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Solution of Two-Parameers Singulrly Perturbed Boundary Value Problem using Completely Exponential Fitted Modified Upwind Finite Difference Method

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Abstract:- In this paper, we build a completely exponentially fitted modified upwind finite difference method for solving two-parameters singularly perturbed boundary value problems on a uniform mesh. Because the problem has dual layers, we divide the domain into two subintervals and develop the discretization equation for the problem using two fitting factors that take care of the problem's two parameters. We solve the discretization equation by using discrete invariant imbedding. We establish the convergence of the method and tabulate the maximum absolute errors with comparisons for the standard examples selected from the literature to demonstrate the method's efficiency**.**

Keywords: *Singularly perturbed two-parameter problem, Exponentially fitted, Boundary layer*

1. Introduction

Two-parameter singularly perturbed boundary value problems (TPSPBVPs) are a distinct type of differential equations that feature two small parameters, which have a significant impact on the solution's behavior. These problems are particularly fascinating because they model intricate phenomena such as boundary layers and rapid variations in the solution. These types of problems arise in various fields, including physics, chemistry, biology, chemical reactor theory, mechanics, lubrication theory, and DC motor theory [1, 2, 5, 6, 17, 30]. In recent years, significant research has been conducted on single-parameter convection-diffusion and reaction-diffusion problems [16, 23, 27]. However, only a limited number of researchers have explored two-parameter singular perturbation problems [7, 15, 25, 26, 28, 30].

O'Malley [18–21] was the first to do a thorough investigation on the asymptotic solutions to TPSPBVPs, to the best of our knowledge. O'Malley examined the behavior of these problems in two scenarios: $\frac{\mu^2}{\epsilon} \to 0$ as $\epsilon \to 0$ and $\frac{\epsilon}{\mu^2} \to 0$ as $\mu \to 0$ and determined adequate criteria for convergence.

Kadalbajoo and Yadaw [11] have introduced a B-spline collocation approach to derive the approximate solution for singularly perturbed two parameter boundary value problems. Kadalbajoo and Yadaw [12] presented the Ritz-Galerkin finite element approach utilizing a Shishkin mesh to address two-parameter boundary value problems. Kumar [13] has investigated the finite difference method on a non-uniform grid for TPSPBVPs. Kumar et al. [14] investigated a parameter uniform technique utilizing asymptotic expansion to address TPSPBVPs. Linb and Roos [15] have examined the analysis of a finite difference scheme for a singular perturbation involving two parameters. Kadalbajoo and Yadaw [17] conducted a comparative analysis of the finite difference, finite element, and B-spline collocation methods for TPSPBVPs. Pandit and Kumar [25] have devised a Haar wavelet method for addressing

second-order singly perturbed boundary value problems with two parameters. Patidar [22] has introduced a fitted operator finite difference approach for a TPSPBVP. Zahra and Mhlawy [31] have examined the exponential spline method for the numerical resolution of TPSPBVPs.
The article is structured as follows: Section 2 provides a description of the situation. The formulation of the numerical scheme is presented in Section 3. Section 4 presents a comprehensive analysis of convergence. Numerical experiments are conducted in Section 5. The final section addresses discussions and conclusions.

2. Description of the Problem

We have Consider two parameters singularly perturbed two-point boundary value problem of the form:

$$\varepsilon\theta''(t)+\mu p(t)\theta'(t)-q(t)\theta(t)=f(t) \qquad (1)$$

with boundary conditions $\theta(0)=\xi$ and $\theta(1)=\eta$ (2)

where $0<\varepsilon<<1$ and $0<\mu<<1$are two small parameters. The functions $p(t), q(t), f(\mathrm{t})$are sufficiently smooth with $p(t)\geq\tilde{p}>0$ and $q(t)\geq\tilde{q}>0$, $\frac{q(t)}{p(t)}\geq\tilde{c}>0$.

The solution to Equation (1.1) may be determined by finding the roots of the characteristic equation

$$\varepsilon\lambda(t)^2+\mu p(t)\lambda(t)-q(t)=0$$

This equation yields two continuous functions

$$\lambda_1(t)=-\frac{\mu p(t)}{2\varepsilon}-\sqrt{\left(\frac{\mu p(t)}{2\varepsilon}\right)^2+\frac{q(t)}{\varepsilon}} \qquad (3)$$

$$\lambda_2(t)=-\frac{\mu p(t)}{2\varepsilon}+\sqrt{\left(\frac{\mu p(t)}{2\varepsilon}\right)^2+\frac{q(t)}{\varepsilon}} \qquad (4)$$

The function $\lambda_1<0$ characterises the boundary layer at the left end point $t=0$, while $\lambda_2>0$ describes boundary layer at the right end $t=1$. Put $\theta_1:=\max_{t\in[0,1]}\lambda_1(t)<-\frac{\mu}{\varepsilon}\leq 0$ and $\theta_2:=\min_{t\in[0,1]}\lambda_2(t)$.

