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Certain subclasses of schlicht functions of Mittag-Leffler-type Poisson distribution series with complex order.

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Abstract

In this paper,we obtain the membership conditions for Mittag-Leffler (M-L)type poisson distribution series to be in the sub-class of analytic functions $S(\vartheta,\eta,\delta)$ and $R(\vartheta,\eta,\delta)$ with negative coefficients, $P(\eta,\vartheta)$ and $R(\eta,\vartheta)$ with positive coefficients defined in the open unit disc.Some special cases of these results are also discussed.

KeyWords: Analytic functions, Univalent functions, Mittag-Leffler, Poisson distribution series, Alexander integral operator.

1 Preliminary

Let $\mathcal{A}$, denote the class of functions of the form

$$f(\zeta) = \zeta + \sum_{l=2}^{\infty} a_l \zeta^l \tag{1.1}$$

and analytic in the open unit disc $\mathcal{U} = \{\zeta \in \mathbb{C}; |\zeta| < 1\}$ satisfying the condition $f(0) = f'(0) - 1 = 0$. $\mathcal{S}$ be the subclass of $\mathcal{A}$ which are univalent(schlicht) in $\mathcal{U}$.

A function $f \in \mathcal{A}$ is an element of $\mathcal{S}^*(\vartheta)$, the class of starlike functions of complex order $\vartheta \in \mathbb{C}^*$ [6] iff

$$\frac{f(\zeta)}{\zeta} \neq 0 \quad \text{and} \quad Re\left(1 + \frac{1}{\vartheta}\left(\frac{\zeta f'(\zeta)}{f(\zeta)} - 1\right)\right) > 0, (\zeta \in \mathcal{U}). \tag{1.2}$$

A function $f \in \mathcal{A}$ is an element of $\mathcal{C}(\vartheta)$, the class of convex functions of complex order $\vartheta \in \mathbb{C}^*$ [11] iff

$$f'(\zeta) \neq 0 \quad \text{and} \quad Re\left(1 + \frac{1}{\vartheta}\left(\frac{\zeta f''(\zeta)}{f'(\zeta)}\right)\right) > 0, (\zeta \in \mathcal{U}). \tag{1.3}$$

We note that $f \in \mathcal{C}(\vartheta) \Leftrightarrow \zeta f' \in \mathcal{S}^*(\vartheta)$.

A function $f \in \mathcal{A}$, is a member of $\mathcal{R}(\vartheta)$, the class of close-to-convex functions of complex order $\vartheta \in \mathbb{C}^*$, iff

$$Re\left(1 + \frac{1}{\vartheta}(f'(\zeta) - 1)\right) > 0, (\zeta \in \mathcal{U}). \tag{1.4}$$

The class $\mathcal{R}(\vartheta)$ was discussed by Halim[4] and Owa [7]. Denote $\mathcal{T}$, a subclass of $\mathcal{S}$ containing the functions are represented by

$$f(\zeta) = \zeta - \sum_{l=2}^{\infty} a_l \zeta^l, (a_l \geq 0) \quad (1.5)$$

The subclasses of $\mathcal{A}(n)$ defined by Altintas et al,[1] which consist of the function of the form

$$f(\zeta) = \zeta - \sum_{l=n+1}^{\infty} a_l \zeta^l, (a_l \geq 0) \quad (1.6)$$

A function f of $\mathcal{A}(n)$ belongs to the subclass $S_n(\vartheta, \eta, \delta)$ if it satisfies,

$$\left| \frac{1}{\vartheta} \left(\frac{\zeta f'(\zeta) + \eta \zeta^2 f''(\zeta)}{(1-\eta) f(\zeta) + \eta \zeta f'(\zeta)} - 1 \right) \right| < \delta, (\zeta \in \mathcal{U}, \vartheta \in \mathbb{C}^*, 0 \leq \eta \leq 1, 0 < \delta \leq 1) \quad (1.7)$$

A function f of $\mathcal{A}(n)$ belongs to the subclass $R_n(\vartheta, \eta, \delta)$ if it satisfies,

$$\left| \frac{1}{\vartheta} (f'(\zeta) + \eta \zeta f''(\zeta) - 1) \right| < \delta, (\zeta \in \mathcal{U}, \vartheta \in \mathbb{C}^*, 0 \leq \eta \leq 1, 0 < \delta \leq 1) \quad (1.8)$$

Note that, $S_1(\vartheta, \eta, \delta) = S(\vartheta, \eta, \delta)$ and $R_1(\vartheta, \eta, \delta) = R(\vartheta, \eta, \delta)$.

