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Euclidean Spaces with Hamming Mininorm

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Abstract. A mininorm on a vector space X over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is a function w from X to $\mathbb{R}$ satisfying the properties of a norm $\| \|$ with the property $\|\alpha x\| = |\alpha| \|x\|$, $\alpha \in \mathbb{K}, x \in X$ replaced by the following property: $\|\alpha x\| = \|x\|$ for all $x \in X$, $\alpha \neq 0$. There are several mininorms on the Euclidean spaces $\mathbb{R}^k$. One such mini- norm is the Hamming weight function. In this paper, we discuss certain basic properties of Euclidean spaces with the Hamming mininorm and also some structural properties of these spaces. Mathematic Subject Calssification 2010: 46B99, 46A19

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1. Introduction

In coding theory, the Hamming weight, of a code word x, denoted by $w_H(x)$, isdefined to be the number of non-zero coordinates of x.

$w_H(x)$ satisfies the following properties:

(i) $w_H(x) \geq 0$ for all code words x and $w_H(x) = 0$ iff $x = 0$

(ii) $w_H(x + y) \leq w_H(x) + w_H(y)$ for all code words x and y

Also, w_H satisfies an additional property $w_H(\alpha x) = w_H(x)$ for all $x \in \mathbb{R}^n$

and $\alpha \in \mathbb{R}$, $\alpha \neq 0$. (1)

Thus, w_H satisfies the conditions of a norm([2], [3], [6]):

But the condition $w_H(\alpha x) = |\alpha|\ w_H(x)$.

Instead, it satisfies

$w_H(\alpha x) = w_H(x)$ for all, $\alpha \neq 0$.

We may call such a function a mininorm.

2. Mini Norms on Vector Spaces

Definition :2.1

A mininorm or weight function on a vector space X over $\mathbb{K} = \mathbb{R}$ or

$\mathbb{C}$ is a function $w : X \to \mathbb{R}$ satisfying the following:

$w(x) \geq 0$ for all $x \in X$ and (2)

$w(x) = 0$ if and only if $x = 0$.

$w(\alpha x) = w(x)$ for all $x \in X$, $\alpha \neq 0$. (3)

$w(x + y) \leq w(x) + w(y)$ for all $x, y \in X$. (4)

Here X with the mininorm w is called a mininormed space and is denoted by (X, w). If the mininorm w is clear from the context, it can be written as X instead of (X, w).

We call $w(x)$ the mininorm or weight of x.

Example: 2.2 Let X = $\mathbb{R}^k$ over $\mathbb{R}$. Then X is a mininormed space with the mininorm $w = w_H$

Note: The same definition works for X = $\mathbb{C}^k$ over $\mathbb{C}$ as well.

We call this mininorm on X = $\mathbb{R}^k$ or $\mathbb{C}^k$, the **Hamming mininorm, or the standard mininorm** and denote it by w_H.

Thus, $w_H(x)$ = number of non-zero co-ordinates of x.

Remark 2.3

If we define $\rho_w(x,y) = w(x - y)$ for $x, y \in X$, then ρ_w defines a metric on X. Thus, every mininormed space is a metric space. ρ_w is called the metric induced by w.

This ρ_w satisfies the conditions

(i) $\rho_w(x + z, y + z) = \rho_w(x, y)$ and

(ii) $\rho_w(\alpha x, \alpha y) = \rho_w(x, y)$, where $\alpha \neq 0$.

(i) is obvious:

For (ii),consider $\rho_w(\alpha x, \alpha y) = w(\alpha x - \alpha y)$

$= w(\alpha(x - y))$

$= w(x - y)$

$= \rho_w(x, y)$. (5)

Note: We shall denote the metric induced by w_H on $\mathbb{R}^k$ or $\mathbb{C}^k$ by ρ_H . Thus,

$\rho_H(x, y)$ = number of non-zero coordinates of $x - y$.

Definition 2.4 A mininormed space (X, w) which is complete with respect to the metric induced by w is called a mini Banach space.

For example, $(\mathbb{R}^n, w_H)$ is a mini Banach space.

Note: A proof for this fact is given in the latter part.

Remark 2.5 Let $x, y \in X$, where (X, w) is a mininormed space. Then,

$w(x) = w(x - y + y) \leq w(x - y) + w(y)$

So, $w(x) - w(y) \leq w(x - y)$. (6)

Interchanging x and y, we get,

$w(y) - w(x) \leq w(y - x) = w(x - y)$.

