^1 \cdot f^2$ $\Delta^F(f^1, f^2) = \min\{f^1, f^2\}$.

2.20. Remark: For any $f^1, f^2 \in (0,1)$ with $f^1 > f^2$ there exists $f^3, f^4 \in (0,1)$ such that $\Delta^F(f^1, f^3) \geq f^2$ and

for any $f^4 \in (0,1)$ there exists $f^5, f^6 \in (0,1)$ such that $\Delta^F(f^5, f^6) \geq f^4$, $\Delta^F(f^1, f^3) \geq f^2$

2.21. Definition: Let F_L linear space over a fuzzy field F a real valued function $\|\cdot, \dots, \cdot\|: F_L \times F_L \rightarrow [0,1]$ and it satisfies the following properties

(1) $\|v_{r1}, v_{r2}\| = 0$ if and only if v_{r1}, v_{r2} are linearly independent over F .

(2) $\|v_{r1}, v_{r2}\| = \|v_{r2}, v_{r1}\|$

(3) $\|v_{r1}, kv_{r2}\| = k\|v_{r1}, v_{r2}\|$

(4) $\|v_{r1}, v_{r2} + v_{r3}\| \leq \|v_{r1}, v_{r2}\| + \|v_{r1}, v_{r3}\|$

Is called the soft m-norm on F_L and the pair $(F_L, \|\cdot, \dots, \cdot\|)$ is called the fuzzy 2-normed linear space.

2.22. Definition: Let F_R be a real vector space over with dimension m over a field F a real valued

function $\|\cdot, \dots, \cdot\|: F_R \rightarrow [0,1]$ and it satisfies the following properties

(1) $\|v_{r1}, v_{r2}, v_{r3}, \dots, v_{rm-1}, v_{rm}\| = 0$, if and only if $v_{r1}, v_{r2}, v_{r3}, \dots, v_{rm-1}, v_{rm}$ are linearly independent over F .

(2) $\|v_{r1}, v_{r2}, v_{r3}, \dots, v_{rm-1}, v_{rm}\|$ is invariant under any permutation.

(3) $\|v_{r1}, v_{r2}, v_{r3}, \dots, v_{rm-1}, kv_{rm}\| = k\|v_{r1}, v_{r2}, v_{r3}, \dots, v_{rm-1}, v_{rm}\|$

(4) $\|v_{r1}, v_{r2}, v_{r3}, \dots, v_{rm-1}, v_{rm} + v'_{rm}\| \leq \|v_{r1}, v_{r2}, v_{r3}, \dots, v_{rm-1}, v_{rm}\| + \|v_{r1}, v_{r2}, v_{r3}, \dots, v_{rm-1}, v'_{rm}\|$ is called the m -norm on F_R and the pair $(F_R, \|\cdot, \dots, \cdot\|)$ is called the m -normed linear space.

2.23. Definition : Let F_V be a linear space in a field F , Fuzzy m -normed linear space is defined as a fuzzy subset F_N of $F_V \times F_V \times \dots \times F_V \times F_V$ (m -times) $\times (-\infty, \infty)$ and the pair (F_V, F_N) .

$F_V \times F_V \times \dots \times F_V \times F_V$ (m -times) $\times (-\infty, \infty)$ is referred to as a fuzzy m -norm on F_V if and only if.

(1) $F_N(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm}, f_t) \geq 0$, for all $f_t \in (-\infty, \infty)$.

(2) $F_N(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm}, f_t) = 0$ if and only if $f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm}$, are linearly dependent if for all $f_t \geq 0$, and $f_t \in (-\infty, \infty)$

(3) $F_N(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm}, f_t)$ is invariant under any permutation of $f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm}, f_t$.

(4) $F_N(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm}, f_t) = F_N(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm}, f_t \cdot \frac{f_t}{|f_t|})$, where $f_t \in F$

(5) $F_N(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm1} + f_{vm2}, f_{t1} + f_{t2}) \geq \min\{F_N(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm1}, f_{t1}), F_N(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm2}, f_{t2})\}$

(6) $F(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm}, f_t)$ is a left continuous and non-decreasing function of $f_t \in (-\infty, \infty)$ such that

$$F(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm}, f_t) = 0.$$

2.24. Definition: Let $(F_V, \|\cdot\|, \dots, \|\cdot\|)$ be a fuzzy m -normed linear space and a sequence $\{f_{vr}\}_{r=1}^\infty$ in $(F_V, \|\cdot\|, \dots, \|\cdot\|)$ is said to convergence to $f_v \in F_V$ if for every $\epsilon > 0$ there exists appositve number M such that

$$\lim_{r \rightarrow \infty} \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vr-1}, f_{vr} - f_v)\| = 0$$

That is $\lim_{r \rightarrow \infty} \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vr-1}, f_{vr})\| = f_v$

2.25. Definition: In a fuzzy m -normed linear space $(F_V, \|\cdot\|, \dots, \|\cdot\|)$ a sequence $\{f_{vr}\}_{r=1}^\infty$ is said to be Cauchy sequence if if for every $\epsilon > 0$ there exists appositve number M such that $\|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vr-1}, f_{vr} - f_{v\ell})\| < \epsilon$, whenever $r, \ell \geq M$. And it is represented by $\lim_{r, \ell \rightarrow \infty} \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vr-1}, f_{vr} - f_{v\ell})\| = 0$.

2.26. Definition: The fuzzy m -normed linear space $(F_V, \|\cdot\|, \dots, \|\cdot\|)$ is said to be complete if every Cauchy sequence is convergent in it.

2.27. Definition: Let E_V be subset of fuzzy m -normed linear space $(F_V, \|\cdot\|, \dots, \|\cdot\|)$ and is said to be bounded if there exists a positive real constant η such that

$$\|(e_{v1}, e_{v2}, e_{v3}, \dots, e_{vr-1}, e_{vr})\| \leq \eta \text{ for all } e_{v1}, e_{v2}, e_{v3}, \dots, e_{vr-1}, e_{vr} \in E_V.$$

2.28. Definition: Let $(F_V, \|\cdot\|, \dots, \|\cdot\|)$ be a fuzzy m -normed linear space for any $(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm}) \in F_V$, then set $\{(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm}) \in F_V / \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm} - f_{v0})\| < \rho\}$ is called an open ball center at f_{v0} with radius ρ and it is represented as $B_\rho(f_{v0}) = \{(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm}) \in F_V / \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm} - f_{v0})\| < \rho\}$

Similarly we can define a closed ball

$$B_\rho(f_{v0}) = \{(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm}) \in F_V / \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm} - f_{v0})\| \leq \rho\}.$$

2.29. Definition: In the fuzzy m -normed linear space $(F_V, \|\cdot\|, \dots, \|\cdot\|)$ a sequence $\{f_{vr}\}_{r=1}^\infty$ is called bounded or norm bounded if there is a constant η such that $\|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vr-1}, f_{vr})\| \leq \eta$, for all r .

2.30. Remark: Every convergent sequence is Cauchy and norm bounded in the fuzzy m -normed linear space $(F_V, \|\cdot\|, \dots, \|\cdot\|)$.

2.31. Theorem: Let $(F_V, \|\cdot\|, \dots, \|\cdot\|)$ be a fuzzy m -normed linear space and let $\{f_{vr}\}_{r=1}^\infty$ and $\{f'_{vr}\}_{r=1}^\infty$ be two Cauchy sequences in $(F_V, \|\cdot\|, \dots, \|\cdot\|)$ such that $\|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vr-1}, f_{vr})\| \rightarrow f_v$,

$\|(f'_{v1}, f'_{v2}, f'_{v3}, \dots, f'_{vr-1}, f'_{vr})\| \rightarrow f'_v$, as $r \rightarrow \infty$. Let $\{k_{vr}\}_{r=1}^\infty$ be a sequence in $\mathcal{K}_V$ where $\mathcal{K}_V$ is being the field of scalars be such that $k_{vr} \rightarrow k_{v0}$ as $r \rightarrow \infty$ then the following axioms holds:

$$(F-1) \|(f_{v1} \pm f'_{v1}, f_{v2} \pm f'_{v2}, f_{v3} \pm f'_{v3}, \dots, f_{vr} \pm f'_{vr})\| \rightarrow f_v \pm f'_v, \text{ as } r \rightarrow \infty$$

$$(F-2) \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vr-1}, k_{vr} f_{vr})\| \rightarrow k_v \cdot f_v, \text{ as } r \rightarrow \infty.$$

(F-3) Let $\{f_{vr}\}_{r=1}^\infty$ and $\{f'_{vr}\}_{r=1}^\infty$ be two Cauchy sequences in $(F_V, \|\cdot\|, \dots, \|\cdot\|)$ and $\{k_{vr}\}_{r=1}^\infty$ be a Cauchy sequence in then $\{f_{vr} + f'_{vr}\}_{r=1}^\infty$ and $\{k_{vr} f_{vr}\}_{r=1}^\infty$ are also Cauchy sequence in $(F_V, \|\cdot\|, \dots, \|\cdot\|)$.

Proof: (1) We have

$$\|(f_{v1} \pm f'_{v1}, f_{v2} \pm f'_{v2}, f_{v3} \pm f'_{v3}, \dots, f_{vr} \pm f'_{vr}) - (f_v \pm f'_v)\| \leq (\|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vr} - f_v)\| +$$

$$\|(f'_{v1}, f'_{v2}, f'_{v3}, \dots, f'_{vr} - f'_v)\|) \rightarrow 0, \text{ as } r \rightarrow \infty,$$

Since $\lim_{r \rightarrow \infty} \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vr-1}, f_{vr} - f_v)\| = 0$ and

$$\lim_{r \rightarrow \infty} \|(f'_{v1}, f'_{v2}, f'_{v3}, \dots, f'_{vr-1}, f'_{vr} - f'_v)\| = 0$$

Hence $\|(f_{v1} \pm f'_{v1}, f_{v2} \pm f'_{v2}, f_{v3} \pm f'_{v3}, \dots, f_{vr} \pm f'_{vr})\| \rightarrow f_v \pm f'_v$, as $r \rightarrow \infty$.