In 2000, Altintas and Ozkan gave the membership conditions for the function belonging to the classes $S(\vartheta, \eta, \delta)$ and $R(\vartheta, \eta, \delta)$, which are stated as follows.

Lemma 1.1 *[1] The function* $f \in S(\vartheta, \eta, \delta)$ *if and only if*

$$\sum_{l=2}^{\infty} [\eta(l-1) + 1](l + \delta|\vartheta| - 1) a_l \leq \delta|\vartheta|, where f \in \mathcal{A}(n). \quad (1.9)$$

Lemma 1.2 *[1] The function* $f \in R(\vartheta, \eta, \delta)$ *if and only if*

$$\sum_{l=2}^{\infty} l[\eta(l-1) + 1] a_l \leq \delta|\vartheta|, where f \in \mathcal{A}(n). \quad (1.10)$$

Let $P(\eta, \vartheta)$ of $\mathcal{A}$ contains the functions of the form (1.1) which satisfy

$$Re\left(1 + \frac{1}{\vartheta}\left(\frac{\zeta f'(\zeta) + \eta \zeta^2 f''(\zeta)}{(1-\eta) f(\zeta) + \eta \zeta f'(\zeta)} - 1\right)\right) > 0, (\zeta \in \mathcal{U}, \vartheta \in \mathbb{C}^*, 0 \leq \eta \leq 1). \quad (1.11)$$

Also, the subclass, $R(\eta, \vartheta)$ of $\mathcal{A}$ contains the functions of the form (1.1) which satisfy

$$Re\left(1 + \frac{1}{\vartheta}(f'(\zeta) + \eta \zeta f''(\zeta) - 1)\right) > 0, (\zeta \in \mathcal{U}, \vartheta \in \mathbb{C}^*, 0 \leq \eta \leq 1). \quad (1.12)$$

$P(\eta, \vartheta)$ and $R(\eta, \vartheta)$ were discussed by Altintas et al. [2] and Aouf [3], they determined the membership conditions for function belonging to the classes $P(\eta, \vartheta)$ and $R(\eta, \vartheta)$, which are stated in the following lemmas.

Lemma 1.3 *[3] the function* $f \in P(\eta, \vartheta)$*, if it satisfies the condition,*

$$\sum_{l=2}^{\infty} [\eta(l-1) + 1][(-1 + l) + |2\vartheta - 1 + l|]|a_l| \leq 2|\vartheta|, where f \in \mathcal{A}. \quad (1.13)$$

Lemma 1.4 *[3] the function* $f \in R(\eta, \vartheta)$*, if it satisfies the condition,*

$$\sum_{l=2}^{\infty} l[\eta(l-1) + 1]|a_l| \leq |\vartheta|, where f \in \mathcal{A}. \quad (1.14)$$

In 1903, (M-L) [5] introduced a function defined by

$$E_{\mu}(\zeta)=\sum_{l=0}^{\infty}\frac{\zeta^{l}}{\Gamma(\mu l+1)},(\zeta\in\mathbb{C},Re(\mu)>0)\tag{1.15}$$

By introducing a parameter, Wiman [12] generalised the function (1.15) as follows,

$$E_{\mu,\xi}(\zeta)=\sum_{l=0}^{\infty}\frac{\zeta^{l}}{\Gamma(\mu l+\xi)},(\zeta,\mu,\xi\in\mathbb{C},Re(\mu)>0,Re(\xi)>0)\tag{1.16}$$

Let $\psi_{\mu,\xi}^{m}(\zeta)\in\mathcal{A}$ with the power series of the form (1.1) and $P(X=s)=\frac{m^{s}}{\Gamma(\mu s+\xi)E_{\mu,\xi}(m)},m>0,s=0,1,2,3,,,,,$ the probability mass function of the (M-L)-type poisson distribution, then

$$\psi_{\mu,\xi}^{m}(\zeta)=\zeta+\sum_{l=2}^{\infty}\frac{m^{l-1}}{\Gamma(\mu(l-1)+\xi)E_{\mu,\xi}(m)}\zeta^{l}.$$

