A Numerical Simulation of Time Fractional Rosenau-Hyman Equation

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Abstract: In this paper, weimplement a recently developed analytical technique, the modified fractional reduced differential transform method (MFRDTM),coupled with Adomian polynomials for solving the fractional Rosenau-Hyman equation.Here the fractional derivatives are described in the Caputo sense. A numerical comparison between the exact and numerical solutions obtained by MFRDTM reveals that the present techniqueis effective, straightforward and simple.

Keywords: *Rosenau Hyman equation, Adomian polynomials, fractional derivatives, fractional differential transform, Reduced differential transform method.*

1. Introduction

Recently, many mathematical models related to the theory of fractional differential equations have received considerable attention from researchers and scientists because of their wide application in science and engineering. Various definitions of fractional differentiation and integration are available in the literature [1-3].

In this study, we consider the time fractional Rosenau Hyman equation:

$$D_t^\alpha v = \tau v v_x + \mu (v^2)_{xxx} \tag{1}$$

subject to the initial condition

$$v(x,0) = f(x) \tag{2}$$

Whereτ and $\mu$are the parameters of convection and nonlinear dispersion terms, respectively.

$\alpha$is the order of fractional derivative$(0 < \alpha \leq 1)$, t is the time and x is the spatial coordinate.Such anequation has been widely used to study the effect of nonlinear dispersion forming patterns in liquid drops when $\alpha = 1$. Many research articles [4-10]discuss the exact and approximate analytical methods of the Rosenau-Hyman equation. Little attention has been given to the time fractional Rosenau-Hyman equation [11-14], and the application of the time-space fractional Rosenau-Hyman equation is missing in the literature.

2. Fractional derivative and integration

2.1 Definition The Riemann-Liouville integral operator J^α of order $\alpha \geq 0$ of a function $f \in C_\delta,\ \delta \geq -1$ is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma\alpha}\int_0^x (x-\tau)^{\alpha-1} f(\tau)\, d\tau, \quad \alpha > 0\,,\ \ \alpha \in R \tag{3}$$

$$J^0 J[f(x)] = 0$$

Here we mention some essential properties of the operatorJ^α.

If $f \in C_\delta,\ \ \delta \geq -1, \alpha, \gamma \geq 0, \beta > -\frac{1}{2}$

$(i) J^\alpha J^\gamma f(x) = J^{\alpha+\gamma} f(x)$

$(ii) J^\alpha J^\gamma f(x) = J^\gamma J^\alpha f(x)$

$(iii) J^{\alpha} x^{\beta} = \dfrac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} x^{\alpha+\beta}$

2.2 Definition The fractional order derivative of $f(t)$ in the Caputo sense is defined as

$$D_t^{\alpha}[f(t)] = I_t^{n-\alpha} D^n[f(t)] = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\theta)^{n-\alpha-1} f^n(\theta)\, d\theta, & n-1<\alpha<n, n \in N \\ D_t^n f(t) & if \;\; \alpha = n, \quad n \in N \end{cases} \tag{4}$$

Where $\Gamma(.)$ denotes the Gamma function.

For the Caputo fractional derivative, we have the following properties:

$(i) D^{\alpha} K = 0$, $K$is a constant

$(ii) D_t^{\alpha}[af(t) + bg(t)] = aD_t^{\alpha} f(t) + bD_t^{\alpha} g(t)$

$$(iii) D^{\alpha} t^{\gamma} = \begin{cases} 0 & \gamma \le \alpha - 1 \\ \dfrac{\Gamma(\gamma+1) t^{\gamma-\alpha}}{\Gamma(\gamma-\alpha+1)} & \gamma > \alpha - 1 \end{cases}$$

Where 'a' and 'b' are the constants.

2.3 Definition The modified Riemann-Liouville derivative of order $'\gamma'$is defined as

$$D_t^{\gamma} f(t) = \begin{cases} \frac{1}{\Gamma(-\gamma)} \int_0^t (t-\theta)^{-\gamma-1} [f(\theta) - f(0)] d\theta & \gamma < 0 \\ \frac{1}{\Gamma(-\gamma)} \frac{d}{dt} \int_0^t (t-\theta)^{-\gamma-1} [f(\theta) - f(0)] d\theta, & 0 < \gamma < 1 \\ \left[f^{(\gamma-m)}(t)\right]^{(m)}, & m \le \gamma \le m+1, \quad m \ge 1 \end{cases} \tag{5}$$

where$f: R \to R$ is a continuous function.