(2) Given that $k_{vr} \rightarrow k_{v0}$ as $r \rightarrow \infty \Rightarrow |k_{vr} - k_{v0}| \rightarrow 0$, as $r \rightarrow \infty$ then there exists a constant λ such that $|k_{vr}| \leq \lambda$ for all r .

Now we consider

$$\|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{v\ell-1}, k_{vr} f_{vr} - k_{v0} f_{v0})\| = \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{v\ell-1}, k_{vr} f_{vr} - k_{v0} f_{v0} + k_{vr} f_{v0} - k_{v0} f_{v0})\| = \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{v\ell-1}, k_{vr} (f_{vr} - f_{v0}) + (f_{vr} - f_{v0}) (k_{vr} - k_{v0}) f_{v0})\|$$

$$\leq \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{v\ell-1}, k_{vr} (f_{vr} - f_{v0}))\| + \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{v\ell-1}, (k_{vr} - k_{v0}) f_{v0})\|$$

$$\leq |k_{vr}| \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{v\ell-1}, f_{vr} - f_{v0})\| + |k_{vr} - k_{v0}| \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{v\ell-1}, f_{v0})\|$$

$$\leq |\lambda| \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{v\ell-1}, f_{vr} - f_{v0})\| + |k_{vr} - k_{v0}| \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{v\ell-1}, f_{v0})\| \rightarrow 0 \text{ as } r \rightarrow \infty.$$

So $\|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{v\ell-1}, k_{vr} f_{vr})\| \rightarrow k_{v0} \cdot f_{v0}$, as $r \rightarrow \infty$.

(3) We have $\|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{v\ell-1}, f_{vr} - f_{v\ell})\| \rightarrow 0$, as $r, \ell \rightarrow \infty$ And

$$\|(f'_{v1}, f'_{v2}, f'_{v3}, \dots, f'_{v\ell-1}, f'_{vr} - f'_{v\ell})\| \rightarrow 0, \text{ as } r, \ell \rightarrow \infty$$

Therefore $\|(f_{v1} + f'_{v1}, f_{v2} + f'_{v2}, f_{v3} + f'_{v3}, \dots, f_{v\ell-1} + f'_{v\ell-1}, f_{vr} + f'_{vr} - (f_{v\ell} + f'_{v\ell}))\|$

$$\leq [\|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{v\ell-1}, f_{vr} - f_{v\ell})\| + \|(f'_{v1}, f'_{v2}, f'_{v3}, \dots, f'_{v\ell-1}, f'_{vr} - f'_{v\ell})\|] \rightarrow 0, \text{ as } r, \ell \rightarrow \infty$$

This implies that $\{f_{vr} + f'_{vr}\}_{r=1}^{\infty}$ is Cauchy sequence in $(F_V, \|\cdot\|, \dots, \|\cdot\|)$

Since $\{k_{vr}\}_{r=1}^{\infty}$ be a Cauchy sequence of scalars in K , the scalar field is complete and $\{k_{vr}\}_{r=1}^{\infty}$ is convergent sequence and hence $\{k_{vr}\}_{r=1}^{\infty}$ is bounded.

Also we consider

$$\|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{v\ell-1}, k_{vr} f_{vr} - k_{v\ell} f_{v\ell})\| = \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{v\ell-1}, k_{vr} f_{vr} - k_{v\ell} f_{v\ell} + k_{v\ell} f_{v\ell} - k_{v\ell} f_{v\ell})\| = \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{v\ell-1}, k_{vr} (f_{vr} - f_{v\ell}))\| + \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{v\ell-1}, (k_{vr} - k_{v\ell}) f_{v\ell})\| \leq [|k_v| + |k_{vr} - k_{v\ell}|] \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{v\ell-1}, f_{v\ell})\| \rightarrow 0, \text{ as } r, \ell \rightarrow \infty$$

Since $\{k_{vr}\}_{r=1}^{\infty}$ is bounded and $\{f_{vr}\}_{r=1}^{\infty}$ is norm bounded

Hence $\{k_{vr} f_{vr}\}_{r=1}^{\infty}$ is a Cauchy sequence in $(F_V, \|\cdot\|, \dots, \|\cdot\|)$.

2.32. Remark: The norm function is continuous as it follows by **2.31. Theorem**.

2.33. Example: Let $(F_V, \|\cdot\|, \dots, \|\cdot\|)$ be a fuzzy m-normed linear space. We define continuous t-norm

$$\Delta^F(f^1, f^2) = f^1 \cdot f^2, \quad \Delta^F(f^1, f^2) = \min\{f^1, f^2\}, \text{ for all } f^1, f^2 \in [0, 1].$$

$$F_V(f_{v1}, f_{v2}, f_{v3}, \dots, f_{v\ell-1}, f_{v\ell}, f_t) = \frac{\|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{v\ell-1}, f_{v\ell})\|}{f_t + \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{v\ell-1}, f_{v\ell})\|} \text{ then the set}$$

$S = \{(F_V, F_V(f_{v1}, f_{v2}, f_{v3}, \dots, f_{v\ell-1}, f_{v\ell}, f_t)) / (f_{v1}, f_{v2}, f_{v3}, \dots, f_{v\ell-1}, f_{v\ell}, f_t) \in F_V^M\}$ is a fuzzy m-normed linear space.

Proof: (1) Obviously for every $f_t \in (-\infty, \infty)$ such that $F_V(f_{v1}, f_{v2}, f_{v3}, \dots, f_{v\ell-1}, f_{v\ell}, f_t) \geq 0$.

$$(2) F_V(f_{v1}, f_{v2}, f_{v3}, \dots, f_{v\ell-1}, f_{v\ell}, f_t) = 0 \Leftrightarrow \frac{\|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{v\ell-1}, f_{v\ell})\|}{f_t + \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{v\ell-1}, f_{v\ell})\|} = 0$$

$$\Leftrightarrow \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{v\ell-1}, f_{v\ell})\| = 0$$

$\Leftrightarrow f_{v1}, f_{v2}, f_{v3}, \dots, f_{v\ell-1}, f_{v\ell}$ are linearly dependent if for all $f_t \geq 0$, and $f_t \in (-\infty, \infty)$

$$(3) F_V(f_{v1}, f_{v2}, f_{v3}, \dots, f_{v\ell-1}, f_{v\ell}, f_t) = \frac{\|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{v\ell-1}, f_{v\ell})\|}{f_t + \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{v\ell-1}, f_{v\ell})\|} = \frac{\|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{v\ell-1}, f_{v\ell})\|}{f_t + \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{v\ell}, f_{v\ell-1})\|} = \dots \text{so}$$

on

Hence it is invariant under any permutation of $f_{v1}, f_{v2}, f_{v3}, \dots, f_{v\ell-1}, f_{v\ell}, f_t$.

$$\begin{aligned}
 (4) \quad F_V(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm}, f_t) &= \frac{\|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{\Delta} f_{vm})\|}{f_t + \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{\Delta} f_{vm})\|} \\
 &= \frac{f_{\Delta} \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm})\|}{f_t + f_{\Delta} \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm})\|} = \frac{\|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm})\|}{\frac{f_t}{f_{\Delta}} + \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm})\|} \\
 &= F_V(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm}, f_t, \frac{f_t}{|f_{\Delta}|}), \text{ where } f_{\Delta} \in F
 \end{aligned}$$

(5) Without loss of generality we consider that

$$\begin{aligned}
 F_V(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm1}, f_{t1}) &\leq F_V(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm2}, f_{t2}) \\
 \Rightarrow \frac{\|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm1})\|}{f_{t1} + \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm1})\|} &\leq \frac{\|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm2})\|}{f_{t2} + \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm2})\|} \\
 \Rightarrow f_{t2} \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm1})\| &+ [\|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm1})\| \cdot \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm2})\|] \\
 \leq f_{t1} \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm2})\| &+ [\|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm1})\| \cdot \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm2})\|] \\
 \Rightarrow f_{t2} \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm1})\| &\leq f_{t1} \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm2})\| \\
 \Rightarrow f_{t1} \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm2})\| - f_{t2} \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm1})\| &\geq 0 \quad \dots\dots(1)
 \end{aligned}$$

Now we consider,

$$\begin{aligned}
 &F_V(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm1} + f_{vm2}, f_{t1} + f_{t2}) - F_V(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm1}, f_{t1}) \\
 \Leftrightarrow &\frac{\|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm1} + f_{vm2})\|}{f_{t1} + f_{t2} + \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm1} + f_{vm2})\|} - \frac{\|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm1})\|}{f_{t1} + \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm1})\|} \\
 \Leftrightarrow &[\|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm1} + f_{vm2})\|][f_{t1} + \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm1})\|] - \\
 &[\|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm1})\|][f_{t1} + f_{t2} + \|(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm1} + f_{vm2})\|] \\
 \Leftrightarrow &f_{t1} \square (f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm2}) \square - f_{t2} \square (f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm1}) \square \dots\dots(2)
 \end{aligned}$$

From (1) and (2) we obtain

$$F_V(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm1} + f_{vm2}, f_{t1} + f_{t2}) - F_V(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm1}, f_{t1}) \geq 0$$

Hence $F_V(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm1} + f_{vm2}, f_{t1} + f_{t2}) \geq \min\{F_V(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm1}, f_{t1}), F_V(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm2}, f_{t2})\}$

(6) Clearly $F(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm}, f_t)$ is a left continuous and non-decreasing function of

$f_t \in (-\infty, \infty)$ such that

$$\lim_{m \rightarrow \infty} F_V(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm}, f_t) = \lim_{m \rightarrow \infty} \frac{\| (f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm}) \|}{f_t + \| (f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm}) \|} = 0.$$

2.34. Definition: A sequence $\{f_{vr}\}_{r=1}^{\infty}$ in fuzzy m-normed linear space S is said to be convergent to f_v if given that $\delta > 0$, $\delta \in (0, 1)$ and $f_t > 0$ there exists appositve number M such that

$$F_N(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm} - f_v, f_t) < \delta \text{ for all } m \geq M.$$

that is $\lim_{m \rightarrow \infty} F_N(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm} - f_v, f_t) < \delta$ for all $m \geq M$.