The depletion of the solution in the boundary layer area is determined by the variables by θ_1 and θ_2.

$$\text{For } \frac{\varepsilon}{\mu^2}\leq 1, |\theta_1|=O\left(\frac{\mu}{\varepsilon}\right) \text{ and } |\theta_2|=O\left(\frac{1}{\mu}\right),$$

$$\frac{\mu^2}{\varepsilon}\leq 1, |\theta_1|=O\left(\frac{1}{\sqrt{\varepsilon}}\right) \text{ and } |\theta_2|=O\left(\frac{1}{\sqrt{\varepsilon}}\right).$$

At $t=0$ the layer is controlled by the term $e^{-\theta_1 t}$ and at $t=1$ the layer is controlled by $e^{-\theta_2(1-t)}$. From [7], we have

$$\theta_1=\begin{cases}\frac{\sqrt{\gamma\tilde{a}}}{\sqrt{\varepsilon}}, \frac{\mu^2}{\varepsilon}\leq\frac{\gamma}{\tilde{a}}\\ \frac{\tilde{a}\mu}{\varepsilon}, \frac{\mu^2}{\varepsilon}\geq\frac{\gamma}{\tilde{a}}\end{cases}, \qquad \theta_2=\begin{cases}\frac{\sqrt{\gamma\tilde{a}}}{2\sqrt{\varepsilon}}, \frac{\mu^2}{\varepsilon}\leq\frac{\gamma}{\tilde{a}}\\ \frac{\gamma}{2\mu}, \frac{\mu^2}{\varepsilon}\geq\frac{\gamma}{\tilde{a}}\end{cases}$$

where, $\tilde{a}=\min_{t\in[0,1]}p(t)$ and $\gamma=\min_{t\in[0,1]}\frac{q(t)}{p(t)}$.

3. Numerical Scheme

Discretize the interval [0, 1] into N equal subintervals of mesh size $h=\frac{1}{N}$, so that $t_i=t_0+ih,\ , i=0,1,2,\dots,N$ are the nodal points with $0=t_0, 1=t_N$. Since, the problem exhibits two boundary layers at $t=0$ and $t=1$, we divide the domain [0, 1] into two subintervals $[0,t_m]$ and $[t_m, 1]$ where $t_m=\frac{1}{2}$. Here, in $[0,t_m]$ the layer will be at the left end $t=0$ and in $[t_m, 1]$ the layer is at right end $t=1$.

We consider the difference scheme

$$\varepsilon\sigma_i(\rho)D_+D_-\theta_i + p(t_i)\mu\tau_i(\rho)\widetilde{D}_+\theta_i - q(t_i)\theta_i = f(t_i) \quad for \quad i = 1,2,\dots,m$$
(5)

$$\varepsilon\sigma_i(\rho)D_+D_-\theta_i + p(t_i)\mu\tau_i(\rho)\widetilde{D}_-\theta_i - q(t_i)\theta_i = f(t_i) \text{ for } i = m+1, m+2,\dots,N-1 \tag{6}$$

with $\theta_0 = \xi, \quad \theta_N = \eta$
(7)

The values of $\sigma_i(\rho)$ and $\tau_i(\rho)$ are selected in such a way that the solution of the homogeneous differential equation matches exactly with the solution of the corresponding homogeneous difference equation, as given in Eq. (5) and Eq. (6).

Here $D_+D_-\theta_i \approx \frac{\theta_{i-1}-2\theta_i+\theta_{i+1}}{h^2}$e, $\widetilde{D}_+\theta_i \approx \frac{\theta_{i+1}-\theta_i}{h} - \frac{h}{2}\theta_i''$, $\widetilde{D}_-\theta_i \approx \frac{\theta_i-\theta_{i-1}}{h} + \frac{h}{2}\theta_i''$ and $\rho = \frac{h}{\varepsilon}$.

