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Some Fixed Point Results on Generalized $\tau-\psi$-Contraction Mappings in Partial Metric Spaces with Application

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Abstract: In this work, we extend generalized $\tau-\psi$ −contraction mappings in the setting of partial metric spaces. These results substantially generalize the results of Baiz et al. [7, 8] and Kumam et al. [5]. We give some consequences of the established result and provide an example in support of our result. An application of main result to the existence of solution of system of integral equations is also presented.

Keywords: $\tau-\psi$-contraction mapping, generalized $\tau-\psi$-contraction mapping, partial metric spaces.

1. Introduction

Fixed point theory is one of the most important topic in the development of nonlinear analysis. The Banach contraction principle [1] is a well-known and one of the most useful theorems in nonlinear analysis. The Banach contraction principle has been widely generalized and extended. Samet et al. [4] extended and generalized the Banach contraction principle by introducing a new class of contractive type mappings known as $\alpha-\psi$ contractive type mappings. Further, Kumam et al. [5] introduced the notion of weak $\alpha-\psi$-contractive mappings and established fixed point results for this class of mappings. On the other hand, Baiz et al. [7, 8] introduced a new generalization of contraction mappings as $\tau-\psi$-contraction and generalized $\tau-\psi$-contraction mappings and established results in rectangular quasi b-metric spaces and rectangular M-metric spaces. Matthews [2,3] introduced a very interesting generalization of the metric space known as partial metric space in which the self distance not required to be zero and proved the partial metric version of Banach fixed point theorem. Many researchers worked on this interesting space. For more, the reader can refer to [6, 9, 14–17] and the references therein. Very recently, Baiz et al. [7] introduced the concepts of $\tau-\psi$-contraction mappings as follows:

Definition 1.1. Let (ϕ_p,∂) be a rectangular quasi b-metric space and $\Gamma:\Phi_p\to\Phi_p$ be a self mapping. Γ is said to be a generalized τ-ψ-contractive mapping if there exists $\psi\in\Psi$ and $\tau>1$ such that $\tau\partial\left(\Gamma\xi_p,\Gamma\eta_p\right)\leq\psi(\Sigma(\xi_p,\eta_p))$
For all, where $\xi_p,\eta_p\in\ \Phi_p$

$$\Sigma(\xi_p,\eta_p)=\max\left\{\partial(\xi_p,\eta_p),\frac{\partial(\xi_p,\Gamma\xi_p)\partial(\xi_p,\Gamma\eta_p)}{\left(1+\partial(\xi_p,\Gamma\eta_p)+\partial(\eta_p,\Gamma\xi_p)\right)},\partial(\xi_p,\Gamma\xi_p),\partial(\eta_p,\Gamma\eta_p)\right\}\quad(2)$$

And $\Psi=\{\psi:\mathbb{R}^+\to\mathbb{R}^+,\ \psi$ is nondecreasing, continuous $\sum_{k=1}^{\infty}s^k\psi^k(t)<\infty,\ s\psi(t)<t$ and $\psi(0)=0$ if and only if if t=0, where ψ^k is the k^{th} iterate of $\psi,s\geq1\}$.

Now, we give some basic properties and results on the concept of partial metric space.

Definition 1.2. [2] Let Φ_p be a non empty set. A function $\partial_p : \Phi_p \times \Phi_p \to [0,\infty)$ is said to be partial metric on Φ_p if the following conditions hold:

1. $\xi_p = \eta_p \Leftrightarrow \partial_p(\xi_p, \xi_p) = \partial_p(\eta_p, \eta_p) = \partial_p(\xi_p, \eta_p)$;
2. $\partial_p(\xi_p, \xi_p) \leq \partial_p(\xi_p, \eta_p)$;
3. $\partial_p(\xi_p, \eta_p) = \partial_p(\eta_p, \xi_p)$;
4. $\partial_p(\xi_p, \eta_p) \leq \partial_p(\xi_p, \zeta_p) + \partial_p(\zeta_p, \eta_p) - \partial_p(\zeta_p, \zeta_p)$. For all $\xi_p, \eta_p, \zeta_p \in \Phi_p$.