2.35. Definition: A sequence $\{f_{vr}\}_{r=1}^{\infty}$ in fuzzy m-normed linear space S is said to be Cauchy sequence if given that $\delta > 0$, and $\delta \in (0, 1)$ and $f_t > 0$ there exists appositve number M such that

$$F_V(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm} - f_{v\ell}, f_t) < \delta \text{ for all } m, \ell \geq M.$$

that is $\lim_{m, \ell \rightarrow \infty} F_N(f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm} - f_{v\ell}, f_t) < \delta$ for all $m, \ell \geq M$.

2.36. Theorem: In a fuzzy m-normed linear space S, every convergent sequence is Cauchy sequence.

2.37. Definition: A fuzzy m-normed linear space S is said to be complete if every Cauchy sequence in S is convergent.

3. MAIN RESULTS AND DISCUSSIONS:

3.1. Definition: Let F_V be a fuzzy right Gamma linear space over F_D a real valued function

$\|, \dots, \|: F_V \times F_V \rightarrow [0, \infty)$ is called fuzzy Gamma-2-normed linear space over F_D . It is denoted by $(F_V, \|, \dots, \|)$. if it satisfies the following properties for every $f_{v1}, f_{v2} \in F_V$, $f_{d1}, f_{d2}, f_{d3} \in F_D$ and $f_\gamma \in F_\Gamma$

- (1) $\| f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2} \| = 0 \Leftrightarrow f_{v1}, f_{v2}$ linearly independent over F_D
- (2) $\| f_{v1} f_\gamma f_{d1}, f_\sigma (f_{v2} f_\gamma f_{d2}) \| = f_\sigma \| f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2} \|$, for any $f_\sigma \in F_\Gamma$
- (3) $\| f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2} + f_{v3} f_\gamma f_{d3} \| \leq \| f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2} \| + \| f_{v1} f_\gamma f_{d1}, f_{v3} f_\gamma f_{d3} \|^{\frac{1}{2}}$

3.2. Definition: Let F_V be a fuzzy right Gamma linear space over F_D a real valued function

$\|, \dots, \|: F_V \times F_V \times F_V \times \dots \times F_V$ (m-times) $\times [0, \infty)$ is called fuzzy right Gamma-m-normed linear space over F_D if it satisfies the following properties, for any $f_{v1}, f_{v2}, \dots, f_{vm-1}, f_{vm} \in F_V$, $f_{d1}, f_{d2}, f_{d3}, \dots, f_{dm-1}, f_{dm} \in F_D$ and $f_\gamma \in F_\Gamma$

- (1) $\| f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2}, \dots, f_{vm-1} f_\gamma f_{dm-1}, f_{vm} f_\gamma f_{dm} \| = 0 \Leftrightarrow f_{v1}, f_{v2}, \dots, f_{vm-1}, f_{vm}$ are linearly independent over F_D .
- (2) $\| f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2}, \dots, f_{vm-1} f_\gamma f_{dm-1}, f_{vm} f_\gamma f_{dm} \| = 0$, is invariant under any permutation of $f_{v1}, f_{v2}, \dots, f_{vm-1}, f_{vm}$.
- (3) $\| f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2}, \dots, f_{vm-1} f_\gamma f_{dm-1}, f_\sigma (f_{vm} f_\gamma f_{dm}) \| = f_\sigma \| f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2}, \dots, f_{vm-1} f_\gamma f_{dm-1}, f_{vm} f_\gamma f_{dm} \|$,

for any $f_\sigma \in F_\Gamma$

- (4) $\| f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2}, \dots, f_{vm-1} f_\gamma f_{dm-1}, f_{vm1} f_\gamma f_{dm1} + f_{vm2} f_\gamma f_{dm2} \| \leq \| f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2}, \dots, f_{vm-1} f_\gamma f_{dm-1}, f_{vm1} f_\gamma f_{dm1} \| + \| f_{v1} f_\gamma f_{d1}, f_{v2} f_\gamma f_{d2}, \dots, f_{vm-1} f_\gamma f_{dm-1}, f_{vm2} f_\gamma f_{dm2} \|^{\frac{1}{2}}$

It is represented by $(F_V, \|, \dots, \|)$ likewise fuzzy left Gamma-m-normed linear space over F_D can be defined a similar manner.

3.3. Definition: Over F_D let F_V be a fuzzy Gamma linear space a real valued function

$\|, \dots, \|: F_V \times F_V \times F_V \dots \times F_V$ (m-times) $\times [0, \infty)$ is called fuzzy Gamma m-normed linear space over F_D if it is either fuzzy right Gamma-m-normed linear space over F_D or fuzzy left Gamma-m-normed linear space over F_D

3.4. Definition: Let $(F_V, \|, \dots, \|)$ be a fuzzy Gamma-m-normed linear space and a sequence $\{f_{vr} f_{\gamma} f_{dr}\}_{r=1}^{\infty}$ in $(F_V, \|, \dots, \|)$ is said to convergence to $f_v f_{\gamma} f_d \in F_V$ if for every $\varepsilon > 0$ there exists appositve number M such that $\|f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vr-1} f_{\gamma} f_{dr-1}, f_{vr} f_{\gamma} f_{dr} - f_v f_{\gamma} f_d\| < \varepsilon$

That is $\lim_{r \rightarrow \infty} \| (f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, f_{v3} f_{\gamma} f_{d3}, \dots, f_{vr-1} f_{\gamma} f_{dr-1}, f_{vr} f_{\gamma} f_{dr} - f_v f_{\gamma} f_d) \| = 0$

Which implies that $\lim_{r \rightarrow \infty} \| (f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, f_{v3} f_{\gamma} f_{d3}, \dots, f_{vr-1} f_{\gamma} f_{dr-1}, f_{vr} f_{\gamma} f_{dr}) \| = f_v f_{\gamma} f_d$

Where $f_{v1}, f_{v2}, f_{v3}, \dots, f_{vr-1}, f_{vr}$ and $f_v \in F_V, f_{d1}, f_{d2}, f_{d3}, \dots, f_{dr-1}, f_{dr}$, and $f_d \in F_D, f_{\gamma} \in F_{\Gamma}$

3.5. Definition: In a fuzzy Gamma-m-normed linear space $(F_V, \|, \dots, \|)$ a

Sequence $\{f_{vr} f_{\gamma} f_{dr}\}_{r=1}^{\infty}$ is said to be Cauchy sequence if for every $\varepsilon > 0$ there exists appositve number M such that $\| (f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, f_{v3} f_{\gamma} f_{d3}, \dots, f_{vr-1} f_{\gamma} f_{dr-1}, f_{vr} f_{\gamma} f_{dr} - f_{v\ell} f_{\gamma} f_{d\ell}) \| < \varepsilon$

Whenever $r, \ell \geq M$. and it is represented by

$\lim_{r, \ell \rightarrow \infty} \| (f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, f_{v3} f_{\gamma} f_{d3}, \dots, f_{vr-1} f_{\gamma} f_{dr-1}, f_{vr} f_{\gamma} f_{dr} - f_{v\ell} f_{\gamma} f_{d\ell}) \| = 0,$

Where $f_{vr}, f_{v\ell}, f_v \in F_V, f_{dr}, f_{d\ell}, f_d \in F_D$ for $r=1, 2, 3, \dots$ and $f_{\gamma} \in F_{\Gamma}$

3.6. Definition: The fuzzy Gamma-m-normed linear space $(F_V, \|, \dots, \|)$ is said to be complete if every Cauchy sequence is convergent in it.

3.7. Definition: Let E_V be subset of fuzzy Gamma-m-normed linear space $(F_V, \|, \dots, \|)$ and is said to be bounded if there exists a positive real constant η_0 such that

$\|(e_{v1} f_{\gamma} f_{d1}, e_{v2} f_{\gamma} f_{d2}, e_{v3} f_{\gamma} f_{d3}, \dots, e_{vr-1} f_{\gamma} f_{dr-1}, e_{vr} f_{\gamma} f_{dr})\| \leq \eta_0,$

for all $e_{v1}, e_{v2}, e_{v3}, \dots, e_{vr-1}, e_{vr} \in E_V, f_{d1}, f_{d2}, f_{d3}, \dots, f_{dr-1}, f_{dr}$, and $f_d \in F_D$ and $f_{\gamma} \in F_{\Gamma}$

3.8. Definition: Let $(F_V, \|, \dots, \|)$ be a fuzzy Gamma-m-normed linear space for any $(f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vm-1} f_{\gamma} f_{dm-1}, f_{vm} f_{\gamma} f_{dm}) \in F_V$, then set $\{(f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vm-1} f_{\gamma} f_{dm-1}, f_{vm} f_{\gamma} f_{dm}) \in F_V / \|(f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vm-1} f_{\gamma} f_{dm-1}, f_{vm} f_{\gamma} f_{dm} - f_{v0} f_{\gamma} f_{d0})\| < \rho_0\}$ is called an open ball center at $f_{v0} f_{\gamma} f_{d0}$ with radius ρ_0 and it is represented as

$B_{\rho_0}(f_{v0} f_{\gamma} f_{d0}) = \{(f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vm-1} f_{\gamma} f_{dm-1}, f_{vm} f_{\gamma} f_{dm}) \in F_V / \|(f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vm-1} f_{\gamma} f_{dm-1}, f_{vm} f_{\gamma} f_{dm} - f_{v0} f_{\gamma} f_{d0})\| < \rho_0\}$

Similarly we can define a closed ball $B_{\rho_0}(f_{v0} f_{\gamma} f_{d0}) = \{(f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vm-1} f_{\gamma} f_{dm-1}, f_{vm} f_{\gamma} f_{dm}) \in F_V / \|(f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vm-1} f_{\gamma} f_{dm-1}, f_{vm} f_{\gamma} f_{dm} - f_{v0} f_{\gamma} f_{d0})\| \leq \rho_0\}$ Where $f_{v1}, f_{v2}, f_{v3}, \dots, f_{vm-1}, f_{vm}$ and $f_{v0} \in F_V, f_{d1}, f_{d2}, f_{d3}, \dots, f_{dm-1}, f_{dm}$, and $f_{d0} \in F_D$ and $f_{\gamma} \in F_{\Gamma}$

3.9. Definition: In the fuzzy Gamma-m-normed linear space $(F_V, \|, \dots, \|)$ a sequence $\{f_{vr} f_{\gamma} f_{dr}\}_{r=1}^{\infty}$ is said to be bounded or norm bounded if there is a constant η_0 such that that $\|(f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vr-1} f_{\gamma} f_{dr-1}, f_{vr} f_{\gamma} f_{dr})\| \leq \eta_0$, for all $f_{vi}, f_{vj}, f_v \in F_V, f_{di}, f_{dj}, f_d \in F_D$ for $i, j=1, 2, \dots, r$, and $f_{\gamma} \in F_{\Gamma}$

3.10. Remark: Every convergent sequence is Cauchy and norm bounded in the fuzzy Gamma-m-normed linear space $(F_V, \|, \dots, \|)$.