Substituting Eq. (3) and Eq. (4) in the corresponding homogeneous difference Eq. (5) and Eq. (6), we can determine the fitting factors

$$\sigma_i(\rho) = -\frac{q(t_i)h}{4\left[\rho-\frac{\mu p_i}{2}\right]}\left(\frac{e^{-\left(\frac{\mu p(t_i)h}{2\varepsilon}\right)}}{sinh\left(\frac{\lambda_1(t_i)h}{2}\right)sinh\left(\frac{\lambda_2(t_i)h}{2}\right)}\right) for\ i = 1,2,\dots,m$$
(8a)

$$\sigma_i(\rho) = -\frac{q(t_i)h}{4\left[\rho+\frac{\mu p_i}{2}\right]}\left(\frac{e^{\left(\frac{\mu p(t_i)h}{2\varepsilon}\right)}}{sinh\left(\frac{\lambda_1(t_i)h}{2}\right)sinh\left(\frac{\lambda_2(t_i)h}{2}\right)}\right), i = m+1, m+2,\dots N-1 \tag{8b}$$

$$\tau_i(\rho) = \frac{q(t_i)h}{2\mu p(t_i)}\left(coth\left(\frac{\lambda_1(t_i)h}{2}\right) + coth\left(\frac{\lambda_2(t_i)h}{2}\right)\right) \text{ for } i = 1,2,\dots N-1$$
(9)

The system of tridiagonal of Eq. (1.5) and Eq. (1.6) is

$$\left(\frac{\sigma_i}{h^2}\left[\varepsilon - \frac{\mu p_i h}{2}\right]\right)\theta_{i-1} - \left(\left(\frac{2\sigma_i}{h^2}\left[\varepsilon - \frac{\mu p_i h}{2}\right]\right) + \frac{p_i\mu\tau_i}{h} + q_i\right)\theta_i + \left(\frac{\sigma_i}{h^2}\left[\varepsilon - \frac{\mu p_i h}{2}\right] + \frac{p_i\mu\tau_i}{h}\right)\theta_{i+1} = f_i \tag{10}$$

for $i = 1,2,\dots,m$

$$\left(\frac{\sigma_i}{h^2}\left[\varepsilon + \frac{\mu p_i h}{2}\right] - \frac{p_i\mu\tau_i}{h}\right)\theta_{i-1} - \left(\left(\frac{2\sigma_i}{h^2}\left[\varepsilon - \frac{\mu p_i h}{2}\right]\right) - \frac{p_i\mu\tau_i}{h} + q_i\right)\theta_i + \left(\frac{\sigma_i}{h^2}\left[\varepsilon + \frac{\mu p_i h}{2}\right]\right)\theta_{i+1} = f_i \tag{11}$$

for $i = m+1, m+2 \dots N-1$.

We solve the system of Eq. (10) and (11) by using Thomas algorithm using the boundary conditions Eq. (7).

4. Conergence Analysis

Writing the tridiagonal system Eq. (10) in matrix-vector form, we get

$$AY = C \tag{12}$$

where $A = (m_{ij})$, $1 \le i,j \le m\text{-}1$ is a tridiagonal matrix of order (m-1), with

$$m_{i\,i-1} = \frac{\sigma_i}{h^2}\left[\varepsilon - \frac{\mu p_i h}{2}\right]$$

$$m_{i\,i} = -\left(\left(\frac{2\sigma_i}{h^2}\left[\varepsilon - \frac{\mu p_i h}{2}\right]\right) + \frac{p_i\mu\tau_i}{h} + q_i\right)$$

$$m_{i\,i+1} = \left(\frac{\sigma_i}{h^2}\left[\varepsilon + \frac{\mu p_i h}{2}\right]\right) + \frac{p_i\mu\tau_i}{h} \tag{13}$$

and $C = (d_i)$ is a column vector with $d_i = f_i$ where $i = 1,2,\dots,m-1$ with local truncation error

$$T_i(h) = h\left(\frac{\tau\mu p_i}{2}\right)\theta_i'' + h^2\left(\frac{\tau\mu p_i}{6}\theta_i''' + \frac{\sigma_i}{12}\left[\varepsilon - \frac{\mu p_i h}{2}\right]\theta'^{''''}_i\right) + Oh^3 \tag{13}$$

i.e., truncation error in the difference scheme is of $O(h)$.