Further,We define the following series

$$\varphi_{\mu,\xi}^{m}(\zeta)=2\zeta-\psi_{\mu,\xi}^{m}(\zeta)=\zeta-\sum_{l=2}^{\infty}\frac{m^{l-1}}{\Gamma(\mu(l-1)+\xi)E_{\mu,\xi}(m)}\zeta^{l}$$

Motivated by the works of Saurabh Porwal,Nanjundan Magesh [9], we obtain the membership conditions for the (M-L)-type poisson distribution series $\varphi_{\mu,\xi}^{m}(\zeta)$ to be in $S(\vartheta,\eta,\delta)$ and $R(\vartheta,\eta,\delta)$ and also membership conditions for the (M-L)-type poisson distribution series $\psi_{\mu,\xi}^{m}(\zeta)$ to be in $P(\eta,\vartheta)$ and $R(\eta,\vartheta)$. In addition, we establish the integral operators for the series $\varphi_{\mu,\xi}^{m}(\zeta)$ and $\psi_{\mu,\xi}^{m}(\zeta)$.

2. Main results

Theorem 2.1 *If* $\mu,m>0$ *and* $\xi>2$, *then* $\varphi_{\mu,\xi}^{m}(\zeta)\in S(\vartheta,\eta,\delta)$ *if and only if*

$$\frac{1}{E_{\mu,\xi}(m)}[\frac{\eta}{\mu^{2}}\left(E_{\mu,\xi-2}(m)-\frac{1}{\Gamma(\xi-2)}\right)$$
$$+\left(\frac{\mu(\eta\delta|\vartheta|+1)+\eta(3-2\xi)}{\mu^{2}}\right)\left(E_{\mu,\xi-1}(m)-\frac{1}{\Gamma(\xi-1)}\right)$$
$$+\left(\frac{\eta}{\mu^{2}}(1-\xi)^{2}+\frac{(\eta\delta|\vartheta|+1)(1-\xi)}{\mu}+\delta|\vartheta|\right)\left(E_{\mu,\xi}(m)-\frac{1}{\Gamma(\xi)}\right)]\leq\delta|\vartheta|.$$

Proof. From lemma(1.1), it is enough to show that

$$\sum_{l=2}^{\infty}[\eta(l-1)+1][l+\delta|\vartheta|-1]\left(\frac{m^{l-1}}{\Gamma(\mu(l-1)+\xi)E_{\mu,\xi}(m)}\right)\leq\delta|\vartheta|\tag{2.1}$$

Consider,

$$\sum_{l=2}^{\infty}[\eta(l-1)+1][l+\delta|\vartheta|-1]\frac{m^{l-1}}{\Gamma(\mu(l-1)+\xi)E_{\mu,\xi}(m)}$$

$$=\sum_{l=1}^{\infty}[\eta l^{2}+(\eta\delta|\vartheta|+1)l+\delta|\vartheta|](\frac{m^{l}}{\Gamma\mu l+\xi)E_{\mu,\xi}(m)})$$

$$=\sum_{l=1}^{\infty}[\frac{\eta}{\mu^{2}}[(l\mu-1+\xi)(l\mu-2+\xi)]$$

$$+\frac{(l\mu+\xi-1)[\eta(3-2\xi)+\mu(\eta\delta|\vartheta|+1)}{\mu^2}+\frac{\eta}{\mu^2}(1-\xi)^2$$

$$+\frac{(\eta\delta|\vartheta|+1)(1-\xi)}{\mu}+\delta|\vartheta|]\left(\frac{m^l}{\Gamma(\mu l+\xi)E_{\mu,\xi}(m)}\right)$$

$$=\frac{1}{E_{\mu,\xi}(m)}[\frac{\eta}{\mu^2}\sum_{l=1}^{\infty}\frac{m^l}{\Gamma(\mu l+\xi-2)}$$

$$+\left(\frac{\eta(3-2\xi)+\mu(\eta\delta|\vartheta|+1)}{\mu^2}\right)\sum_{l=1}^{\infty}\frac{m^l}{\Gamma(\mu l+\xi-1)}$$

$$+\left(\frac{\eta(1-\xi)^2}{\mu^2}+\frac{(\eta\delta|\vartheta|+1)(1-\xi)}{\mu}+\delta|\vartheta|\right)\sum_{l=1}^{\infty}\frac{m^l}{\Gamma(\mu l+\xi)}]$$