The set Φ_p equipped with the metric ∂_p defined above is called a partial metric space and it is denoted by (Φ_p, ∂_p) (in short PMS).

Example 1.3. [12] Let $\Phi_p = \{[a,b] : a, b \in \mathbb{R}, a \leq b\}$ and define $\partial_p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$. Then (Φ_p, ∂_p) is a partial metric space.

Example 1.4 [12] Let $\Phi_p = [0,\infty)$ and define $\partial_p(\xi_p, \eta_p) = \max\{\xi_p, \eta_p\}$. Then (Φ_p, ∂_p) is partial metric space.

Lemma 1.5. [2,5] Let (Φ_p, ∂_p)be a partial metric space.

(a) A sequence $\{\xi_{p_i}\}$ in (Φ_p, ∂_p) converges to a point $\xi_p \in \Phi_p$ if

$$\partial_p(\xi_p, \xi_p) = \lim_{i\to\infty} \partial_p(\xi_{p_i}, \xi_p),$$

(b) A sequence $\{\xi_{p_i}\}$ in (Φ_p, ∂_p) is a Cauchy sequence if $\lim_{j,i\to\infty} \partial_p(\xi_{p_i}, \xi_{p_j})$ exists and finite,

(c) (Φ_p, ∂_p) is complete if every Cauchy $\{\xi_{p_i}\}$ in Φ_P converges to a point $\xi_p \in \Phi_p$ such that

$$\partial_p(\xi_p, \xi_p) = \lim_{j,i\to\infty} \partial_p(\xi_{p_i}, \xi_{p_j}) = \lim_{i\to\infty} \partial_p(\xi_{p_i}, \xi_p) = \partial_p(\xi_p, \xi_p).$$

Lemma 1.6. [2,3,11] Let ∂_p be a partial metric on Φ_p, then the functions $d_{p_k}, d_{p_m} : \Phi_p \times \Phi_p \to \mathbb{R}^+$ such that

$$d_{p_k}(\xi_p, \eta_p) = 2\partial_p(\xi_p, \eta_p) - \partial_p(\xi_p, \xi_p) - \partial_p(\eta_p, \eta_p)$$

And

$$d_{p_m}(\xi_p, \eta_p) = \max\{\partial_p(\xi_p, \eta_p) - \partial_p(\xi_p, \xi_p), \partial_p(\xi_p, \eta_p) - \partial_p(\eta_p, \eta_p)\}$$
$$= 2\partial_p(\xi_p, \eta_p) - \min\{\partial_p(\xi_p, \xi_p), \partial_p(\eta_p, \eta_p)\}$$

Are metric on Φ_p. Further (Φ_p, d_{p_k}) and (Φ_p, d_{p_m}) are metric spaces. It is clear that d_{p_k} and d_{p_m} are equivalent.

Let (Φ_p, ∂_p) be a partial metric space. Then

1. A sequence $\{\xi_{p_i}\}$ in (Φ_p, ∂_p) is a Cauchy sequence $\Leftrightarrow$ $\{\xi_{p_i}\}$ is a Cauchy sequence in (Φ_p, d_{p_k}),
2. (Φ_p, ∂_p) is complete $\Leftrightarrow$ (Φ_p, d_{p_k}) is complete. Moreover

$$\lim_{i\to\infty} d_{p_k}(\xi_{p_i}, \xi_p) = 0 \Leftrightarrow \partial_p(\xi_p, \xi_p) = \lim_{i\to\infty} \partial_p(\xi_{p_i}, \xi_p) = \lim_{i,j\to\infty} \partial_p\left(\xi_{p_i}, \xi_{p_j}\right)$$

Lemma 1.7. [10] Assume that $\xi_{p_i} \to \zeta$ as $i \to \infty$ in a partial metric (Φ_p, ∂_p) such that $\partial_p(\zeta, \zeta) = 0$. Then $\lim_{i\to\infty} \partial_p(\xi_{p_i}, \sigma_p) = \partial_p(\zeta, \sigma_p)$ for every $\sigma_p \in \Phi_p$.

Lemma 1.8. [13] Let (Φ_p, ∂_p) be a partial metric space.