Writing the tridiagonal system Eq. (11) in matrix-vector form, we get

$$AY = C$$

(14)

where $A = (m_{ij})$, $m+1 \le i,j \le N\text{-}1$ is a tri-diagonal matrix with

$$m_{i\,i-1} = \left(\frac{\sigma_i}{h^2}\left[\varepsilon + \frac{\mu p_i h}{2}\right] - \frac{p_i \mu \tau_i}{h}\right)$$

$$m_{i\,i} = -\left(\left(\frac{2\sigma_i}{h^2}\left[\varepsilon - \frac{\mu p_i h}{2}\right]\right) - \frac{p_i \mu \tau_i}{h} + q_i\right)$$

$$m_{i\,i+1} = \left(\frac{\sigma_i}{h^2}\left[\varepsilon + \frac{\mu p_i h}{2}\right]\right)$$

and $C = (d_i)$ is a column vector with $d_i = f_i$, where $i = m+1, m+2, \ldots, N-1$ with truncation error $T_i(h) = h\left(-\frac{\tau \mu p_i}{2}\right)\theta_i{}'' + h^2\left(\frac{\tau \mu p_i}{6}\theta_i''' + \frac{\sigma_i}{12}\left[\varepsilon + \frac{\mu p_i h}{2}\right]\theta'{}_i'''\right)$.

We also have $$A\overline{\Theta} - T(h) = C$$
(15)

where $\overline{\Theta} = \left(\overline{\theta}_0, \overline{\theta}_1, \ldots, \overline{\theta}_N\right)^t$ the actual solution, $T(h) = (T_0(h), T_1(h), \ldots, T_N(h))^t$ is the local truncation error.

Using Eq. (12), Eq. (14) and Eq. (15), we get

$$A\left(\overline{\Theta} - Y\right) = T(h) \quad (16)$$

Hence, the error equation is $$AE = T(h) \quad (17)$$

where $E = \overline{\Theta} - Y = (e_0, e_1, e_2, \ldots, e_N)^t$.

Clearly, we have

$$S_i = \sum_{j=1}^{N-1} m_{ij} = -\frac{\sigma}{h^2}\left(\varepsilon - \frac{\mu p_i h}{2}\right) + q_i \quad for\ i = 1$$

$$S_i = \sum_{j=1}^{N-1} m_{ij} = 2q_i = B_{i_0} \quad for\ i = 2,3,\ldots\ldots\ldots, N-2$$

$$S_i = \sum_{j=1}^{N-1} m_{ij} = -\frac{\sigma}{h^2}\left(\varepsilon + \frac{\mu p_i h}{2}\right) + q_i \quad for\ i = N-1$$

Since $0 < \varepsilon << 1$ and $0 < \mu << 1$, the matrix A is irreducible and monotone. Then, it follows that A^{-1} exists and its elements are nonnegative.

Hence, using Eq. (17), we get

$$E = A^{-1}T(h) \quad (18)$$

and $$\|E\| \le \|A^{-1}\|.\|T(h)\|$$
(19)

Let $\bar{m}_{ki}$ be the $(ki)^{th}$ element of A^{-1}. Since $\bar{m}_{ki} \ge 0$, from the theory of matrices we have

$$\sum_{i=1}^{N-1} \overline{m}_{k,i} S_i = 1, \quad k = 1, 2, \ldots., N\text{-}1 \quad (20)$$

Therefore, $$\sum_{i=1}^{N-1} \overline{m}_{k,i} \le \frac{1}{\min\limits_{1\le i\le N-1} S_i} = \frac{1}{B_{i_0}} \le \frac{1}{|B_{i_0}|} \quad (21)$$

for some i_0 between 1 and N-1 and $B_{i_0} = 2b_i$.

Define $\|A^{-1}\| = \max_{1\le k\le N-1} \sum_{i=1}^{N-1} |\bar{m}_{ki}|$ and $\|T(h)\| = \max_{1\le i\le N-1} |T_i(h)|$. Using Eq. (13), Eq. (18) and Eq. (21), we get

$$e_j = \sum_{i=1}^{N-1} \bar{m}_{ki}\ T_i(h),\quad j = 1, 2, 3, \ldots, N\text{-}1$$

which implies $$e_j \le \frac{kh}{/b_i/},\qquad j = 1\ (1)\ N\text{-}1$$

(22)

where $k = \frac{\tau\mu p_i \theta_i''}{4}$ is a constant.

Therefore, using Eq. (22), we have $\|E\| = O(h)$ i.e., the proposed scheme is a first order convergent.

5. Numerical Experiments

Four boundary value problems of types Eq. (1) and Eq. (2) are considered to check the applicability of the proposed method. We selected these problems due to their extensive discussion in the literature and the availability of exact solutions for comparison. Since the considered problems have an exact solution, we estimate the maximum absolute errors using $E_{N,\varepsilon,\mu} = \max_{0\le i\le N} |\theta(t_i) - \theta_i|$ where $\theta(t_i)$ is the exact solution and θ_i is the computed solution.