$$=\frac{1}{E_{\mu,\xi}(m)}[\frac{\eta}{\mu^2}(E_{\mu,\xi-2}(m)-\frac{1}{\Gamma(\xi-2)})$$

$$+\left(\frac{\mu(\eta\delta|\vartheta|+1)+\eta(3-2\xi)}{\mu^2}\right)\left(E_{\mu,\xi-1}(m)-\frac{1}{\Gamma(\xi-1)}\right)$$

$$+(\frac{\eta}{\mu^2}(1-\xi)^2+\frac{(\eta\delta|\vartheta|+1)(1-\xi)}{\mu}+\delta|\vartheta|)(E_{\mu,\xi}(m)-\frac{1}{\Gamma(\xi)})]$$

$$\leq\delta|\vartheta|.$$

Theorem 2.2 *If* $\mu, m > 0$ *and* $\xi > 2$, *then* $\varphi_{\mu,\xi}^{m}(\zeta) \in R(\vartheta,\eta,\delta)$ *if and only if*

$$\frac{1}{E_{\mu,\xi}(m)}[\frac{\eta}{\mu^2}(E_{\mu,\xi-2}(m)-\frac{1}{\Gamma(\xi-2)})$$

$$+(\frac{\mu(\eta+1)+\eta(3-2\xi)}{\mu^2})(E_{\mu,\xi-1}(m)-\frac{1}{\Gamma(\xi-1)})$$

$$+(\frac{\eta}{\mu^2}(1-\xi)^2+\frac{(1+\eta)(1-\xi)}{\mu}+1)(E_{\mu,\xi}(m)-\frac{1}{\Gamma(\xi)})]\leq\delta|\vartheta|.$$

Proof. From lemma(1.2), It is sufficient to show that,

$$\sum_{l=2}^{\infty} l[\eta(l-1)+1]\left(\frac{m^{l-1}}{\Gamma(\mu(l-1)+\xi)E_{\mu,\xi}(m)}\right)\leq\delta|\vartheta|$$

Consider,

$$\sum_{l=2}^{\infty} l[\eta(-1+l)+1]\left(\frac{m^{l-1}}{\Gamma(\mu(l-1)+\xi)E_{\mu,\xi}(m)}\right)$$

$$=\sum_{l=1}^{\infty}(l+1)[\eta l+1]\left(\frac{m^l}{\Gamma(\mu l+\xi)E_{\mu,\xi}(m)}\right)$$

$$=\sum_{l=1}^{\infty}[\eta l^2+(\eta+1)l+1]\left(\frac{m^l}{\Gamma(\mu l+\xi)E_{\mu,\xi}(m)}\right)$$

$$=\frac{1}{E_{\mu,\xi}(m)}[\frac{\eta}{\mu^2}\sum_{l=1}^{\infty}\frac{m^l}{\Gamma(\mu l+\xi-2)}+\left(\frac{\eta(3-2\xi)+\mu(\eta+1)}{\mu^2}\right)\sum_{l=1}^{\infty}\frac{m^l}{\Gamma(\mu l+\xi-1)}$$

$$+(\frac{\eta}{\mu^2}(1-\xi)^2+\frac{(1+\eta)(1-\xi)}{\mu}+1)\sum_{l=1}^{\infty}\frac{m^l}{\Gamma(\mu l+\xi)}]$$

$$=\frac{1}{E_{\mu,\xi}(m)}[\frac{\eta}{\mu^2}(E_{\mu,\xi-2}(m)-\frac{1}{\Gamma(\xi-2)})$$

$$+(\frac{\mu(\eta+1)+\eta(3-2\xi)}{\mu^2})(E_{\mu,\xi-1}(m)-\frac{1}{\Gamma(\xi-1)})$$

$$+(\frac{\eta}{\mu^2}(1-\xi)^2+\frac{(1+\eta)(1-\xi)}{\mu}+1)(E_{\mu,\xi}(m)-\frac{1}{\Gamma(\xi)})]$$

$$\leq \delta|\vartheta|.$$

Theorem 2.3 *If* $\mu, m > 0$ *and* $\xi > 2$, *then* $\psi_{\mu,\xi}^{m}(\zeta) \in P(\eta,\vartheta)$ *iff*

$$\frac{2}{E_{\mu,\xi}(m)}[\frac{\eta}{\mu^2}(E_{\mu,\xi-2}(m)-\frac{1}{\Gamma(\xi-2)})$$

$$+\left(\frac{\mu(\eta|\vartheta|+1)+\eta(3-2\xi)}{\mu^2}\right)(E_{\mu,\xi-1}(m)-\frac{1}{\Gamma(\xi-1)})$$

$$+\left(\frac{\eta}{\mu^2}(1-\xi)^2+\frac{(\eta|\vartheta|+1)(1-\xi)}{\mu}+|\vartheta|\right)(E_{\mu,\xi}(m)-\frac{1}{\Gamma\xi})] \leq 2|\vartheta|.$$