Below we list some significant properties for the modified Riemann-Liouville derivatives:

$(i) D_t^{\gamma}(a) = 0, \gamma > 0, a$ is a constant

$(ii) D_t^{\gamma}[cf(t)] = cD_t^{\gamma} f(t), \gamma > 0$

$(iii) D_t^{\gamma} t^{\beta} = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\gamma)} t^{\beta-\gamma}, \;\; \beta > \gamma > 0$

$(i) D_t^{\gamma}[f(t)g(t)] = [D_t^{\gamma} f(t)] g(t) + f(t)[D_t^{\gamma} g(t)]$

$(v) D_t^{\gamma}[f(h(t))] = f_h'(h(t)) D_t^{\gamma} h(t)$

3. Fractional Reduced Differential Transform Method (FRDTM)

Consider the function of two variables $v(x,t)$and assume that it can be expressed as $v(x,t) = h(x)g(t)$. Based on the properties of one-dimensional differential transform, the function $v(x,t)$[15, 16, 17] can be represented as

$$v(x,t) = \sum_{k=0}^{\infty} V_k(x)\, t^{\alpha k} \tag{6}$$

where $V_k(x)$ is called the t-dimensional spectrum function of $v(x,t)$and α is the fractional order.

3.1 Definition If $v(x,t)$ is continuously differentiable with respect to t and xin the domain of interest, then

$$V_k(x) = \frac{1}{\Gamma(k\alpha+1)} [(D_t^{\alpha})^k v(x,t)]_{t=0} \tag{7}$$

where $(D_t^{\alpha})^k = D_t^{\alpha} D_t^{\alpha} \dots D_t^{\alpha}$, the k-times Caputo fractional order derivative

3.2 Definition The differential inverse transform of $V_k(x)$ is defined as

$$v(x,t) = \sum_{k=0}^{\infty} V_k(x) t^{k\alpha} \tag{8}$$

We combine the Eqns. (6) and (7), we can write

$$v(x,t)=\sum_{k=0}^{\infty}\left(\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}v(x,t)\right]_{t=0}\right)t^{k\alpha} \qquad 0<\alpha\le 1 \tag{9}$$

We summarized some standard theorems of fractional reduced differential transform [19, 20].

3.3 Properties If $V_k(x)$ is the fractional reduced differential transform of the function $v(x,t)$ then

(i)If $v(x,t)=af(x,t)\pm bg(x,t)$ then $V_k(x)=aF_k(x)\pm bG_k(x), \quad a,b\in R$

(ii)If $v(x,t)=f(x,t)g(x,t)$ then $V_k(x)=\sum_{k_1=0}^{k}F_{k_1}(x)G_{k-k_1}(x)$

(iii) If $v(x,t)=x^m f(x,t)$ then $V_k(x)=x^m F_k(x)$

(iv) If $v(x,t)=\frac{\partial f(x,t)}{\partial x}$ then $V_k(x)=\frac{\partial F_k(x)}{\partial x}$

(v) If $v(x,t)=D_t^{n\alpha}u(x,t)$ then $V_k(x)=\frac{\Gamma\alpha(k+n)+1}{\Gamma(\alpha k+1)}F_{k+n}(x), \quad n\in N, \quad \alpha\in R$

4. Basic Idea of the Modified FRDTM

To illustrate the basic idea of the Modified FRDTM, we consider the following fractional differential equation:

$$D_t^{n\alpha}v(x,t)+L[v(x,t)]+N[v(x,t)]=h(x,t), \; t>0, \; x\in R, \; (n-1)<\alpha\le n \tag{10}$$

Subject to the initial condition

$$v(x,0)=V_0(x)=f(x)$$

where $D_t^{n\alpha}=\frac{\partial^{n\alpha}}{\partial t^{n\alpha}}$, $L[v(x,t)]$ a general linear term, $N[v(x,t)]$ a general nonlinear term and $h(x,t)$ is a source function.

First, we operate a complex fractional transformation [18] to reduce the fractional differential equation (6) into an ordinary differential equation.

$$D_T^n v(x,t)+L[v(x,t)]+N[v(x,t)]=h(x,t), \qquad T>0, \; x\in I \tag{11}$$

In the above equation, the nonlinear function$N[v(x,t)]$is approximated by the series of Adomian polynomials A_k[14], $k=0,1,2,\ldots$

$$N[v(x,t)]=\sum_{k=0}^{\infty}A_k \tag{12}$$

Now substituting the Eqns.(8) and (12) in to the Eqn.(6) and operating Riemann-Liouville integral to both sides of the Eqn.(6) we have

$$\sum_{k=0}^{\infty}V_k(x)t^{k\alpha}=f(x)-J^{\alpha}R\left[\sum_{k=0}^{\infty}V_k(x)t^{k\alpha}\right]-J^{\alpha}\left[\sum_{k=0}^{\infty}A_k(V_0,V_1,..V_k)t^{k\alpha}\right]-J^{\alpha}\sum_{k=0}^{\infty}H_k(x)t^{k\alpha} \tag{13}$$

where$H_k(x)$ is the differential transform of$h(x,t)$.