Example 1. $\varepsilon\theta'' + \mu\theta' - \theta = -t,\ 0 < t < 1$ with $\theta(0) = 1,\ \theta(1) = 0.$

The exact solution is $\theta(t) = \frac{(1+\mu)+(1-\mu)e^{m_2}}{e^{m_2}-e^{m_1}} e^{m_1 t} + \frac{(1+\mu)+(1-\mu)e^{m_1}}{e^{m_1}-e^{m_2}} e^{m_2 t} + t + \mu$ where $m_1 = \frac{-\mu-\sqrt{\mu^2+4\varepsilon}}{2\varepsilon}$, $m_2 = \frac{-\mu+\sqrt{\mu^2+4\varepsilon}}{2\varepsilon}$.

Tables 1 and 2 represent the MAEs for a range of values of *N*. Figure 1 illustrates the boundary layer behaviour in the solution.

Example 2. $-\varepsilon\theta'' + \mu\theta' + \theta = \cos\pi t,\quad 0 < t < 1$ with $\theta(0) = 0,\ \theta(1) = 0.$

The exact solution of this problem is $\theta(t) = p\cos\pi t + q\sin\pi t + Ae^{\lambda_1 t} + Be^{-\lambda_2(1-t)}$

where $p = \frac{\varepsilon\pi^2+1}{\mu^2\pi^2+(\varepsilon\pi^2+1)^2}$, $q = \frac{\mu\pi}{\mu^2\pi^2+(\varepsilon\pi^2+1)^2}$, $A = -p\,\frac{1+e^{-\lambda_2}}{1-e^{\lambda_1-\lambda_2}}$, $B = p\,\frac{1+e^{\lambda_1}}{1-e^{\lambda_1-\lambda_2}}$, $\lambda_{1,2} = \frac{\mu\mp\sqrt{\mu^2+4\varepsilon}}{2\varepsilon}$

Tables 3,4 and 5 present the MAEs for a range of values of *N*. Figure 2 illustrates the boundary layer behaviour in the solution.

Example 3. $-\varepsilon\theta'' - \mu\theta' + \theta = e^{(1-t)}\quad 0 < t < 1$ with $\theta(0) = 0,\ \theta(1) = 0.$

The exact solution of this problem is

$$\theta(t) = \frac{e^{(m_2+1)}-1}{D} e^{m_1 t} + \frac{1-e^{(m_1+1)}}{D} e^{m_2 t} - \frac{e^{(1-t)}}{\varepsilon(m_1+1)(m_2+1)}$$

where $D = \varepsilon(e^{m_2} - e^{m_1})(m_1+1)(m_2+1)$, $m_1 = \frac{-\mu-\sqrt{\mu^2+4\varepsilon}}{2\varepsilon}$ $\quad m_2 = \frac{-\mu+\sqrt{\mu^2+4\varepsilon}}{2\varepsilon}$.

The MAEs are listed in Tables 6 and 7 for numerous values of N. The layer profile is depicted in Figure 4.

6. Discussion and Conclusion

This chapter examines the use of a completely exponentially modified upwind fitted finite difference approach on a uniform mesh to solve two-parameter singly perturbed boundary value problems with a dual boundary layer structure. We partitioned the domain into two subintervals and formulated the discretization equation for the problem by incorporating two fitting factors that account for the problem's two parameters. We employed Thomas algorithm successfully solve the tridiagonal set of discretization equations. An analysis is conducted to determine the convergence of the suggested approach. We calculate the maximum absolute errors by comparing them with the results in [9], [12], [14], [27] and [33] using MATLAB programming to showcase the effectiveness of the strategy.

The solutions of the examples are shown graphically, and we observed that the numerical solution is in good agreement with the exact solution, and for fixed values, as the width of the left and right boundary layers decreases. This method is uncomplicated and easily executable. One can apply this technique to solve a class of higher-order, multi-parameter singular perturbation problems.