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That is, $-(w(x) - w(y)) \leq w(x - y)$. (7)

From (6) and (7), we get,

$|w(x) - w(y)| \leq w(x - y)$. (8)

3. Basic Properties of Mininorms

Proposition:3.1

Every mininorm is continuous.

Proof

Let w be mininorm on X over $\mathbb{K}$.

Suppose (x_n) is a sequence in X such that $x_n \to x$. That is, $w(x_n - x) \to 0$

Now, $|w(x_n) - w(x)| \leq w(x_n - x) \to 0$ from (8) and (9)

So, $w(x_n) \to w(x)$ and w is continuous.

Remark:3.2

Let (X, w) be a mininormed space. Then, $w(\alpha x) \leq w(x)$ for all $x \in X, \alpha \in K$.

Proof:

If $\alpha \neq 0$, then, $w(\alpha x) = w(x)$.

If $\alpha = 0$, then, $w(\alpha x) = w(0) = 0 \leq w(x)$.

Theorem:3.3

Let X be a finite – dimensional vector space over $\mathbb{K}$, Then, every mininorm on X is a bounded function.

Proof:

Let w be a mininorm on X.

Suppose $dim(X) = n$. Let $\{x_1, x_2, \ldots, x_n\}$ be a basis for X.

Take $x \in X$. Then, there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{K}$ such that $x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$

Now $w(x) \leq w(\alpha_1 x_1) + w(\alpha_2 x_2) + \cdots + w(\alpha_n x_n)$

$\leq w(x_1) + w(x_2) + \cdots + w(x_n)$, using Remark 3.2

Thus, $w(x) \leq \alpha$ for all $x \in X$, where $\alpha = w(x_1) + w(x_2) + \cdots + w(x_n)$.

Proposition:3.4

Let (X, w) be a mininormed space over $\mathbb{K}$. Suppose (x_n) and (y_n) are sequences in X such that $x_n \to x$ and $y_n \to y$ in X. Then,

(i) $(x_n) + (y_n) \to x + y$ and

(ii) $\alpha x_n \to \alpha x, \alpha \in \mathbb{K}$.

Proof:

$$d(x_n + y_n, x + y) = w\big((x_n + y_n) - (x + y)\big)$$
$$\leq w(x_n - x) + w(y_n - y).$$

This proves (i)

Proof of (ii) is also easy.

If $\alpha = 0$, then $\alpha x_n = 0$ for all n and $\alpha x = 0$.

If $\alpha \neq 0$, then $d(\alpha x_n, \alpha x) = w(\alpha x_n - \alpha x) = w\big(\alpha(x_n - x)\big)$

$$= w(x_n - x) \to 0$$

Proposition:3.5

Let w be a mininorm on X, Let Y be a subspace of X. Let $x \in X, y \in Y$ and $\alpha \in \mathbb{K}, \alpha \neq 0$.

We have $w(\alpha x + y) \geq dist(x, Y)$,where $dist(x, Y)$ is the distance from x to Y defined by

$dist(x, Y) = inf\{dist(x, y)/y \in Y\} = inf\{w(x - y)/y \in Y\}$.

Proof:

Consider $w(\alpha x + y) = w\alpha(x + y/\alpha) = w(x + y/\alpha)$

$\geq inf\{w(x - y)/y \in Y\}$, since $y/\alpha \in Y$

$= dist(x, Y)$.

Remark:3.6

Let $x \in X$ and Y be a subspace of X. Then, $dist(\alpha x, Y) = dist(x, Y)$, for any $\alpha \neq 0$.

Proof:

$dist(\alpha x, Y) = inf\{w(\alpha x - y)/y \in Y\}$

$= inf\{w\big(\alpha(x - y/\alpha)\big)y \in Y\}$

$= inf\{w(x - y/\alpha)/y \in Y\}$

$= inf\{w(x - y)/y \in Y\}$, since Y is a subspace

$= dist(x, Y)$.

4. Structural Properties of Mininormed Euclidean Spaces

Theorem:4.1

Every Euclidean space with the standard mininorm w_H is complete.

Proof:

Consider any Euclidean space $\mathbb{R}^k$ with the standard mininorm.

Let $(x_n) = (x_n(1), x_n(2), \dots, x_n(k))$ be a Cauchy sequence[3,6] in $\mathbb{R}^k$

Then, for every $\epsilon > 0$ there exists a positive integer n_0, such that

$w_H(x_n - x_m) < \epsilon$ for all $n, m \geq n_0$.