We now successfully obtain thefollowing recursive relation

$V_0(x)=f(x)$and

$$V_{k+1}(x)=-R\left[V_k(x)\frac{\Gamma(\alpha k+1)}{\Gamma\alpha(k+1)+1}\right]-A_k(x)\frac{\Gamma(\alpha k+1)}{\Gamma\alpha(k+1)+1}-H_k(x)\frac{\Gamma(\alpha k+1)}{\Gamma\alpha(k+1)+1}, \quad k\ge 0 \tag{14}$$

which leads to the determination of the components $V_1(x), V_2(x), \ldots..$

Finally, summing these components, we obtain the approximate solution $v(x,t)$ by the truncated series

$$v(x,t) = \lim_{m\to\infty} \bar{v}_m(x,t) = \sum_{p=0}^{m} V_p(x) t^{p\alpha} \quad (15)$$

where p is the order of convergence of the above series

5. Numerical Simulation

To test the efficiency and simplicity of the present numerical method, we consider the fractional order Rosenau-Hyman equation.

$$D_t^\alpha v = v v_{xxx} + v v_x + 3 v_x v_{xx} \quad (16)$$

Withthe initial the condition

$$v(x,0) = V_0(x) = -\frac{8}{3} k \cos^2\left(\frac{x}{4}\right) \quad (17)$$

According to MFRDTM, we can construct the following recursive relation for the Eqn. (16),

$$V_{k+1}(x) = \frac{\Gamma k\alpha + 1}{\Gamma(k+1)\alpha + 1} A_k$$

where$A_k = v_k v_{k_{xxx}} + v_k v_{k_x} + 3 v_{k_x} v_{k_{xx}}$

The first few Adomian polynomials are given by

$$A_0 = v_0 v_{0_{xxx}} + v_0 v_{0_x} + 3 v_{0_x} v_{0_{xx}}$$

$$A_1 = v_1 v_{0_{xxx}} + v_0 v_{1_{xxx}} + v_1 v_{0_x} + v_0 v_{1_x} + 3 v_{1_x} v_{0_{xx}} + 3 v_{0_x} v_{1_{xx}}$$

$$A_2 = v_0 v_{2_{xxx}} + v_1 v_{1_{xxx}} + v_2 v_{0_{xxx}} + v_2 v_{0_x} + v_1 v_{1_x} + v_0 v_{2_x} + 3 v_{2_x} v_{0_{xx}} + 3 v_{1_x} v_{1_{xx}} + 3 v_{0_x} v_{2_{xx}}$$

$$A_3 = v_3 v_{0_{xxx}} + v_2 v_{1_{xxx}} + v_1 v_{2_{xxx}} + v_0 v_{3_{xxx}} + v_3 v_{0_x} + v_2 v_{1_x} + v_1 v_{2_x} + v_0 v_{3_x} +$$

$$3 v_{3_x} v_{0_{xx}} + 3 v_{2_x} v_{1_{xx}} + 3 v_{1_x} v_{2_{xx}} + 3 v_{0_x} v_{3_{xx}}$$

$$A_4 = v_4 v_{0_{xxx}} + v_3 v_{1_{xxx}} + v_2 v_{2_{xxx}} + v_1 v_{3_{xxx}} + v_0 v_{4_{xxx}} + v_4 v_{0_x} + v_3 v_{1_x} + v_2 v_{2_x} +$$

$$v_1 v_{3_x} + v_0 v_{4_x} + 3 v_{4_x} v_{0_{xx}} + 3 v_{3_x} v_{1_{xx}} + 3 v_{2_x} v_{2_{xx}} + 3 v_{1_x} v_{3_{xx}} + 3 v_{0_x} v_{4_{xx}}$$

Utilizing the recursive relation together with Adomian polynomials, we can obtain the transformed components $V_1, V_2, \ldots$...of the series solution as follows:

$$V_1 = -\frac{2}{3} \mathrm{k}^2 \sin\left(\frac{\mathrm{x}}{2}\right) \frac{1}{\Gamma 1 + \alpha}$$

$$V_2 = \frac{\mathrm{k}^3}{3} \cos\left(\frac{\mathrm{x}}{2}\right) \frac{1}{\Gamma 1 + 2\alpha}$$

$$V_3 = \frac{\mathrm{k}^4}{6} \sin\left(\frac{\mathrm{x}}{2}\right) \frac{1}{\Gamma 1 + 3\alpha}$$

$$V_4 = -\frac{k^5}{12} \cos\left(\frac{x}{2}\right) \frac{1}{\Gamma 1 + 4\alpha}$$ and so on.