Proof. From lemma (1.3), It is enough to show that,

$$\sum_{l=2}^{\infty}[1+\eta(-1+l)][-1+l+|2\vartheta-1+l|]\left|\frac{m^{l-1}}{\Gamma(\mu(l-1)+\xi)E_{\mu,\xi}(m)}\right| \leq 2|\vartheta|$$

Consider,

$$\sum_{l=2}^{\infty}[1+\eta(-1+l)][-1+l+|2\vartheta-1+l|]\left|\frac{m^{l-1}}{\Gamma(\mu(l-1)+\xi)E_{\mu,\xi}(m)}\right|$$

$$\leq \sum_{l=1}^{\infty}(1+\eta l)(2l+2|\vartheta|)\frac{m^l}{\Gamma(\mu l+\xi)E_{\mu,\xi}(m)}$$

$$=\frac{2}{E_{\mu,\xi}(m)}\sum_{l=1}^{\infty}[\eta l^2+(\eta|\vartheta|+1)l+|\vartheta|]\frac{m^l}{\Gamma(\mu l+\xi)}$$

$$=\frac{2}{E_{\mu,\xi}(m)}[\frac{\eta}{\mu^2}\sum_{l=1}^{\infty}\frac{m^l}{\Gamma(\mu l+\xi-2)}$$

$$+\left(\frac{\eta(3-2\xi)+\mu(\eta|\vartheta|+1)}{\mu^2}\right)\sum_{l=1}^{\infty}\frac{m^l}{\Gamma(\mu l+\xi-1)}$$

$$+\left(\frac{\eta}{\mu^2}(1-\xi)^2+\frac{(1-\xi)(\eta|\vartheta|+1)}{\mu}+|\vartheta|\right)\sum_{l=1}^{\infty}\frac{m^l}{\Gamma(\mu l+\xi)}]$$

$$=\frac{2}{E_{\mu,\xi}(m)}[\frac{\eta}{\mu^2}(E_{\mu,\xi-2}(m)-\frac{1}{\Gamma(\xi-2)})$$

$$+\left(\frac{\mu(\eta|\vartheta|+1)+\eta(3-2\xi)}{\mu^2}\right)(E_{\mu,\xi-1}(m)-\frac{1}{\Gamma(\xi-1)})$$

$$+\left(\frac{\eta}{\mu^2}(1-\xi)^2+\frac{(1-\xi)(\eta|\vartheta|+1)}{\mu}+|\vartheta|\right)(E_{\mu,\xi}(m)-\frac{1}{\Gamma\xi})]$$

$$\leq 2|\vartheta|.$$

Theorem 2.4 *If* $\mu, m > 0$ *and* $\xi > 2$, *then* $\psi_{\mu,\xi}^{m}(\zeta) \in R(\eta,\vartheta)$ *iff*

$$\frac{1}{E_{\mu,\xi}(m)}[\frac{\eta}{\mu^2}(E_{\mu,\xi-2}(m)-\frac{1}{\Gamma(\xi-2)})$$

$$+(\frac{\mu(\eta+1)+\eta(3-2\xi)}{\mu^2})(E_{\mu,\xi-1}(m)-\frac{1}{\Gamma\xi-1})$$

$$+(\frac{\eta}{\mu^2}(1-\xi)^2+\frac{(1+\eta)(1-\xi)}{\mu}+1)(E_{\mu,\xi}(m)-\frac{1}{\Gamma\xi})] \leq |\vartheta|.$$

Proof. From lemma (1.4), It is enough to show that,

$$\sum_{l=2}^{\infty} l[\eta(l-1)+1]\left|\frac{m^{l-1}}{\Gamma(\mu(l-1)+\xi)E_{\mu,\xi}(m)}\right| \leq |\vartheta|$$