3.11. Theorem: Let $(F_V, \|, \dots, \|)$ be a fuzzy Gamma-m-normed linear space and let $\{f_{vr} f_{\gamma} f_{dr}\}_{r=1}^{\infty}$ and $\{f'_{vr} f_{\gamma} f'_{dr}\}_{r=1}^{\infty}$ be two Cauchy sequences in $(F_V, \|, \dots, \|)$ such that

$$\|(f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vr-1} f_{\gamma} f_{dr-1}, f_{vr} f_{\gamma} f_{dr})\| \rightarrow f_v f_{\gamma} f_d,$$

$$\|(f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vr-1} f_{\gamma} f_{dr-1}, f_{vr} f_{\gamma} f_{dr})\| \rightarrow f_v f_{\gamma} f_d, \text{ as } r \rightarrow \infty.$$

Let $\{k_{vr}\}_{r=1}^{\infty}$ be a sequence in $\mathcal{V}$ where $\mathcal{K}_{\mathcal{V}}$ is being the field of scalars be such that $k_{vr} \rightarrow k_v$ as $r \rightarrow \infty$ then the following axioms holds:

$$(1) \|(f_{v1} f_{\gamma} f_{d1} \pm f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2} \pm f_{v2} f_{\gamma} f_{d2}, f_{v3} f_{\gamma} f_{d3} \pm f_{v3} f_{\gamma} f_{d3}, \dots, f_{vr} f_{\gamma} f_{dr} \pm f_{vr} f_{\gamma} f_{dr})\| \rightarrow f_v f_{\gamma} f_d \pm f_v f_{\gamma} f_d, \text{ as } r \rightarrow \infty$$

$$(2) \|(f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vr-1} f_{\gamma} f_{dr-1}, k_{vr}(f_{vr} f_{\gamma} f_{dr}))\| \rightarrow k_v(f_v f_{\gamma} f_d), \text{ as } r \rightarrow \infty.$$

(3) Let $\{f_{vr} f_{\gamma} f_{dr}\}_{r=1}^{\infty}$ and $\{f'_{vr} f_{\gamma} f'_{dr}\}_{r=1}^{\infty}$ be two Cauchy sequences in $(F_V, \|\cdot, \dots, \cdot\|)$ and $\{k_{vr}\}_{r=1}^{\infty}$ be a Cauchy sequence in F_V then $\{f_{vr} f_{\gamma} f_{dr} + f'_{vr} f_{\gamma} f'_{dr}\}_{r=1}^{\infty}$ and $\{k_{vr} f_{vr} f_{\gamma} f_{dr}\}_{r=1}^{\infty}$ are also Cauchy sequence in $(F_V, \|\cdot, \dots, \cdot\|)$.

Proof: (1) We have $\lim_{r \rightarrow \infty} \|(f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, f_{v3} f_{\gamma} f_{d3}, \dots, f_{vr-1} f_{\gamma} f_{dr-1}, f_{vr} f_{\gamma} f_{dr} - f_v f_{\gamma} f_d)\| = 0$ and

$$\lim_{r \rightarrow \infty} \|(f'_{v1} f_{\gamma} f'_{d1}, f'_{v2} f_{\gamma} f'_{d2}, f'_{v3} f_{\gamma} f'_{d3}, \dots, f'_{vr-1} f_{\gamma} f'_{dr-1}, f'_{vr} f_{\gamma} f'_{dr} - f_v f_{\gamma} f_d)\| = 0$$

Then which implies that

$$\|(f_{v1} f_{\gamma} f_{d1} \pm f'_{v1} f_{\gamma} f'_{d1}, f_{v2} f_{\gamma} f_{d2} \pm f'_{v2} f_{\gamma} f'_{d2}, f_{v3} f_{\gamma} f_{d3} \pm f'_{v3} f_{\gamma} f'_{d3}, \dots, f_{vr} f_{\gamma} f_{dr} \pm f'_{vr} f_{\gamma} f'_{dr} - (f_v f_{\gamma} f_d \pm f_v f_{\gamma} f_d))\|$$

$$\leq \|(f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, f_{v3} f_{\gamma} f_{d3}, \dots, f_{vr-1} f_{\gamma} f_{dr-1}, f_{vr} f_{\gamma} f_{dr} - f_v f_{\gamma} f_d)\| + \|(f'_{v1} f_{\gamma} f'_{d1}, f'_{v2} f_{\gamma} f'_{d2}, f'_{v3} f_{\gamma} f'_{d3}, \dots, f'_{vr-1} f_{\gamma} f'_{dr-1}, f'_{vr} f_{\gamma} f'_{dr} - f_v f_{\gamma} f_d)\| \rightarrow 0$$

as $r \rightarrow \infty$

$$\text{Hence } \|(f_{v1} f_{\gamma} f_{d1} \pm f'_{v1} f_{\gamma} f'_{d1}, f_{v2} f_{\gamma} f_{d2} \pm f'_{v2} f_{\gamma} f'_{d2}, f_{v3} f_{\gamma} f_{d3} \pm f'_{v3} f_{\gamma} f'_{d3}, \dots, f_{vr} f_{\gamma} f_{dr} \pm f'_{vr} f_{\gamma} f'_{dr})\| \rightarrow f_v f_{\gamma} f_d \pm f_v f_{\gamma} f_d$$

as $r \rightarrow \infty$.

(2) Given that $k_{vr} \rightarrow k_v$ as $r \rightarrow \infty \Rightarrow |k_{vr} - k_v| \rightarrow 0$, as $r \rightarrow \infty$ then there exists a constant λ such that $|k_{vr}| \leq \lambda$ for all r .

$$\text{Now we consider } \|(f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vr-1} f_{\gamma} f_{dr-1}, k_{vr}(f_{vr} f_{\gamma} f_{dr}) - k_{v0}(f_v f_{\gamma} f_d))\|$$

$$= \|(f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vr-1} f_{\gamma} f_{dr-1}, k_{vr}(f_{vr} f_{\gamma} f_{dr}) - k_{vr}(f_{v0} f_{\gamma} f_{d0}) + k_{vr}(f_{v0} f_{\gamma} f_{d0}) - k_{v0}(f_v f_{\gamma} f_d))\|$$

$$\leq \|(f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vr-1} f_{\gamma} f_{dr-1}, k_{vr}(f_{vr} f_{\gamma} f_{dr} - f_{v0} f_{\gamma} f_{d0}))\| + \|(f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vr-1} f_{\gamma} f_{dr-1}, (k_{vr} - k_{v0})f_{v0} f_{\gamma} f_{d0})\|$$

$$\leq |k_{vr}| \|(f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vr-1} f_{\gamma} f_{dr-1}, f_{vr} f_{\gamma} f_{dr} - f_{v0} f_{\gamma} f_{d0})\| + |k_{vr} - k_{v0}| \|(f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vr-1} f_{\gamma} f_{dr-1}, f_{v0} f_{\gamma} f_{d0})\| \leq \lambda \|(f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vr-1} f_{\gamma} f_{dr-1}, f_{vr} f_{\gamma} f_{dr} - f_{v0} f_{\gamma} f_{d0})\| +$$

$$|k_{vr} - k_{v0}| \|(f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vr-1} f_{\gamma} f_{dr-1}, f_{v0} f_{\gamma} f_{d0})\| \rightarrow 0, \text{ as } r \rightarrow \infty.$$

$$\text{So } \|(f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vr-1} f_{\gamma} f_{dr-1}, k_{vr}(f_{vr} f_{\gamma} f_{dr}))\| \rightarrow k_{v0}(f_v f_{\gamma} f_d), \text{ as } r \rightarrow \infty.$$

(3) We have $\{f_{vr} f_{\gamma} f_{dr}\}_{r=1}^{\infty}$ and $\{f'_{vr} f_{\gamma} f'_{dr}\}_{r=1}^{\infty}$ be two Cauchy sequences in $(F_V, \|\cdot, \dots, \cdot\|)$ then $\|(f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vr-1} f_{\gamma} f_{dr-1}, f_{vr} f_{\gamma} f_{dr} - f_{v\ell} f_{\gamma} f_{d\ell})\| \rightarrow 0$, as $r, \ell \rightarrow \infty$ and

$$\|(f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vr-1} f_{\gamma} f_{dr-1}, f_{vr} f_{\gamma} f_{dr} - f_{v\ell} f_{\gamma} f_{d\ell})\| \rightarrow 0, \text{ as } r, \ell \rightarrow \infty$$

$$\text{Therefore } \|(f_{v1} f_{\gamma} f_{d1} + f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2} + f_{v2} f_{\gamma} f_{d2}, f_{v3} f_{\gamma} f_{d3} + f_{v3} f_{\gamma} f_{d3}, \dots, f_{vr} f_{\gamma} f_{dr} + f_{vr} f_{\gamma} f_{dr} - (f_{v\ell} f_{\gamma} f_{d\ell} \pm f_{v\ell} f_{\gamma} f_{d\ell}))\|$$

$$\leq \|(f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vr-1} f_{\gamma} f_{dr-1}, f_{vr} f_{\gamma} f_{dr} - f_{v\ell} f_{\gamma} f_{d\ell})\| + \|(f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vr-1} f_{\gamma} f_{dr-1}, f_{vr} f_{\gamma} f_{dr} - f_{v\ell} f_{\gamma} f_{d\ell})\| \rightarrow 0, \text{ as } r, \ell \rightarrow \infty$$

This implies that $\{f_{vr} f_{\gamma} f_{dr} + f'_{vr} f_{\gamma} f'_{dr}\}_{r=1}^{\infty}$ is Cauchy sequence in $(F_V, \|\cdot, \dots, \cdot\|)$.