Table 1. MAEs in the solution of Example 1 for $\varepsilon = 10^{-3}$

μ / N	2^6	2^7	2^8	2^9	2^{10}
Proposed method					
10^{-2}	2.0091(-4)	5.0703(-5)	1.2706(-5)	3.1783(-6)	7.9469(-7)
10^{-3}	2.0099(-5)	5.0708(-6)	1.2706(-6)	3.1783(-7)	7.9469(-8)
10^{-4}	2.0099(-6)	5.0708(-7)	1.2706(-7)	3.1783(-8)	7.9470(-9)
Results in [12]					
10^{-2}	3.6590(-3)	1.1005(-3)	2.7573(-4)	6.8812(-5)	1.7196(-5)
10^{-3}	3.0262(-3)	7.4023(-4)	1.8406(-4)	4.5953(-5)	1.1484(-5)
10^{-4}	2.9008(-3)	7.0989(-4)	1.7654(-4)	4.4076(-5)	1.1015(-5)

Table 2. MAEs in solution of Example 1 for $\mu = 10^{-4}$

ε / N	2^6	2^7	2^8	2^9	2^{10}
Proposed method					
10^{-1}	1.2312(-8)	3.0783(-9)	7.6959(-10)	1.9241(-10)	4.9152(-11)
10^{-2}	2.0046(-7)	5.0162(-8)	1.2543(-8)	3.1360(-9)	7.8402(-10)
10^{-3}	2.0099(-6)	5.0708(-7)	1.2706(-7)	3.1783(-8)	7.9470(-9)
Results in [12]					
10^{-1}	1.5752(-5)	3.9408(-6)	9.8514(-7)	2.4628(-7)	6.1570(-8)
10^{-2}	2.8064(-4)	7.0125(-5)	1.7522(-5)	4.3807(-6)	1.0952(-6)
10^{-3}	2.9008(-3)	7.0989(-4)	1.7654(-4)	4.4076(-5)	1.1015(-5)

Table 3. MAEs in the solution of Example 2 for N = 128

ε / μ	10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-10}

Proposed method					
10^{-3}	1.6525 (-4)	4.5638 (-5)	4.1508 (-5)	4.5102(-5)	4.5102(-5)
10^{-4}	1.5338 (-3)	5.1058 (-5)	4.7762 (-5)	4.7736(-5)	4.7736(-5)
10^{-5}	7.9426 (-3)	1.2653 (-4)	3.7890 (-5)	3.7802(-5)	3.7801(-5)
10^{-6}	1.0389 (-2)	3.0641 (-4)	1.0101 (-5)	9.6195(-6)	9.6188(-6)
10^{-7}	1.0655 (-2)	3.1416 (-4)	3.2928 (-6)	9.8743(-7)	9.8667(-7)
Results in [9]					
10^{-3}	8.3832(-5)	9.4446(-3)	1.3075(-2)	1.8164(-2)	1.8359(-2)
10^{-4}	8.2686(-5)	9.0436(-3)	9.4539(-3)	1.3076(-2)	1.8163(-2)
10^{-5}	8.2572(-5)	9.0036(-3)	9.0525(-3)	9.4540(-3)	1.3076(-2)
10^{-6}	8.2561(-5)	8.9996(-3)	9.0124(-3)	9.0526(-3)	9.4540(-3)
10^{-7}	8.2559(-5)	8.9992(-3)	9.0084(-3)	9.0125(-3)	9.0526(-3)

Table 4. Comparison of MAEs for Example 2

μ	$\varepsilon = 10^{-2}$, $N = 128$			
	Kadalbajoo and Yadaw [12]	Zahra and El Mhlawy [31]	Sapna Pandit and Manoj Kumar [25]	Our method
10^{-3}	8.3832(-5)	4.1924(-5)	4.2303(-5)	3.1808(-5)
10^{-4}	8.2686(-5)	4.1296(-5)	4.1318(-5)	3.1107(-5)
10^{-5}	8.2572(-5)	4.1232(-5)	4.1220(-5)	3.1037(-5)
10^{-6}	8.2561(-5)	4.1226(-5)	4.1210(-5)	3.1030(-5)
10^{-7}	8.2559(-5)	4.1225(-5)	4.1209(-5)	3.1029(-5)

Table 5. Comparison of MAEs for Example 2

μ	$\varepsilon = 10^{-4}$, $N = 128$			
	Kadalbajoo and Yadaw [12]	Zahra and El Mhlawy [31]	Sapna Pandit and Manoj Kumar [25]	Our method

10^{-3}	9.4446(-3)	4.7598(-3)	5.1964(-3)	1.6232(-4)
10^{-4}	9.0436(-3)	4.2856(-3)	4.1710(-3)	5.1058(-5)
10^{-5}	9.0036(-3)	4.2295(-3)	4.0754(-3)	4.8001(-5)
10^{-6}	8.9996(-3)	4.2238(-3)	4.0659(-3)	4.7762(-5)
10^{-7}	8.9992(-3)	4.2232(-3)	4.0650(-3)	4.7738(-5)