By choosing $\epsilon = 1$, we get an integer n_0 such that $w_H(x_n - x_m) < 1$ for all $n, m \geq n_0$

So, $w_H(x_n - x_m) = 0$ for all $n, m \geq n_0$, as w_H takes only integral values.

Hence $x_n = x_m$ for all $n, m \geq n_0$.

This implies that (x_n) is a constant sequence except for a finite number of terms.

In fact, $(x_n) = \big(x_1, x_2, \dots, x_{n_0}, x_{n_0}, x_{n_0}, \dots\big)$.

Thus (x_n) is convergent to x_{n_0}. So $\mathbb{R}^k$ is complete.

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Definition:4.2 A mininormed space which is complete in the induced metric is called a mini Banach space.

Remark:4.3 $\mathbb{R}_H^k$ are mini Banach spaces.

Let us denote the Euclidean space $\mathbb{R}^k$ with the standard mininorm w_H by $\mathbb{R}_H^k$.

Theorem:4.4

Every subset E of $\mathbb{R}_H^k$ is open.

Proof:

Let E be an arbitrary non-empty subset of $\mathbb{R}^k$

For $x \in E$, consider $B(x,1) = \{y \in \mathbb{R}^k / d_H(x,y) < 1\}$

$= \{y \in \mathbb{R}_H^k / d_H(x,y) = 0\}$, as there is no d_H value between 0 and 1

$= \{x\} \subset E$

Thus, every point of E is an interior point and hence E is open.

Corollary:4.5

Every subset of $\mathbb{R}_H^k$ is closed.

Proof:

For any subset E of $\mathbb{R}_H^k$, E^c is open, by the above theorem. Hence E is closed.

Remark:4.6

Every subset of $\mathbb{R}_H^k$ is both open and closed.

Theorem:4.7

$\mathbb{R}_H^k$ is not connected.

Proof:

Let E be any non-empty proper subset of $\mathbb{R}_H^k$. Then, both E and E^c are open.

Now, $\mathbb{R}_H^k$ is the union of the disjoint open sets E and E^c.

So, $\mathbb{R}_H^k$ cannot be connected.

Theorem:4.8

Only finite subsets of $\mathbb{R}_H^k$ are compact.

Proof:

Let E be a compact subset of $\mathbb{R}_H^k$

As every subset of $\mathbb{R}_H^k$ open, every singleton set is open.

So, $S = \{\{x\}/x \in E\}$ is an open covering for E.

Since E is compact, S has a finite sub covering for E,say, $\{\{x_1\},\{x_2\},\dots,\{x_n\}\}$, as E is compact[4]

Hence $E \subset \{x_1, x_2, \dots, x_n\}$ and so E is a finite set.

In a normed space $(X, \| \ \ \|)$, a closed ball of radius r is defined as $B_r = \{x \in X / \|x\| \leq r\}$

It is a fact that Balls are convex sets in normed spaces [3].

(A set A is convex if $\alpha x + (1-\alpha)y \in A$ for all $x, y \in A$ and $0 \leq \alpha \leq 1$)

But this result does not hold in $\mathbb{R}_H^k$

For example, consider $\mathbb{R}_H^3$

Take a ball $B_1 = \{x \in \mathbb{R}^3 / w_H(x) \leq 1\}$

Now $x = (1,0,0)$ and $y = (0,2,0) \in B_1$.

Let $\alpha = \frac{1}{2}$ .Then, $\alpha x + (1-\alpha)y = \frac{1}{2}x + \frac{1}{2}y = (\frac{1}{2}, 1,0)$.

So, $w_H(\alpha x + (1-\alpha)y) = 2$.

So, $\alpha x + (1-\alpha)y \notin B_1$.

Thus B_1 is not convex.

References

[1] J.Justesan and Hoholdt – A course in Error Correcting codes. Hindustan Book Agency, New Delhi, 2004.
[2] E. Kreyszig – Introductory Functional Analysis with Applications. John Wiley & Sons, New York, 1978.
[3] B.V. Limaya – Functional Analysis. New Age International Publishers, New Delhi, 1996.
[4] M. Melna Frincy and J.R.V. Edward – Extension of the δ – function to $\mathbb{R}^n$. Turkish Online Journal of Qualitative Inquiry . Volume 12, Issue 6, June 2021: 714-718.
[5] M. Melna Frincy and J.R.V. Edward – Relationship Between Weight Function and 1 – norm . Ratio Mathematica. Volume 44, Issue 914, ISSN 1592 -7415 eISSN: 2282-8214
[6] G.F.Simmons – Introduction to Topology and Modern Analysis McGraw-Hill, Tokyo, 1963.