Since $\{k_{vr}\}_{r=1}^{\infty}$ be a Cauchy sequence of scalars in F_V the scalar field is complete and $\{k_{vr}\}_{r=1}^{\infty}$ is convergent sequence and hence $\{k_{vr}\}_{r=1}^{\infty}$ is bounded.

Also we consider $\| (f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vr-1} f_{\gamma} f_{dr-1}, k_{vr} (f_{vr} f_{\gamma} f_{dr}) - k_{v\ell} (f_{v\ell} f_{\gamma} f_{d\ell})) \|$

$$= \| (f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vr-1} f_{\gamma} f_{dr-1}, k_{vr}(f_{vr} f_{\gamma} f_{dr}) - k_{v\ell} (f_{v\ell} f_{\gamma} f_{d\ell}) + k_{vr} (f_{v\ell} f_{\gamma} f_{d\ell}) - k_{v\ell} (f_{v\ell} f_{\gamma} f_{d\ell})) \|$$

$$\leq \| (f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vr-1} f_{\gamma} f_{dr-1}, k_{vr}(f_{vr} f_{\gamma} f_{dr} - f_{v\ell} f_{\gamma} f_{d\ell})) \| +$$

$$\| (f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vr-1} f_{\gamma} f_{dr-1}, (k_{vr} - k_{v\ell})(f_{v\ell} f_{\gamma} f_{d\ell})) \|$$

$$\leq |k_{vr}| \| (f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vr-1} f_{\gamma} f_{dr-1}, f_{vr} f_{\gamma} f_{dr} - f_{v\ell} f_{\gamma} f_{d\ell}) \| + |k_{vr} - k_{v\ell}| \| (f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vr-1} f_{\gamma} f_{dr-1}, f_{v\ell} f_{\gamma} f_{d\ell}) \| \rightarrow 0,$$

as $r, \ell \rightarrow \infty$

Since $\{k_{vr}\}_{r=1}^{\infty}$ is bounded and $\{f_{vr} f_{\gamma} f_{dr}\}_{r=1}^{\infty}$ is norm bounded.

Hence $\{k_{vr} f_{vr} f_{\gamma} f_{dr}\}_{r=1}^{\infty}$ is a Cauchy sequence in $(F_V, \|\cdot\|, \dots, \|\cdot\|)$.

3.12. Remark: The norm function is continuous as it follows by **3.11.Theorem**.

3.13. Definition: let F_V Gamma linear space over a field F . A fuzzy subset F_{Γ} of

$F_V \times F_V \times \dots \times F_V$ F_V (m-times) $\times (-\infty, \infty)$ and the pair (F_V, F_{Γ}) is called fuzzy Gamma-m-normed linear space and $F_V \times F_V \times \dots \times F_V \times F_V$ (m-times) $\times (-\infty, \infty)$ is called as a fuzzy m-norm on F_V if and only if

(1) $F_{\Gamma}(f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vm-1} f_{\gamma} f_{dm-1}, f_{vm} f_{\gamma} f_{dm}, f_t) \geq 0$, for all $f_t \in (-\infty, \infty)$.

(2) $F_{\Gamma}(f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vm-1} f_{\gamma} f_{dm-1}, f_{vm} f_{\gamma} f_{dm}, f_t) = 0$ if and only if $f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vm-1} f_{\gamma} f_{dm-1}, f_{vm} f_{\gamma} f_{dm}$ are linearly dependent if for all $f_t \geq 0$, and $f_t \in (-\infty, \infty)$.

(3) $F_{\Gamma}(f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vm-1} f_{\gamma} f_{dm-1}, f_{vm} f_{\gamma} f_{dm}, f_t)$ is invariant under any permutation of

$f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vm-1} f_{\gamma} f_{dm-1}, f_{vm} f_{\gamma} f_{dm}, f_t$.

(4) $F_{\Gamma}(f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vm-1} f_{\gamma} f_{dm-1}, f_{\Delta} (f_{vm} f_{\gamma} f_{dm}), f_t) = F_{\Gamma}(f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vm-1} f_{\gamma} f_{dm-1}, f_{vm} f_{\gamma} f_{dm}, \frac{f_t}{|f_{\Delta}|})$ where $f_{\Delta} \in F$.

(5) $F_{\Gamma}(f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vm-1} f_{\gamma} f_{dm-1}, f_{vm1} f_{\gamma} f_{dm1} + f_{vm2} f_{\gamma} f_{dm2}, f_{t1} + f_{t2}) \geq \min\{F_{\Gamma}(f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots,$

$f_{vm-1} f_{\gamma} f_{dm-1}, f_{vm1} f_{\gamma} f_{dm1}, f_{t1}), F_{\Gamma}(f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vm-1} f_{\gamma} f_{dm-1}, f_{vm2} f_{\gamma} f_{dm2}, f_{t2})$

(6) $F_{\Gamma}(f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vm-1} f_{\gamma} f_{dm-1}, f_{vm} f_{\gamma} f_{dm}, f_t)$ is a left continuous and non-decreasing function of $f_t \in (-\infty, \infty)$ such that $\lim_{r \rightarrow 0} F_{\Gamma}(f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, f_{v3} f_{\gamma} f_{d3}, \dots, f_{vm-1} f_{\gamma} f_{dm-1}, f_{vm} f_{\gamma} f_{dm}, f_t) = 0$

3.14. Example: Let $(F_V, \|\cdot\|, \dots, \|\cdot\|)$ be a fuzzy Gamma-m-normed linear space. We define continuous t-norm Δ^F $(f^1, f^2) = f^1 \cdot f^2$, $\Delta^F(f^1, f^2) = \min\{f^1, f^2\}$, for all $f^1, f^2 \in [0, 1]$.

$$F_{\Gamma}(f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vm-1} f_{\gamma} f_{dm-1}, f_{vm} f_{\gamma} f_{dm}, f_t) = \frac{\| (f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vm-1} f_{\gamma} f_{dm-1}, f_{vm} f_{\gamma} f_{dm}) \|}{f_t + \| (f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vm-1} f_{\gamma} f_{dm-1}, f_{vm} f_{\gamma} f_{dm}) \|}$$

then the set $S = \{(F_V, F_{\Gamma}(f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vm-1} f_{\gamma} f_{dm-1}, f_{vm} f_{\gamma} f_{dm}, f_t)) / (f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vm-1} f_{\gamma} f_{dm-1}, f_{vm} f_{\gamma} f_{dm}, f_t) \in F_V^M\}$ or the set is (F_V, F_{Γ}) a fuzzy Gamma-m-normed linear space.

Proof: (1) Obviously for every $f_t \in (0, \infty)$

$$\| (f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vm-1} f_{\gamma} f_{dm-1}, f_{vm} f_{\gamma} f_{dm}) \| \geq 0 \Leftrightarrow \left[\frac{\| (f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vm-1} f_{\gamma} f_{dm-1}, f_{vm} f_{\gamma} f_{dm}) \|}{f_t + \| (f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vm-1} f_{\gamma} f_{dm-1}, f_{vm} f_{\gamma} f_{dm}) \|} \right] \geq 0$$

$$\Leftrightarrow F_{\Gamma}(f_{v1} f_{\gamma} f_{d1}, f_{v2} f_{\gamma} f_{d2}, \dots, f_{vm-1} f_{\gamma} f_{dm-1}, f_{vm} f_{\gamma} f_{dm}, f_t) \geq 0.$$

(2) We have

$$F_{\Gamma}(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, f_{vm}f_{\gamma}f_{dm}, f_t) = 0 \Leftrightarrow \frac{\|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, f_{vm}f_{\gamma}f_{dm})\|}{f_t + \|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, f_{vm}f_{\gamma}f_{dm})\|} = 0$$

$$\Leftrightarrow \|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, f_{vm}f_{\gamma}f_{dm})\| = 0$$

$$\Leftrightarrow f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, f_{vm}f_{\gamma}f_{dm} \text{ are linearly dependent if for all } f_t \geq 0, \text{ and } f_t \text{ in } (-\infty, \infty).$$

$$(3) \quad F_{\Gamma}(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, f_{vm}f_{\gamma}f_{dm}, f_t) = \frac{\|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, f_{vm}f_{\gamma}f_{dm})\|}{f_t + \|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, f_{vm}f_{\gamma}f_{dm})\|}$$

$$= \frac{\|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, f_{vm}f_{\gamma}f_{dm})\|}{f_t + \|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, f_{vm}f_{\gamma}f_{dm})\|} = F_{\Gamma}(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm}f_{\gamma}f_{dm}, f_{vm-1}f_{\gamma}f_{dm-1}, f_t) = \dots \text{so on.}$$

This proves that $F_{\Gamma}(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, f_{vm}f_{\gamma}f_{dm}, f_t)$ is invariant under any permutation of $f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, f_{vm}f_{\gamma}f_{dm}$.