Table 6. MAEs in the solution of Example 3 for $\mu = 2^{-5}$

N/ε	2^{-10}	2^{-11}	2^{-12}	2^{-13}	2^{-14}	2^{-15}	
2^7	3.759(-4)	7.849(-4)	1.591(-3)	4.440(-3)	5.244(-3)	7.330(-3)	8.547(-3)
2^8	9.426(-5)	1.978(-4)	4.063(-4)	8.165(-4)	4.440(-3)	2.692(-3)	3.763(-3)
2^9	2.358(-5)	4.954(-5)	1.021(-4)	2.075(-4)	4.143(-4)	8.881(-4)	1.364(-3)
2^{10}	5.897(-6)	1.239(-5)	2.557(-5)	5.211(-5)	1.051(-4)	2.089(-4)	4.440(-4)

Table 7. MAEs in the solution of Example 3 for $\varepsilon = 10^{-16}$

N/μ	2^{-32}	2^{-36}	2^{-40}	2^{-44}	2^{-48}	2^{-52}
2^7	1.115(-5)	1.115(-5)	1.115(-5)	1.115(-5)	1.115(-5)	1.115(-5)
2^8	3.205(-6)	3.205(-6)	3.205(-6)	3.205(-6)	3.205(-6)	3.205(-6)
2^9	8.318(-7)	8.318(-7)	8.318(-7)	8.318(-7)	8.318(-7)	8.318(-7)
2^{10}	2.099(-7)	2.099(-7)	2.099(-7)	2.099(-7)	2.099(-7)	2.099(-7)

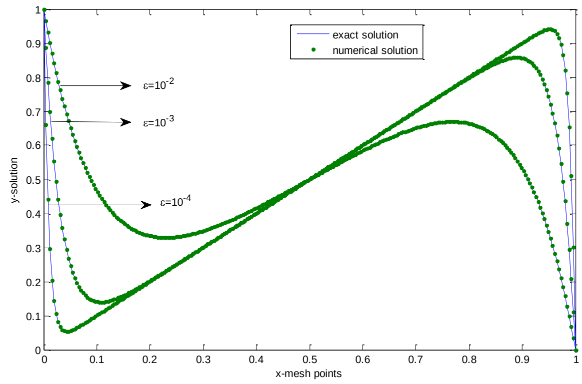


Fig 1. Graphical representation of solution in Example 1 with $\mu = 10^{-3}$

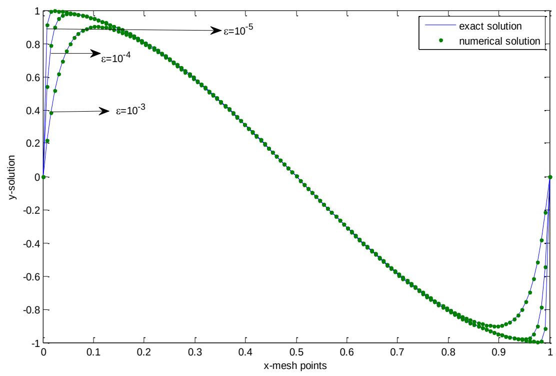


Fig 2. Graphical representation of solution in Example 2 with $\mu = 10^{-4}$

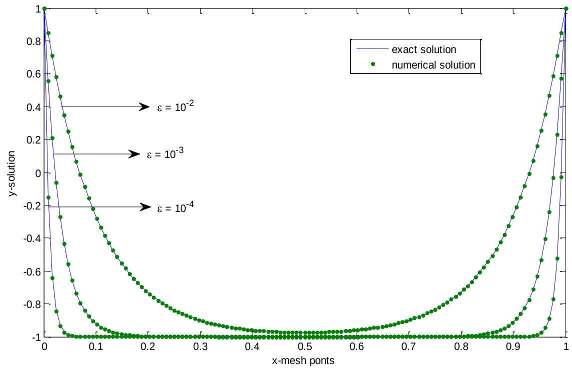
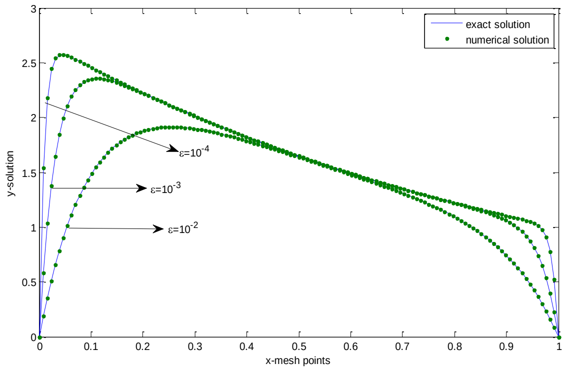


Fig 3. Graphical representation of solution in Example 3 with $\mu = 10^{-3}$



ig 4. Graphical representation of solution in Example 4 with $\mu = 10^{-3}$

References

[1] E. Bohl (1981), Finite Modele gewohnlicher Randwertaufgaben, Vieweg Teubner Verlag, NY.