1. If $\partial_p(\xi_p, \eta_p) = 0$ then $\xi_p = \eta_p$,
2. If $\xi_p \neq \eta_p$ then $\partial_p(\xi_p, \eta_p) > 0$.

Definition1.9. [5] Let Ψ be the family of functions $\psi : [0,\infty) \to [0,\infty)$ such that

(i) ψ is nondecreasing;

(ii) $\psi(t) > 0$ flor each $t > 0$;

(iii) $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$, where ψ^n is the n^{th} iterate of ψ.

Lemma 1.10 [5] For every function $\psi \in \Psi$ one has $\psi(t) < t$ for any $t > 0$.

2. Main Results

Definition 2.1. Let (Φ_p, ∂_p) be a partial metric space and $\Gamma: \Phi_p \to \Phi_p$ be a given self map. We say that Γ is generalized $\tau - \psi -$contraction mapping on Φ_p, if there exists $\psi \in \Psi$ and $\tau > 1$ such that for all $\xi_p, \eta_p \in \Phi_p$ we have

$$\tau\, \partial_p(\Gamma\xi_p, \Gamma\eta_p) \le \psi\left(\max\left\{\partial_p\,(\xi_p, \eta_p), \frac{\partial_p(\xi_p, \Gamma\xi_p) + \partial_p(\eta_p,\ \Gamma\eta_p)}{2}\right\}\right) \qquad (1)$$

Theorem 2.1: Let (Φ_p, ∂_p) be a complete partial metric space and $\Gamma: \Phi_p \to \Phi_p$ be a given self map. Suppose that

(i) Γ is generalized $\tau - \psi -$contraction mapping;

(ii) There exists $\xi_{p_i} \in \Phi_p$ such that $\xi_{p_i} = \Gamma\xi_{p_{i-1}} = \Gamma^{i+1}\xi_{p_0}$ for all $i \in \mathbb{N}$;

(iii) Γ is continuous.

Then Γ has a unique fixed point in Φ_p.

Proof. From (ii) we have a sequence $\{\xi_{p_i}\}$ in Φ_p such that $\xi_{p_{i+1}}{=}\Gamma\xi_{p_i}$ for all $i \in \mathbb{N}$.

If $\xi_{p_{i+1}}{=}\xi_{p_i}$ for some $i \in \mathbb{N}$, then ξ_{p_i} is a fixed point of Γ and the existence part of the proof is finished. Suppose that $\xi_{p_{i+1}} \neq \xi_{p_i}$ for every $i \in \mathbb{N}$.

Now, from (i) and (ii) we get

$$\partial_p(\xi_{p_i}, \xi_{p_{i+1}}) = \partial_p(\Gamma\xi_{p_{i-1}}, \Gamma\xi_{p_i}) \le \tau\, \partial_p(\Gamma\xi_{p_{i-1}}, \xi_{p_i})$$

$$\le \psi\left(\max\left\{\partial_p\,(\xi_{p_{i-1}}, \xi_{p_i}), \frac{\partial_p(\xi_{p_{i-1}}, \Gamma\xi_{p_{i-1}}) + \partial_p(\xi_{p_i},\ \Gamma\xi_{p_i})}{2}\right\}\right)$$

$$= \psi\left(\max\left\{\partial_p\,(\xi_{p_{i-1}}, \xi_{p_i}), \frac{\partial_p(\xi_{p_{i-1}}, \xi_{p_i}) + \partial_p(\xi_{p_i},\ \xi_{p_{i+1}})}{2}\right\}\right)$$

$$\le \psi(\max\{\partial_p\,(\xi_{p_i}, \xi_{p_{i+1}}), \partial_p\,(\xi_{p_{i-1}}, \xi_{p_i}),\}) \qquad (2)$$

Now, if

$$\partial_p\,(\xi_{p_i}, \xi_{p_{i+1}}) > \partial_p\,(\xi_{p_{i-1}}, \xi_{p_i})$$

Then

$$\partial_p\,(\xi_{p_i}, \xi_{p_{i+1}}) \le \psi(\,\partial_p\,(\xi_{p_i}, \xi_{p_{i+1}})) < \partial_p(\xi_{p_i}, \xi_{p_{i+1}})$$