Consider, $\sum_{l=2}^{\infty} l[\eta(l-1)+1]\left|\frac{m^{l-1}}{\Gamma(\mu(l-1)+\xi)E_{\mu,\xi}(m)}\right|$

$$=\sum_{l=1}^{\infty}(1+l)[\eta l+1]\left(\frac{m^{l}}{\Gamma(\mu l+\xi)E_{\mu,\xi}(m)}\right)$$

$$=\sum_{l=1}^{\infty}[\eta l^2+(\eta+1)l+1]\left(\frac{m^{l}}{\Gamma(\mu l+\xi)E_{\mu,\xi}(m)}\right)$$

$$=\frac{1}{E_{\mu,\xi}(m)}[\frac{\eta}{\mu^2}\sum_{l=1}^{\infty}\frac{m^l}{\Gamma(\mu l+\xi-2)}$$

$$+\left(\frac{\eta(3-2\xi)+\mu(\eta+1)}{\mu^2}\right)\sum_{l=1}^{\infty}\frac{m^l}{\Gamma(\mu l+\xi-1)}$$

$$+(\frac{\eta}{\mu^2}(1-\xi)^2+\frac{(1+\eta)(1-\xi)}{\mu}+1)\sum_{l=1}^{\infty}\frac{m^l}{\Gamma(\mu l+\xi)}]$$

$$=\frac{1}{E_{\mu,\xi}(m)}[\frac{\eta}{\mu^2}(E_{\mu,\xi-2}(m)-\frac{1}{\Gamma(\xi-2)})$$

$$+(\frac{\mu(\eta+1)+\eta(3-2\xi)}{\mu^2})(E_{\mu,\xi-1}(m)-\frac{1}{\Gamma\xi-1})$$

$$+(\frac{\eta}{\mu^2}(1-\xi)^2+\frac{(1+\eta)(1-\xi)}{\mu}+1)(E_{\mu,\xi}(m)-\frac{1}{\Gamma\xi})]$$

$$\leq|\vartheta|.$$

Remark 2.1 *If we take* $\mu=\xi=1$ *in theorems (2.1),(2.2),(2.3),(2.4),then we find the corresponding results of poisson distribution series.*

3. Special cases

Corollary 3.1 *Let* $\mu, m>0$ *and* $\xi>1$, *then* $\varphi_{\mu,\xi}^{m}(\zeta)\in S(\vartheta,0,1)$ *if and only if*

$$\frac{1}{E_{\mu,\xi}(m)}[\frac{1}{\mu}(E_{\mu,\xi-1}(m)-\frac{1}{\Gamma(\xi-1)})+\left(\frac{(1-\xi)}{\mu}+|\vartheta|\right)(E_{\mu,\xi}(m)-\frac{1}{\Gamma\xi})]\leq|\vartheta|.$$

Corollary 3.2 *If* $\mu, m>0$ *and* $\xi>1$, *then* $\varphi_{\mu,\xi}^{m}(\zeta)\in R(\vartheta,0,1)$ *if and only if*

$$\frac{1}{E_{\mu,\xi}(m)}[\frac{1}{\mu}(E_{\mu,\xi-1}(m)-\frac{1}{\Gamma(\xi-1)})+\left(\frac{(1-\xi)}{\mu}+1\right)(E_{\mu,\xi}(m)-\frac{1}{\Gamma(\xi)})]\leq|\vartheta|.$$

Corollary 3.3 *Let* $\mu, m>0$ *and* $\xi>2$, *then* $\varphi_{\mu,\xi}^{m}(\zeta)\in S(\vartheta,1,1)$ *if and only if*

$$\frac{1}{E_{\mu,\xi}(m)}[\frac{1}{\mu^2}(E_{\mu,\xi-2}(m)-\frac{1}{\Gamma(\xi-2)})$$

$$+\left(\frac{\mu(|\vartheta|+1)+(3-2\xi)}{\mu^2}\right)(E_{\mu,\xi-1}(m)-\frac{1}{\Gamma(\xi-1)})$$

$$+\left(\frac{1}{\mu^2}(1-\xi)^2+\frac{(|\vartheta|+1)(1-\xi)}{\mu}+|\vartheta|\right)(E_{\mu,\xi}(m)-\frac{1}{\Gamma(\xi)})]\leq|\vartheta|.$$