(4) Now we assume that LHS

$$F_{\Gamma}(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, k_{\Delta}(f_{vm}f_{\gamma}f_{dm}), f_t) = \frac{\|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, k_{\Delta}(f_{vm}f_{\gamma}f_{dm}))\|}{f_t + \|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, k_{\Delta}(f_{vm}f_{\gamma}f_{dm}))\|}$$

$$= \frac{k_{\Delta} \|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, (f_{vm}f_{\gamma}f_{dm}))\|}{f_t + k_{\Delta} \|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, (f_{vm}f_{\gamma}f_{dm}))\|} = \frac{\|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, (f_{vm}f_{\gamma}f_{dm}))\|}{\frac{f_t}{|k_{\Delta}|} + \|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, (f_{vm}f_{\gamma}f_{dm}))\|}$$

$$= F_{\Gamma}(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, f_{vm}f_{\gamma}f_{dm}, \frac{f_t}{|k_{\Delta}|}), \text{ where } k_{\Delta} \text{ in } = \text{RHS}$$

(5) Without loss of generality we consider that

$$F_{\Gamma}(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, f_{vm1}f_{\gamma}f_{dm1}, f_s) \leq F_{\Gamma}(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, f_{vm2}f_{\gamma}f_{dm2}, f_t)$$

$$\Leftrightarrow \frac{\|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, f_{vm1}f_{\gamma}f_{dm1})\|}{f_s + \|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, f_{vm1}f_{\gamma}f_{dm1})\|} \leq \frac{\|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, f_{vm2}f_{\gamma}f_{dm2})\|}{f_t + \|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, f_{vm2}f_{\gamma}f_{dm2})\|}$$

$$\Leftrightarrow \|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, f_{vm1}f_{\gamma}f_{dm1})\| (f_t + \|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, f_{vm2}f_{\gamma}f_{dm2})\|)$$

$$\leq \|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, f_{vm2}f_{\gamma}f_{dm2})\| (f_s + \|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, f_{vm1}f_{\gamma}f_{dm1})\|)$$

$$\Leftrightarrow f_t \|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, f_{vm1}f_{\gamma}f_{dm1})\| \leq f_s \|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, f_{vm2}f_{\gamma}f_{dm2})\|$$

$$\Leftrightarrow f_t \|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, f_{vm1}f_{\gamma}f_{dm1})\| - f_s \|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, f_{vm2}f_{\gamma}f_{dm2})\| \leq 0$$

$$\Leftrightarrow f_s \|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, f_{vm2}f_{\gamma}f_{dm2})\| - f_t \|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, f_{vm1}f_{\gamma}f_{dm1})\| \geq 0 \dots (1)$$

Consider again without loss of generality $F_{\Gamma}(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, f_{vm1}f_{\gamma}f_{dm1} + f_{vm2}f_{\gamma}f_{dm2}, f_s + f_t) - F_{\Gamma}(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, f_{vm1}f_{\gamma}f_{dm1}, f_s) =$

$$\frac{\|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, f_{vm1}f_{\gamma}f_{dm1} + f_{vm2}f_{\gamma}f_{dm2})\|}{f_s + f_t + \|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, f_{vm1}f_{\gamma}f_{dm1} + f_{vm2}f_{\gamma}f_{dm2})\|} - \frac{\|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, f_{vm1}f_{\gamma}f_{dm1})\|}{f_s + \|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dm-1}, f_{vm1}f_{\gamma}f_{dm1})\|}$$

$$\begin{aligned}
& \| (f_{v1} f_{\gamma d1}, f_{v2} f_{\gamma d2}, \dots, f_{vm-1} f_{\gamma d_{m-1}}, f_{vm1} f_{\gamma d_{m1}} + f_{vm2} f_{\gamma d_{m2}}) \| (f_s + \| (f_{v1} f_{\gamma d1}, f_{v2} f_{\gamma d2}, \dots, f_{vm-1} f_{\gamma d_{m-1}}, f_{vm1} f_{\gamma d_{m1}}) \|) - \| (f_{v1} f_{\gamma d1}, f_{v2} f_{\gamma d2}, \dots, \\
& f_{vm-1} f_{\gamma d_{m-1}}, f_{vm1} f_{\gamma d_{m1}}) \| ((f_s + f_t) + \| (f_{v1} f_{\gamma d1}, f_{v2} f_{\gamma d2}, \dots, f_{vm-1} f_{\gamma d_{m-1}}, f_{vm1} f_{\gamma d_{m1}} + f_{vm2} f_{\gamma d_{m2}}) \|) \\
& = \frac{(f_s + \| (f_{v1} f_{\gamma d1}, f_{v2} f_{\gamma d2}, \dots, f_{vm-1} f_{\gamma d_{m-1}}, f_{vm1} f_{\gamma d_{m1}}) \|) ((f_s + f_t) + \| (f_{v1} f_{\gamma d1}, f_{v2} f_{\gamma d2}, \dots, f_{vm-1} f_{\gamma d_{m-1}}, f_{vm1} f_{\gamma d_{m1}} + f_{vm2} f_{\gamma d_{m2}}) \|)}{(f_s + \| (f_{v1} f_{\gamma d1}, f_{v2} f_{\gamma d2}, \dots, f_{vm-1} f_{\gamma d_{m-1}}, f_{vm1} f_{\gamma d_{m1}}) \|) ((f_s + f_t) + \| (f_{v1} f_{\gamma d1}, f_{v2} f_{\gamma d2}, \dots, f_{vm-1} f_{\gamma d_{m-1}}, f_{vm1} f_{\gamma d_{m1}} + f_{vm2} f_{\gamma d_{m2}}) \|)} \\
& \Leftrightarrow \frac{f_s \| (f_{v1} f_{\gamma d1}, f_{v2} f_{\gamma d2}, \dots, f_{vm-1} f_{\gamma d_{m-1}}, f_{vm1} f_{\gamma d_{m1}}) \| - f_t \| (f_{v1} f_{\gamma d1}, f_{v2} f_{\gamma d2}, \dots, f_{vm-1} f_{\gamma d_{m-1}}, f_{vm1} f_{\gamma d_{m1}}) \|}{(f_s + \| (f_{v1} f_{\gamma d1}, f_{v2} f_{\gamma d2}, \dots, f_{vm-1} f_{\gamma d_{m-1}}, f_{vm1} f_{\gamma d_{m1}}) \|) ((f_s + f_t) + \| (f_{v1} f_{\gamma d1}, f_{v2} f_{\gamma d2}, \dots, f_{vm-1} f_{\gamma d_{m-1}}, f_{vm1} f_{\gamma d_{m1}} + f_{vm2} f_{\gamma d_{m2}}) \|)} \geq 0 \\
& \frac{\| (f_{v1} f_{\gamma d1}, f_{v2} f_{\gamma d2}, \dots, f_{vm-1} f_{\gamma d_{m-1}}, f_{vm1} f_{\gamma d_{m1}} + f_{vm2} f_{\gamma d_{m2}}) \|}{f_s + f_t + \| (f_{v1} f_{\gamma d1}, f_{v2} f_{\gamma d2}, \dots, f_{vm-1} f_{\gamma d_{m-1}}, f_{vm1} f_{\gamma d_{m1}} + f_{vm2} f_{\gamma d_{m2}}) \|} - \frac{\| (f_{v1} f_{\gamma d1}, f_{v2} f_{\gamma d2}, \dots, f_{vm-1} f_{\gamma d_{m-1}}, f_{vm1} f_{\gamma d_{m1}}) \|}{f_s + \| (f_{v1} f_{\gamma d1}, f_{v2} f_{\gamma d2}, \dots, f_{vm-1} f_{\gamma d_{m-1}}, f_{vm1} f_{\gamma d_{m1}}) \|} \geq 0 \\
& \Leftrightarrow \frac{\| (f_{v1} f_{\gamma d1}, f_{v2} f_{\gamma d2}, \dots, f_{vm-1} f_{\gamma d_{m-1}}, f_{vm1} f_{\gamma d_{m1}} + f_{vm2} f_{\gamma d_{m2}}) \|}{f_s + f_t + \| (f_{v1} f_{\gamma d1}, f_{v2} f_{\gamma d2}, \dots, f_{vm-1} f_{\gamma d_{m-1}}, f_{vm1} f_{\gamma d_{m1}} + f_{vm2} f_{\gamma d_{m2}}) \|} \geq \frac{\| (f_{v1} f_{\gamma d1}, f_{v2} f_{\gamma d2}, \dots, f_{vm-1} f_{\gamma d_{m-1}}, f_{vm1} f_{\gamma d_{m1}}) \|}{f_s + \| (f_{v1} f_{\gamma d1}, f_{v2} f_{\gamma d2}, \dots, f_{vm-1} f_{\gamma d_{m-1}}, f_{vm1} f_{\gamma d_{m1}}) \|} \\
& \Leftrightarrow F_{\Gamma}(f_{v1} f_{\gamma d1}, f_{v2} f_{\gamma d2}, \dots, f_{vm-1} f_{\gamma d_{m-1}}, f_{vm1} f_{\gamma d_{m1}} + f_{vm2} f_{\gamma d_{m2}}, f_s + f_t) \geq F_{\Gamma}(f_{v1} f_{\gamma d1}, f_{v2} f_{\gamma d2}, \dots, f_{vm-1} f_{\gamma d_{m-1}}, f_{vm1} f_{\gamma d_{m1}}, \\
& f_s) \dots \dots (2)
\end{aligned}$$