Which is a contradiction since $\partial_p(\xi_{p_i}, \xi_{p_{i+1}}) > 0$ by Lemma 1.8, therefore

$$\partial_p\,(\xi_{p_i}, \xi_{p_{i+1}}) \le \psi(\,\partial_p\,(\xi_{p_i}, \xi_{p_{i-1}}))$$

for all i. Continuing this process and using induction, we obtain

$$\partial_p\left(\xi_{p_i},\xi_{p_{i+1}}\right) \leq \psi^i(\partial_p\left(\xi_{p_0},\xi_{p_1}\right)) \qquad (3)$$

Using definition of ψ and letting $i \to \infty$ in (3) we get that

$$\lim_{i\to\infty} \partial_p\left(\xi_{p_i},\xi_{p_{i+1}}\right) = 0$$

Fix $\epsilon > 0$ and let $i(\epsilon) \in \mathbb{N}$ such that

$$\partial_p\left(\xi_{p_j},\xi_{p_{j+1}}\right) < \epsilon - \psi(\epsilon),$$

For all $j > i(\epsilon)$. We show that

$$\partial_p\left(\xi_{p_j},\xi_{p_{j+1}}\right) < \epsilon \qquad (4)$$

For all $i \geq j$.

Note that (4) holds for $i = j$. Assume (4) holds for some $i > j$, then

$$\partial_p\left(\xi_{p_j},\xi_{p_{i+2}}\right) \leq \partial_p\left(\xi_{p_j},\xi_{p_{j+1}}\right) + \partial_p\left(\xi_{p_{j+1}},\xi_{p_{i+2}}\right) - \partial_p\left(\xi_{p_{j+1}},\xi_{p_{j+1}}\right)$$

$$\leq \partial_p\left(\xi_{p_j},\xi_{p_{j+1}}\right) + \partial_p\left(\Gamma\xi_{p_j},\Gamma\xi_{p_{i+1}}\right)$$

$$\leq \partial_p\left(\xi_{p_j},\xi_{p_{j+1}}\right) + \tau\,\partial_p\left(\Gamma\xi_{p_j},\Gamma\xi_{p_{i+1}}\right)$$

$$\leq \partial_p\left(\xi_{p_j},\xi_{p_{j+1}}\right) + \psi(\max\left\{\partial_p\left(\xi_{p_j},\xi_{p_{i+1}}\right), \frac{\partial_p\left(\xi_{p_j},\Gamma\xi_{p_j}\right) + \partial_p\left(\xi_{p_{i+1}},\Gamma\xi_{p_{i+1}}\right)}{2}\right\}$$

$$\leq \partial_p\left(\xi_{p_j},\xi_{p_{j+1}}\right) + \psi(\max\left\{\partial_p\left(\xi_{p_j},\xi_{p_{i+1}}\right), \frac{\partial_p\left(\xi_{p_j},\xi_{p_{j+1}}\right) + \partial_p\left(\xi_{p_{i+1}},\xi_{p_{i+2}}\right)}{2}\right\}$$

$$\leq \epsilon - \psi(\epsilon) + \psi(\epsilon) \qquad (5)$$

This implies that (4) holds for $i \geq j$ and hence we get

$$\lim_{j,i\to\infty} \partial_p\left(\xi_{p_i},\xi_{p_j}\right) = 0 \qquad (6)$$

This implies that $\{\xi_{p_i}\}$ is a Cauchy sequence in the metric space (Φ_p,∂_p) and hence in (Φ_p, d_{p_k}) which is complete. Therefore the sequence $\{\xi_{p_i}\}$ is convergent in the space (Φ_p, d_{p_k}). This implies that there exists $\xi_p^* \in \Phi_p$ such that $\lim_{i\to\infty} d_{p_k}(\xi_{p_i},\xi_p^*) = 0$. Again from Lemma 1.6 and (6), we get

$$\partial_p(\xi_p^*,\xi_p^*) = \lim_{i\to\infty} \partial_p(\xi_{p_i},\xi_p^*) = \lim_{j,i\to\infty} \partial_p\left(\xi_{p_i},\xi_{p_j}\right) = 0$$