Table 1. Approximate solution of R.H.Eqn.obtained by eighth iterate MFRDTM method for $\alpha = 0.5$, 0.7, 1 and absolute error when $k = 0.5$

x	t	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 1$	Absolute Error
$\pi/4$	0.4	-1.312145142	-1.310730062	-1.304979051	1.11022×10^{-15}
	0.6	-1.314290777	-1.316024886	-1.313795239	2.57571×10^{-14}
	0.8	-1.314814942	-1.319101011	-1.320993942	3.42392×10^{-13}
	1.0	-1.314313275	-1.320505078	-1.326557167	2.51865×10^{-12}
$\pi/2$	0.4	-1.209148762	-1.19933163	-1.182778052	1.11022×10^{-15}
	0.6	-1.221190375	-1.216132042	-1.203223634	4.90718×10^{-14}
	0.8	-1.230135568	-1.229789775	-1.222328102	6.51700×10^{-13}
	1.0	-1.237086178	-1.241129521	-1.240043707	4.82902×10^{-12}
$3\pi/4$	0.4	-1.023564402	-1.006839786	-0.982003772	1.44328×10^{-15}
	0.6	-1.043668766	-1.032588076	-1.010966094	6.50590×10^{-14}
	0.8	-1.05967316	-1.054748150	-1.039067846	8.61533×10^{-13}
	1.0	-1.073017881	-1.074297201	-1.066238790	6.40398×10^{-12}
π	0.4	-0.783645596	-0.762559668	-0.733222277	1.99840×10^{-15}
	0.6	-0.808752004	-0.793335893	-0.766292088	7.06101×10^{-14}
	0.8	-0.829379077	-0.820624637	-0.799112887	9.40358×10^{-13}
	1.0	-0.847086295	-0.845406827	-0.831602639	7.00417×10^{-12}

Table 2. Approximate solution of R.H.Eqn.obtained by eighth iterate MFRDTM method for $\alpha = 0.5$, 0.7, 1 and absolute error when $k = 1$

x	t	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 1$	Absolute Error
$\pi/4$	0.4	-2.618349506	-2.641343393	-2.641987884	6.84785×10^{-13}
	0.6	-2.595556740	-2.634428866	-2.660942021	2.56505×10^{-11}
	0.8	-2.567622713	-2.614945895	-2.666631130	3.32512×10^{-10}
	1.0	-2.537051191	-2.586159127	-2.658998367	2.40917×10^{-09}
$\pi/2$	0.4	-2.501283616	-2.489937857	-2.444656205	1.30295×10^{-12}
	0.6	-2.518006617	-2.531191158	-2.512652335	4.95639×10^{-11}
	0.8	-2.522358280	-2.555200131	-2.568865099	6.52883×10^{-10}
	1.0	-2.519102459	-2.566139834	-2.612732835	4.80971×10^{-09}
$3\pi/4$	0.4	-2.206407883	-2.162449766	-2.078135693	1.72306×10^{-12}
	0.6	-2.260100725	-2.245590455	-2.184822022	6.59317×10^{-11}

	0.8	-2.296075577	-2.309436224	-2.283000557	8.73859×10^{-10}
	1.0	-2.320631127	-2.358436926	-2.371690326	6.47801×10^{-09}
π	0.4	-1.778614463	-1.708736214	-1.598225774	1.88071×10^{-12}
	0.6	-1.861102899	-1.821106874	-1.727360275	7.22626×10^{-11}
	0.8	-1.923224095	-1.915069499	-1.852557788	9.61798×10^{-10}
	1.0	-1.971852655	-1.994671288	-1.972567377	7.16010×10^{-09}

6. Conclusion

We presented an approximate analytical method to obtain the numerical solution of the R-H Equation using the Modified fractional reduced differential transform method. Numerical computations are performed using the Mathematicapackage. From the test results, we observed that the present techniquegives the solution in the form of a rapidlyconvergent series withoutperturbation or discretization.

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