From (1) and (2) we obtain

$$F_{\Gamma}(f_{v1} f_{\gamma d1}, f_{v2} f_{\gamma d2}, \dots, f_{vm-1} f_{\gamma d_{m-1}}, f_{vm1} f_{\gamma d_{m1}} + f_{vm2} f_{\gamma d_{m2}}, f_s + f_t) \geq \min\{F_{\Gamma}(f_{v1} f_{\gamma d1}, f_{v2} f_{\gamma d2}, \dots, f_{vm-1} f_{\gamma d_{m-1}}, f_{vm1} f_{\gamma d_{m1}}, f_s), F_{\Gamma}(f_{v1} f_{\gamma d1}, f_{v2} f_{\gamma d2}, \dots, f_{vm-1} f_{\gamma d_{m-1}}, f_{vm2} f_{\gamma d_{m2}}, f_t)\}$$

$$(6) \text{ We have clearly } F_{\Gamma}(f_{v1} f_{\gamma d1}, f_{v2} f_{\gamma d2}, \dots, f_{vm-1} f_{\gamma d_{m-1}}, f_{vm} f_{\gamma d_m}, f_t),$$

is a left continuous and non-decreasing function of f_t in $(-\infty, \infty)$ such that

$$\begin{aligned}
& \lim_{m \rightarrow \infty} F_{\Gamma}(f_{v1} f_{\gamma d1}, f_{v2} f_{\gamma d2}, \dots, f_{vm-1} f_{\gamma d_{m-1}}, f_{vm} f_{\gamma d_m}, f_t) \\
& = \lim_{m \rightarrow \infty} \frac{\| (f_{v1} f_{\gamma d1}, f_{v2} f_{\gamma d2}, \dots, f_{vm-1} f_{\gamma d_{m-1}}, f_{vm} f_{\gamma d_m}) \|}{f_t + \| (f_{v1} f_{\gamma d1}, f_{v2} f_{\gamma d2}, \dots, f_{vm-1} f_{\gamma d_{m-1}}, f_{vm} f_{\gamma d_m}) \|} = 0
\end{aligned}$$

3.15. Definition: Let (F_V, F_{Γ}) be the fuzzy Gamma-m-normed linear space, a sequence $\{f_{vr} f_{\gamma d_r}\}_{r=1}^{\infty}$ in (F_V, F_{Γ}) is convergent to $f_v f_{\gamma d}$ if for every δ in $(0, 1)$ and f_t in $(0, \infty)$ there exists appositve number M such that

$$\begin{aligned}
& F_{\Gamma}(f_{v1} f_{\gamma d1}, f_{v2} f_{\gamma d2}, \dots, f_{vr-1} f_{\gamma d_{r-1}}, f_{vr} f_{\gamma d_r} - f_v f_{\gamma d}, f_t) < \delta, \text{ for all } r \geq M \\
& \Rightarrow \lim_{r \rightarrow \infty} F_{\Gamma}(f_{v1} f_{\gamma d1}, f_{v2} f_{\gamma d2}, \dots, f_{vr-1} f_{\gamma d_{r-1}}, f_{vr} f_{\gamma d_r} - f_v f_{\gamma d}, f_t) = 0
\end{aligned}$$

3.16. Definition: Let (F_V, F_{Γ}) be the fuzzy Gamma-m-normed linear space, a sequence $\{f_{vr} f_{\gamma d_r}\}_{r=1}^{\infty}$ in $((F_V, F_{\Gamma}))$ is said to be Cauchy sequence if for every δ is in $(0, 1)$ and $f_t \in (0, \infty)$ there exists appositve number M such that

$$F_{\Gamma}(f_{v1} f_{\gamma d1}, f_{v2} f_{\gamma d2}, \dots, f_{vr-1} f_{\gamma d_{r-1}}, f_{vr} f_{\gamma d_r} - f_{v\ell} f_{\gamma d_{\ell}}, f_t) < \delta, \text{ for all } r, \ell \geq M,$$

$$\text{That is } \lim_{r, \ell \rightarrow \infty} F_{\Gamma}(f_{v1} f_{\gamma d1}, f_{v2} f_{\gamma d2}, \dots, f_{vr-1} f_{\gamma d_{r-1}}, f_{vr} f_{\gamma d_r} - f_{v\ell} f_{\gamma d_{\ell}}, f_t) = 0$$

3.17. Definition: A fuzzy Gamma-m-normed linear space (F_V, F_{Γ}) is said to be complete if every Cauchy sequence in (F_V, F_{Γ}) is convergence sequence.

3.18. Theorem: Let (F_V, F_{Γ}) be the fuzzy Gamma-m-normed linear space and let $\{f_{vr} f_{\gamma d_r}\}_{r=1}^{\infty}$ be a sequence in it then

(1) A sequence $\{f_{vr} f_{\gamma d_r}\}_{r=1}^{\infty}$ convergence to $f_v f_{\gamma d}$ if and only if $F_{\Gamma}(f_{v1} f_{\gamma d1}, f_{v2} f_{\gamma d2}, \dots, f_{vr-1} f_{\gamma d_{r-1}}, f_{vr} f_{\gamma d_r} - f_v f_{\gamma d}, f_t) \rightarrow 0$, as $r \rightarrow \infty$.

(2) Every convergence sequence is a Cauchy sequence in (F_V, F_{Γ}) .

Proof: Suppose a sequence $\{f_{vr}f_{\gamma}f_{dr}\}_{r=1}^{\infty}$ convergence to $f_v f_{\gamma} f_d$ if fix $f_t > 0$, then for any given δ in $(0,1)$ there is a positive integer $r \geq M$, such that

$$F_{\Gamma}(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vr-1}f_{\gamma}f_{dr-1}, f_{vr}f_{\gamma}f_{dr} - f_v f_{\gamma} f_d, f_t) < \delta, \text{ for all } r \geq M$$

$$\Rightarrow F_{\Gamma}(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vr-1}f_{\gamma}f_{dr-1}, f_{vr}f_{\gamma}f_{dr} - f_v f_{\gamma} f_d, f_t) \rightarrow 0, \text{ as } r \rightarrow \infty.$$

$$\text{that is } \lim_{r \rightarrow \infty} F_{\Gamma}(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vr-1}f_{\gamma}f_{dr-1}, f_{vr}f_{\gamma}f_{dr}, f_t) = f_v f_{\gamma} f_d$$

Conversely if for each $f_t \in (0, \infty)$ $F_{\Gamma}(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vr-1}f_{\gamma}f_{dr-1}, f_{vr}f_{\gamma}f_{dr} - f_v f_{\gamma} f_d, f_t) \rightarrow 0$, as $r \rightarrow \infty$ then for every δ in $(0,1)$ there exists positive integer $r \geq M$, such that $F_{\Gamma}(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vr-1}f_{\gamma}f_{dr-1}, f_{vr}f_{\gamma}f_{dr} - f_v f_{\gamma} f_d, f_t) < \delta$, for all $r \geq M$,

Hence a sequence $\{f_{vr}f_{\gamma}f_{dr}\}_{r=1}^{\infty}$ is convergence to $f_v f_{\gamma} f_d$ in (F_V, F_{Γ})

(2) Suppose a sequence $\{f_{vr}f_{\gamma}f_{dr}\}_{r=1}^{\infty}$ is convergence in (F_V, F_{Γ}) and if it is convergence to $f_v f_{\gamma} f_d$. Let f_t in $(0, \infty)$ and $0 < \varepsilon < 1$ then we choose δ is in $(0,1)$ such that $\delta \Delta^F \delta < \varepsilon$, since $\{f_{vr}f_{\gamma}f_{dr}\}_{r=1}^{\infty}$ is convergence to $f_v f_{\gamma} f_d$

Which implies that there exists positive integer $r \geq M$, such that

$$F_{\Gamma}(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vr-1}f_{\gamma}f_{dr-1}, f_{vr}f_{\gamma}f_{dr} - f_v f_{\gamma} f_d, \frac{f_t}{2}) < \delta, \text{ for all } r \geq M,$$

We consider that

$$F_{\Gamma}(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vr-1}f_{\gamma}f_{dr-1}, f_{vr}f_{\gamma}f_{dr} - f_v f_{\gamma} f_d, f_t)$$

$$= F_{\Gamma}(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vr-1}f_{\gamma}f_{dr-1}, f_{vr}f_{\gamma}f_{dr} - f_v f_{\gamma} f_d + f_v f_{\gamma} f_d - f_v f_{\gamma} f_d, \frac{f_t + f_t}{2})$$

$$\leq \{F_{\Gamma}(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vr-1}f_{\gamma}f_{dr-1}, f_{vr}f_{\gamma}f_{dr} - f_v f_{\gamma} f_d, \frac{f_t}{2}) \Delta^F F_{\Gamma}(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vr-1}f_{\gamma}f_{dr-1}, f_{vr}f_{\gamma}f_{dr} - f_v f_{\gamma} f_d, \frac{f_t}{2})\}$$

$$< \delta \Delta^F \delta$$

$$< \varepsilon, \text{ for all } r, \ell \geq M$$

Therefore $\{f_{vr}f_{\gamma}f_{dr}\}_{r=1}^{\infty}$ is a cauchy sequence in (F_V, F_{Γ}) .

3.19. Remark: The following examples 3.21.Example and 3.22.Example shows that there may exist cauchy sequence in the fuzzy Gamma-m-normed linear space (F_V, F_{Γ}) which is not convergent.

3.20. Remark: For the following examples 3.21.Example and 3.22.Example consider a fuzzy Gamma-m-normed linear space (F_V, F_{Γ}) as in previous 3.14. Example.

3.21.Example: Let (F_V, F_{Γ}) be the fuzzy Gamma-m-normed linear space, a sequence $\{f_{vr}f_{\gamma}f_{dr}\}_{r=1}^{\infty}$ in (F_V, F_{Γ}) then a sequence $\{f_{vr}f_{\gamma}f_{dr}\}_{r=1}^{\infty}$ is a convergence in (F_V, F_{Γ}) a sequence $\{f_{vr}f_{\gamma}f_{dr}\}_{r=1}^{\infty}$ is a convergence in $(F_V, \|\cdot, \dots, \|\cdot)$

Proof: We have a sequence $\{f_{vr}f_{\gamma}f_{dr}\}_{r=1}^{\infty}$ is also a convergence in (F_V, F_{Γ})

$$\lim_{r \rightarrow \infty} F_{\Gamma}(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, f_{v3}f_{\gamma}f_{d3}, \dots, f_{vr-1}f_{\gamma}f_{dr-1}, f_{vr}f_{\gamma}f_{dr} - f_v f_{\gamma} f_d, f_t) = 0$$

$$\Leftrightarrow \lim_{r \rightarrow \infty} \frac{\|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vr-1}f_{\gamma}f_{dr-1}, f_{vr}f_{\gamma}f_{dr} - f_v f_{\gamma} f_d)\|}{f_t + \|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dr-1}, f_{vr}f_{\gamma}f_{dr} - f_v f_{\gamma} f_d)\|} = 0$$

$$\Leftrightarrow \lim_{r \rightarrow \infty} \|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vr-1}f_{\gamma}f_{dr-1}, f_{vr}f_{\gamma}f_{dr} - f_v f_{\gamma} f_d)\| = 0$$

Hence a sequence $\{f_{vr}f_{\gamma}f_{dr}\}_{r=1}^{\infty}$ is a convergence in $(F_V, \|\cdot, \dots, \|\cdot)$.