As Γ is continuous, we have

$$\xi_p^* = \lim_{i\to\infty} \xi_{p_{i+1}} = \lim_{i\to\infty} \Gamma\xi_i = \Gamma\xi_p^*$$

Now, we show that the uniqueness of a fixed point of Γ. Assume that Γ has two distinct fixed point ξ_p^* and η_p^* such that $\Gamma\xi_p^* = \xi_p^*$ and $\Gamma\eta_p^* = \eta_p^*$ then replacing ξ_p by ξ_p^* and η_p by η_p^* in (1) we get

$$\partial_p(\xi_p^*, \eta_p^*) = \partial_p(\Gamma\xi_p^*, \Gamma\eta_p^*) \le \tau\, \partial_p(\Gamma\xi_p^*, \Gamma\eta_p^*)$$

$$\le \psi(\max\left\{\partial_p(\xi_p^*, \eta_p^*), \frac{\partial_p\left(\xi_p^*,\ \Gamma\xi_p^*\right) + \partial_p\left(\eta_p^*,\ \Gamma\eta_p^*\right)}{2}\right\})$$

$$\le \psi(\max\left\{\partial_p(\xi_p^*, \eta_p^*), \frac{\partial_p\left(\xi_p^*,\ \xi_p^*\right) + \partial_p\left(\eta_p^*,\ \eta_p^*\right)}{2}\right\})$$

$$\le \psi\left(\partial_p(\xi_p^*, \eta_p^*)\right) < \partial_p(\xi_p^*, \eta_p^*) \qquad (7)$$

which is a contradiction. Hence Γ has a unique fixed point. This completes the proof.

Now, we state the following fixed point theorem by removing the continuity assumption of Γ from Theorem 2.1.

Theorem 2.2. (Φ_p, ∂_p) be complete partial metric space and $\Gamma: \Phi_p \to \Phi_p$ be self map. Suppose that

(i) Γ is generalized $\tau - \psi$-contraction mapping;

(ii) There exists $\xi_{p_i} \in \Phi_p$ such that $\xi_{p_i} = \Gamma\xi_{p_{i-1}} = \Gamma^{i+1}\xi_{p_0}$ for all $i \in \mathbb{N}$;

(iii) $\{\xi_{p_i}\}$ is a sequence in Φ_p such that $\xi_{p_i} \to \xi_p$ as $i \to \infty$.

Then Γ has a unique fixed point in Φ_p.

Proof. Following the proof of Theorem 2.1 we know that the sequence $\{\xi_{p_i}\}$ given by $\xi_{p_i} = \xi_{p_{i+1}}$ is a Cauchy sequence in the complete partial metric space (Φ_p, ∂_p).

Consequently, there exists $\xi_p \in \Phi_p$ such that

$$\partial_p(\xi_p, \xi_p) = \lim_{i\to\infty} \partial_p(\xi_{p_i}, \xi_p) = \lim_{j,i\to\infty} \partial_p\left(\xi_{p_i}, \xi_{p_j}\right) = 0$$

Therefore, it is sufficient to show that Γ admits a fixed point.

Now, using the triangular inequality and (1) we get

$$\partial_p(\Gamma\xi_p, \xi_p) \le \partial_p(\Gamma\xi_p, \xi_{p_{i+1}}) + \partial_p(\xi_{p_{i+1}}, \xi_p) - \partial_p(\xi_{p_{i+1}}, \xi_{p_{i+1}})$$

$$\le \partial_p(\Gamma\xi_p, \Gamma\xi_{p_i}) + \partial_p(\xi_{p_{i+1}}, \xi_p)$$

$$\le \tau\, \partial_p(\Gamma\xi_p, \Gamma\xi_{p_i}) + \partial_p(\xi_{p_{i+1}}, \xi_p)$$

$$\le \psi\left(\max\left\{\partial_p(\xi_p, \xi_{p_i}), \frac{\partial_p\left(\xi_p,\ \Gamma\xi_p\right) + \partial_p\left(\xi_{p_i},\ \Gamma\xi_{p_i}\right)}{2}\right\}\right) + \partial_p(\xi_{p_{i+1}}, \xi_p) \qquad (8)$$

Taking $i \to \infty$ in (8) we get

$$\partial_p(\Gamma\xi_p, \xi_p) \le \psi\left(\frac{\partial_p\left(\Gamma\xi_p,\ \xi_p\right)}{2}\right) < \frac{\partial_p\left(\Gamma\xi_p,\ \xi_p\right)}{2}$$

This is a contradiction, and so we obtain $\Gamma\xi_p = \xi_p$.