Corollary 3.4 *Let* $\mu, m > 0$ *and* $\xi > 2$, *then* $\varphi_{\mu,\xi}^{m}(\zeta) \in R(\vartheta, 1,1)$ *if and only if*

$$\frac{1}{E_{\mu,\xi}(m)}[\frac{1}{\mu^2}(E_{\mu,\xi-2}(m) - \frac{1}{\Gamma(\xi-2)})$$
$$+\left(\frac{3-2\xi+2\mu}{\mu^2}\right)(E_{\mu,\xi-1}(m) - \frac{1}{\Gamma(\xi-1)})$$
$$+\left(\frac{1}{\mu^2}(1-\xi)^2 + \frac{2(1-\xi)}{\mu} + 1\right)(E_{\mu,\xi}(m) - \frac{1}{\Gamma(\xi)})] \leq |\vartheta|.$$

Corollary 3.5 *If* $\mu, m > 0$ *and* $\xi > 1$, *then* $\psi_{\mu,\xi}^{m}(\zeta) \in P(0, \vartheta)$ *if and only if*

$$\frac{2}{E_{\mu,\xi}(m)}[\frac{1}{\mu}(E_{\mu,\xi-1}(m) - \frac{1}{\Gamma(\xi-1)}) + \left(\frac{(1-\xi)}{\mu} + |\vartheta|\right)(E_{\mu,\xi}(m) - \frac{1}{\Gamma(\xi)})] \leq 2|\vartheta|.$$

Corollary 3.6 *If* $\mu, m > 0$ *and* $\xi > 2$, *then* $\psi_{\mu,\xi}^{m}(\zeta) \in P(1, \vartheta)$ *if and only if*

$$\frac{2}{E_{\mu,\xi}(m)}[\frac{1}{\mu^2}(E_{\mu,\xi-2}(m) - \frac{1}{\Gamma(\xi-2)})$$
$$+\left(\frac{\mu(|\vartheta|+1)+(3-2\xi)}{\mu^2}\right)(E_{\mu,\xi-1}(m) - \frac{1}{\Gamma(\xi-1)})$$
$$+\left(\frac{1}{\mu^2}(1-\xi)^2 + \frac{(|\vartheta|+1)(1-\xi)}{\mu} + |\vartheta|\right)(E_{\mu,\xi}(m) - \frac{1}{\Gamma(\xi)})] \leq 2|\vartheta|.$$

Corollary 3.7 *If* $\mu, m > 0$ *and* $\xi > 1$, *then* $\psi_{\mu,\xi}^{m}(\zeta) \in R(0, \vartheta)$ *if and only if*

$$\frac{1}{E_{\mu,\xi}(m)}[\frac{1}{\mu}(E_{\mu,\xi-1}(m) - \frac{1}{\Gamma(\xi-1)}) + \left(\frac{(1-\xi)}{\mu} + 1\right)(E_{\mu,\xi}(m) - \frac{1}{\Gamma(\xi)})] \leq |\vartheta|.$$

Corollary 3.8 *If* $\mu, m > 0$ *and* $\xi > 2$, *then* $\psi_{\mu,\xi}^{m}(\zeta) \in R(1, \vartheta)$ *if and only if*

$$\frac{1}{E_{\mu,\xi}(m)}[\frac{1}{\mu^2}(E_{\mu,\xi-2}(m) - \frac{1}{\Gamma(\xi-2)})$$
$$+\left(\frac{2\mu+3-2\xi}{\mu^2}\right)(E_{\mu,\xi-1}(m) - \frac{1}{\Gamma(\xi-1)})$$
$$+\left(\frac{1}{\mu^2}(1-\xi)^2 + \frac{2(1-\xi)}{\mu} + 1\right)(E_{\mu,\xi}(m) - \frac{1}{\Gamma(\xi)})] \leq |\vartheta|.$$

4 Alexander integral operator

We derive the membership conditions for the Alexander integral operator $\kappa_{\mu,\xi}^{m}(\zeta)$ belongs to the classes $R(\vartheta, \eta, \delta)$and $R(\eta, \vartheta)$.