Conversely we have a sequence $\{f_{vr}f_{\gamma}f_{dr}\}_{r=1}^{\infty}$ is a convergence in $(F_V, \|\cdot, \dots, \cdot\|)$

$$\begin{aligned} &\Leftrightarrow \lim_{r \rightarrow \infty} \|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vr-1}f_{\gamma}f_{dr-1}, f_{vr}f_{\gamma}f_{dr} - f_{v\ell}f_{\gamma}f_{d\ell})\| = 0 \\ &\Leftrightarrow \lim_{r \rightarrow \infty} \frac{\|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vr-1}f_{\gamma}f_{dr-1}, f_{vr}f_{\gamma}f_{dr} - f_{v\ell}f_{\gamma}f_{d\ell})\|}{f_t + \|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dr-1}, f_{vr}f_{\gamma}f_{dr} - f_{v\ell}f_{\gamma}f_{d\ell})\|} = 0 \\ &\lim_{r \rightarrow \infty} F_{\Gamma}(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, f_{v3}f_{\gamma}f_{d3}, \dots, f_{vr-1}f_{\gamma}f_{dr-1}, f_{vr}f_{\gamma}f_{dr} - f_{v\ell}f_{\gamma}f_{d\ell}, f_t) = 0 \end{aligned}$$

Hence a sequence $\{f_{vr}f_{\gamma}f_{dr}\}_{r=1}^{\infty}$ is also a convergence in (F_V, F_{Γ}) .

3.22.Example: Let (F_V, F_{Γ}) be the fuzzy Gamma-m-normed linear space, a sequence $\{f_{vr}f_{\gamma}f_{dr}\}_{r=1}^{\infty}$ in (F_V, F_{Γ}) then a sequence $\{f_{vr}f_{\gamma}f_{dr}\}_{r=1}^{\infty}$ is a Cauchy sequence in (F_V, F_{Γ}) a sequence $\{f_{vr}f_{\gamma}f_{dr}\}_{r=1}^{\infty}$ is a Cauchy sequence in $(F_V, \|\cdot, \dots, \cdot\|)$.

Proof: Suppose a sequence $\{f_{vr}f_{\gamma}f_{dr}\}_{r=1}^{\infty}$ is a Cauchy sequence in (F_V, F_{Γ})

$$\begin{aligned} &\lim_{r, \ell \rightarrow \infty} F_{\Gamma}(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vr-1}f_{\gamma}f_{dr-1}, f_{vr}f_{\gamma}f_{dr} - f_{v\ell}f_{\gamma}f_{d\ell}, f_t) = 0 \\ &\Leftrightarrow \lim_{r, \ell \rightarrow \infty} \frac{\|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vr-1}f_{\gamma}f_{dr-1}, f_{vr}f_{\gamma}f_{dr} - f_{v\ell}f_{\gamma}f_{d\ell})\|}{f_t + \|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dr-1}, f_{vr}f_{\gamma}f_{dr} - f_{v\ell}f_{\gamma}f_{d\ell})\|} = 0 \\ &\lim_{r, \ell \rightarrow \infty} \|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vr-1}f_{\gamma}f_{dr-1}, f_{vr}f_{\gamma}f_{dr} - f_{v\ell}f_{\gamma}f_{d\ell})\| = 0 \end{aligned}$$

Hence a sequence $\{f_{vr}f_{\gamma}f_{dr}\}_{r=1}^{\infty}$ is a Cauchy sequence in $(F_V, \|\cdot, \dots, \cdot\|)$.

Conversely we prove that a sequence $\{f_{vr}f_{\gamma}f_{dr}\}_{r=1}^{\infty}$ is a Cauchy sequence in (F_V, F_{Γ}) when it is a Cauchy sequence in $(F_V, \|\cdot, \dots, \cdot\|)$ if for every $\varepsilon > 0$ there exists appositve number M such that

$$\begin{aligned} &\Leftrightarrow \|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vr-1}f_{\gamma}f_{dr-1}, f_{vr}f_{\gamma}f_{dr} - f_{v\ell}f_{\gamma}f_{d\ell})\| < \varepsilon, \text{ whenever } r, \ell \geq M \\ &\Leftrightarrow \left| \frac{\|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vr-1}f_{\gamma}f_{dr-1}, f_{vr}f_{\gamma}f_{dr} - f_{v\ell}f_{\gamma}f_{d\ell})\|}{f_t + \|(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vm-1}f_{\gamma}f_{dr-1}, f_{vr}f_{\gamma}f_{dr} - f_{v\ell}f_{\gamma}f_{d\ell})\|} \right| < \varepsilon, \text{ whenever } r, \ell \geq M \\ &\Leftrightarrow |F_{\Gamma}(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vr-1}f_{\gamma}f_{dr-1}, f_{vr}f_{\gamma}f_{dr} - f_{v\ell}f_{\gamma}f_{d\ell}, f_t)| < \varepsilon, \text{ whenever } r, \ell \geq M \end{aligned}$$

Hence a sequence $\{f_{vr}f_{\gamma}f_{dr}\}_{r=1}^{\infty}$ is a Cauchy sequence in (F_V, F_{Γ}) .

3.23.Theorem: In a fuzzy Gamma-m-normed linear space (F_V, F_{Γ}) , every Cauchy sequence has a convergent subsequence is complete.

Proof: Let (F_V, F_{Γ}) be the fuzzy Gamma-m-normed linear space and let a sequence $\{f_{vr}f_{\gamma}f_{dr}\}_{r=1}^{\infty}$ is a Cauchy sequence in (F_V, F_{Γ}) .

Let $\{f_{vr\ell}f_{\gamma}f_{d\ell}\}_{r=1}^{\infty}$ be a subsequence of $\{f_{vr}f_{\gamma}f_{dr}\}_{r=1}^{\infty}$ and it is convergence to $f_v f_{\gamma} f_d$. now we need to prove that the sequence $\{f_{vr}f_{\gamma}f_{dr}\}_{r=1}^{\infty}$ is convergence to $f_v f_{\gamma} f_d$, for this let ε in $(0, 1)$ and $f_t \in (0, \infty)$, choose δ is in $(0, 1)$ such that $\delta \Delta^F \delta < \varepsilon$.

Given that a sequence $\{f_{vr}f_{\gamma}f_{dr}\}_{r=1}^{\infty}$ is a Cauchy sequence in (F_V, F_{Γ}) that is if for every ϵ in $(0, 1)$ and $f_t \in (0, \infty)$ there exists appositve number M such that $F_{\Gamma}(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{vr-1}f_{\gamma}f_{dr-1}, f_{vr}f_{\gamma}f_{dr} - f_{v\ell}f_{\gamma}f_{d\ell}, f_t) < \delta$, for all $r, \ell \geq M$.

We have the sub-sequence $\{f_{v\ell}f_{\gamma}f_{d\ell}\}_{\ell=1}^{\infty}$ is also convergent to $f_v f_{\gamma} f_d$, there exists a $r, \ell > M$ such that $F_{\Gamma}(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{v\ell-1}f_{\gamma}f_{d\ell-1}, f_{v\ell}f_{\gamma}f_{d\ell} - f_v f_{\gamma} f_d, \frac{1}{2}f_t) < \delta$ for all $r, \ell \geq M$.

Now $F_{\Gamma}(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{v\ell-1}f_{\gamma}f_{d\ell-1}, f_{v\ell}f_{\gamma}f_{d\ell} - f_v f_{\gamma} f_d, f_t) = F_{\Gamma}(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{v\ell-1}f_{\gamma}f_{d\ell-1}, f_{v\ell}f_{\gamma}f_{d\ell} - f_v f_{\gamma} f_d, \frac{1}{2}f_t + \frac{1}{2}f_t) \leq F_{\Gamma}(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{v\ell-1}f_{\gamma}f_{d\ell-1}, f_{v\ell}f_{\gamma}f_{d\ell} - f_v f_{\gamma} f_d, \frac{1}{2}f_t) \Delta^F F_{\Gamma}(f_{v1}f_{\gamma}f_{d1}, f_{v2}f_{\gamma}f_{d2}, \dots, f_{v\ell-1}f_{\gamma}f_{d\ell-1}, f_{v\ell}f_{\gamma}f_{d\ell} - f_v f_{\gamma} f_d, \frac{1}{2}f_t) < \delta \Delta^F \delta < \varepsilon$

Therefore a sequence $\{f_{v\ell}f_{\gamma}f_{d\ell}\}_{\ell=1}^{\infty}$ is convergence to $f_v f_{\gamma} f_d$ in (F_v, F_r)

Hence it is complete.

4. Discussion and Conclusion

In this research paper, we convicted the concept of fuzzy gamma ring, fuzzy gamma vector space and using this also introduced the notion of fuzzy Gamma m-normed linear space and produced a detailed axioms with theory of fuzzy n-normed linear space. In fuzzy Gamma m-normed linear space obtained some results on cauchy and convergence sequence. Also provided theorems of completeness sequence and Cauchy sequence in fuzzy Gamma m-normed linear space. This work can be extended to Banach fuzzy gamma linear space by introducing the concepts of completeness in fuzzy Gamma m-normed linear space.

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