Corollary 2.3. (Φ_p, ∂_p) be complete partial metric space and $\Gamma: \Phi_p \to \Phi_p$ be self map satisfying the condition

$$\partial_p(\Gamma\xi_p, \Gamma\eta_p) \le \psi\left(\partial_p(\xi_p, \eta_p)\right) \qquad (9)$$

For all $\xi_p, \eta_p \in \Phi_p, \psi \in \Psi$. Then Γ has a unique fixe point in Φ_p.

Corollary 2.4. (Φ_p, ∂_p) be complete partial metric space and $\Gamma: \Phi_p \to \Phi_p$ be self map satisfying the condition

$$\partial_p(\Gamma\xi_p, \Gamma\eta_p) \le \psi\left(M(\xi_p, \eta_p)\right) \qquad (10)$$

For all $\xi_p, \eta_p \in \Phi_p, \psi \in \Psi$. Where

$$M(\xi_p, \eta_p) = \max\left\{\partial_p(\xi_p, \eta_p), \frac{\partial_p\left(\xi_p, \Gamma\xi_p\right) + \partial_p\left(\eta_p, \Gamma\eta_p\right)}{2}\right\}$$

Then Γ has a unique fixed point in Φ_p.

Example 2.2. Let $\Phi_p = [0,1]$ and $\partial_p(\xi_p, \eta_p) = \max\{\xi_p, \eta_p\}$. Then (Φ_p, ∂_p) is a complete partial metric space. Consider the mapping $\Gamma: \Phi_p \to \Phi_p$ defined by $\Gamma(\xi_p) = \frac{\xi_p}{3}$ for all ξ_p and $\psi: [0, \infty) \to [0, \infty)$ be such that $\psi(t) = \frac{t}{1+t}$ and $\tau = \frac{3}{2}$ without loss of generality we assume that $\xi_p \ge \eta_p$.

Now,

$$\tau\,\partial_p(\Gamma\xi_p, \Gamma\eta_p) = \frac{3}{2}\partial_p\left(\frac{\xi_p}{3}, \frac{\eta_p}{3}\right) = \frac{\xi_p}{2} \qquad (11)$$

On the other side

$$\psi(\max\left\{\partial_p(\xi_p, \eta_p), \frac{\partial_p\left(\xi_p, \Gamma\xi_p\right) + \partial_p\left(\eta_p, \Gamma\eta_p\right)}{2}\right\} = \max\left\{\partial_p(\xi_p, \eta_p), \frac{\partial_p\left(\xi_p, \frac{\xi_p}{3}\right) + \partial_p\left(\eta_p, \frac{\eta_p}{3}\right)}{2}\right\}$$

$$= \psi(\xi_p) = \frac{\xi_p}{1+\xi_p} \qquad (12)$$

Therefore

$$\tau\,\partial_p(\Gamma\xi_p, \Gamma\eta_p) \le \psi\left(\max\left\{\partial_p(\xi_p, \eta_p), \frac{\partial_p\left(\xi_p, \Gamma\xi_p\right) + \partial_p\left(\eta_p, \Gamma\eta_p\right)}{2}\right\}\right) \qquad (13)$$

It is clear that it satisfies all the conditions of Theorem 2.1. Hence Γ has a fixed point, which in this case is 0.

3. Application

This section is influenced by the findings discussed in paper [6] and the aim is to provide an application of Theorem 2.1 as a study of the existence of a unique solution of the following integral equations:

$$\xi_p(t) = f(t) + \lambda \int_0^1 G(t,s)F_n\left(s, \xi_p(s)\right) ds \qquad (14)$$

$$\eta_p(t) = f(t) + \lambda \int_0^1 G(t,s)G_n\left(s, \eta_p(s)\right) ds \qquad (15)$$

For all $t \in [0,1]$ and λ is a real number.