Theorem 4.1 *If* $\mu, m > 0$ *and* $\xi > 1$, *then the function*

$$\kappa_{\mu,\xi}^{m}(\zeta) = \int_0^{\zeta} \frac{\varphi_{\mu,\xi}^{m}(t)}{t} dt$$

is in the class $R(\vartheta, \eta, \delta)$ if and only if

$$\frac{1}{E_{\mu,\xi}(m)}[\frac{\eta}{\mu}\left(E_{\mu,\xi-1}(m) - \frac{1}{\Gamma(\xi-1)}\right) + \left(\frac{\eta(1-\xi)}{\mu} + 1\right)\left(E_{\mu,\xi}(m) - \frac{1}{\Gamma(\xi)}\right)] \leq \delta|\vartheta|$$

Proof. From lemma (1.2), it is enough to show that,

$$\sum_{l=2}^{\infty} l[\eta(-1+l)+1]\left(\frac{m^{l-1}}{\Gamma(\mu(-1+l)+\xi)lE_{\mu,\xi}(m)}\right) \leq \delta|\vartheta|$$

Consider,

$$\sum_{l=2}^{\infty} l[\eta(-1+l)+1]\left(\frac{m^{l-1}}{\Gamma(\mu(-1+l)+\xi)lE_{\mu,\xi}(m)}\right)$$

$$= \sum_{l=1}^{\infty} [\eta l+1]\left(\frac{m^{l}}{\Gamma(\mu l+\xi)\{E_{\mu,\xi}(m)}\right)$$

$$= \frac{1}{E_{\mu,\xi}(m)}[\frac{\eta}{\mu}\sum_{l=1}^{\infty}\frac{m^{l}}{\Gamma(\mu l+\xi-1)} + \left(\frac{\eta(1-\xi)}{\mu}+1\right)\sum_{l=1}^{\infty}\frac{m^{l}}{\Gamma(\mu l+\xi)}]$$

$$= \frac{1}{E_{\mu,\xi}(m)}[\frac{\eta}{\mu}\left(E_{\mu,\xi-1}(m)-\frac{1}{\Gamma(\xi-1)}\right) + \left(\frac{\eta(1-\xi)}{\mu}+1\right)\left(E_{\mu,\xi}(m)-\frac{1}{\Gamma(\xi)}\right)]$$

$$\leq \delta|\vartheta|.$$

Theorem 4.2 *If* $\mu, m > 0$ *and* $\xi > 1$*, then the function* $\kappa_{\mu,\xi}^{m}(\zeta) = \int_0^{\zeta}\frac{\psi_{\mu,\xi}^{m}(t)}{t}dt$ *is in the class* $R(\eta,\vartheta)$ *if and only if*

$$\frac{1}{E_{\mu,\xi}(m)}[\frac{\eta}{\mu}\left(E_{\mu,\xi-1}(m)-\frac{1}{\Gamma(\xi-1)}\right) + \left(\frac{\eta(1-\xi)}{\mu}+1\right)\left(E_{\mu,\xi}(m)-\frac{1}{\Gamma(\xi)}\right)] \leq |\vartheta|$$

Proof. From lemma (1.4), it is enough to show that,

$$\sum_{l=2}^{\infty} l[\eta(-1+l)+1]\left|\frac{m^{l-1}}{\Gamma(\mu(-1+l)+\xi)lE_{\mu,\xi}(m)}\right| \leq |\vartheta|$$

Consider,

$$\sum_{l=2}^{\infty} l[\eta(l-1)+1]\left(\frac{m^{l-1}}{\Gamma(\mu(l-1)+\xi)lE_{\mu,\xi}(m)}\right)$$

$$= \sum_{l=1}^{\infty} [\eta l+1]\left(\frac{m^{l}}{\Gamma(\mu l+\xi)E_{\mu,\xi}(m)}\right)$$

$$= \frac{1}{E_{\mu,\xi}(m)}[\frac{\eta}{\mu}\sum_{l=1}^{\infty}\frac{m^{l}}{\Gamma(\mu l+\xi-1)} + \left(\frac{\eta(1-\xi)}{\mu}+1\right)\sum_{l=1}^{\infty}\frac{m^{l}}{\Gamma(\mu l+\xi)}]$$

$$= \frac{1}{E_{\mu,\xi}(m)}[\frac{\eta}{\mu}\left(E_{\mu,\xi-1}(m)-\frac{1}{\Gamma(\xi-1)}\right) + \left(\frac{\eta(1-\xi)}{\mu}+1\right)\left(E_{\mu,\xi}(m)-\frac{1}{\Gamma(\xi)}\right)]$$

$$\leq |\vartheta|.$$

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