Let $\Phi_p = C([0,1], \mathbb{R})$ be a set of all real valued continuous function on $[0,1]$. Let Φ_p be endowed with partial metric $\partial_p: \Phi_p \times \Phi_p \to [0, \infty)$ defined by

$$\partial_p(\xi_p, \eta_p) = d(\xi_p, \eta_p) + c_n = \sup_{t\in[0,1]}\left|\xi_p(t) - \eta_p(t)\right| + c_n$$

For all $\xi_p, \eta_p \in \Phi_p$ and $\{c_n\}$ is a sequence of positive real numbers such that $\lim_{n\to\infty} c_n = 0$. Now we prove the following Theorem to ensure the existence of solution of system of integral equations.

Theorem 3.1. Assume the following conditions are satisfied:

(i) $F_n, G_n : [0,1] \times \mathbb{R} \to \mathbb{R}$, $G: [0,1] \times [0,1] \to \mathbb{R}^+$ and $f: [0,1] \to \mathbb{R}$ are continuous.

(ii) Define

$$\Gamma\xi_p(t) = f(t) + \lambda \int_0^1 G(t,s) F_n\left(s, \xi_p(s)\right) ds \qquad (16)$$

$$\Gamma\eta_p(t) = f(t) + \lambda \int_0^1 G(t,s) G_n\left(s, \eta_p(s)\right) ds \qquad (17)$$

And when $n \to \infty$

$$\left|F_n\left(t, \xi_p(t)\right) - G_n\left(t, \eta_p(t)\right)\right| \le \psi\left(\left|\xi_p(t) - \eta_p(t)\right|\right)$$

For all $t \in [0,1]$ and $\psi \in \Psi$,

(iii)

$$\sup_{t\in[0,1]} \int_0^1 |G(t,s)| ds \le R < +\infty$$

(iv) $\lambda R \le 1$.

Then the system of integral equations given in (14) and (15) has a solution.

Proof. Following the assumptions of Theorem 3.1, we have

$$\partial_p(\Gamma\xi_p, \Gamma\eta_p) \le d(\Gamma\xi_p, \Gamma\eta_p) + c_n$$

$$= \sup_{t\in[0,1]} \left|\Gamma\xi_p(t) - \Gamma\eta_p(t)\right| + c_n$$

$$= \sup_{t\in[0,1]} \int_0^1 |\,\lambda G(t,s)[F_n\left(s, \xi_p(s)\right) - G_n(s, \eta_p(s))]| ds + c_n$$

Letting $n \to \infty$ we get

$$\partial_p(\Gamma\xi_p, \Gamma\eta_p) \le |\lambda| \sup_{t\in[0,1]} \int_0^1 \psi\left(\left|\xi_p(t) - \eta_p(t)\right|\right) |G(t,s)| ds$$

$$\le |\lambda| \psi(d(\xi_p, \eta_p)) \sup_{t\in[0,1]} \int_0^1 |G(t,s)| ds$$

$$\le |\lambda| R \psi(d(\xi_p, \eta_p))$$

$$\le \psi\left(d(\xi_p, \eta_p)\right) \le \psi\left(\partial_p(\xi_p, \eta_p)\right)$$

Thus

$$\partial_p(\Gamma\xi_p, \Gamma\eta_p) \le \psi(\partial_p(\Gamma\xi_p, \Gamma\eta_p)) \le \psi\left(\max\left\{\partial_p(\xi_p, \eta_p), \frac{\partial_p\left(\xi_p, \Gamma\xi_p\right) + \partial_p\left(\eta_p, \Gamma\eta_p\right)}{2}\right\}\right)$$

Hence

$$\tau(\partial_p(\Gamma\xi_p, \Gamma\eta_p)) \le \psi\left(\max\left\{\partial_p(\xi_p, \eta_p), \frac{\partial_p\left(\xi_p, \Gamma\xi_p\right) + \partial_p\left(\eta_p, \Gamma\eta_p\right)}{2}\right\}\right)$$

Clearly, all the conditions of Theorem 2.1 are satisfied and so Γ has a unique fixed point. Thus the system of integral equations (14) and (15) has a unique solution.

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