[2] J. Bigge, E. Bohl (1985), Deformations of the bifurcation diagram due to discretization, Math Comput., 45, pp. 393–403.

[3] G. Birkhoff, C.D. Boor (1964), Error bounds for spline interpolation, J Math Mech., 13, pp. 827–835.

[4] M. Brdar, H. Zarin (2016), A singularly perturbed problem with two-parameters on a Bakhvalov-type mesh, J Comput Appl Math., 292, pp. 307–319.

[5] J. Chen, R.E. O'Malley Jr (1974), On the asymptotic solution of a two-parameter boundary value problem of chemical reactor theory, SIAM J Appl Math., 26, pp. 717–729.

[6] R.C. Diprima (1968), Asymptotic methods for an infinitely long slider squeeze-film bearing, J Lubric Technol. 90, pp. 173–183.

[7] J.L. Gracia, E. O'Riordan, M.L. Pickett (2006), A parameter robust higher order numerical method for a singularly perturbed twoparameter problem, Appl Numer Math., 56, pp. 962–980.

[8] C.A. Hall (1968), On error bound for spline interpolation, J Approxim Theory., 1, PP. 209–218.

[9] Jin Zhang, Lv. Yanhui (2022), Finite element method for singularly perturbed problems with two parameters on a Bakhvalov-type mesh in 2D, Numerical Algorithms., 90, pp. 447–475.

[10] MK. Kadalbajoo, A.S. Yadaw (2011), Finite difference, Finite element and B-spline collocation methods applied to two-parameter singularly perturbed boundary value problems, J Numer Anal Ind Appl Math., 5, pp. 163–180.

[11] MK. Kadalbajoo, A.S. Yadaw (2008), B-spline collocation method for a two-parameter singularly perturbed convection-diffusion boundary value problem, Appl Math Comput., 201, pp. 504–513.

[12] M.K. Kadalbajoo, Arjun Singh Yadaw (2009), Parameter-Uniform Ritz-Galerkin Finite Element Method for Two Parameter Singularly Perturbed Boundary Value Problems, International Journal of Pure and Applied Mathematics., 55(2), pp. 287-300.

[13] D. Kumar (2012), Finite difference scheme for singularly perturbed convection-diffusion problem with two small parameters, Math Aeterna., 2, pp. 441–458.

[14] D. Kumar, A.S. Yadaw, MK. Kadalbajoo (2013) A parameter-uniform method for two-parameters singularly perturbed boundary value problems via asymptotic expansion. Appl Math Inf Sci 7:1525–1532

[15] T. Linb, H.G. Roos, (2004), Analysis of a finite difference scheme for a singular perturbed problem with two small parameters, J Math Anal Appl., 289, pp. 355–366.

[16] K.W. Morton (1996), Numerical solution of convection-diffusion problems, Applied mathematics and Mathematical Computation, vol 12. Chapman & Hall, London.

[17] A.H. Nayfeh (1993), Introduction to Perturbation Techniques, John Wiley and Sons, New York, USA.

[18] R.E. O'Malley Jr (1974), Introduction to Singular Perturbations, Academic Press, New York.

[19] R.E. O'Malley Jr (1990), Singular Perturbation Methods for Ordinary Differential Equations. Springer, New York.

[20] R.E. O'Malley Jr (1967), Singular perturbations of boundary value problems for linear ordinary differential equations involving twoparameters, J Math Anal Appl., 19, pp. 291–308.

[21] R.E. O'Malley Jr (1967), Two parameter singular perturbation problems for second order equations, J Math Mech.,1 6, pp. 1143–1164.

[22] K.C. Patidar (2008), A robust fitted operator finite difference method for a two-parameter singular perturbation problem, J Differ Equ Appl., 14, pp. 1197–1214.

[23] H.G. Roos, M. Stynes and L. Tobiska (2008), Robust Numerical Methods for Singularly Perturbed Differential Equations, Computational Mathematics., Springer, Heidelberg.

[24] H.G. Roos, Z. Uzelac (2003), The SDFEM for a convection-diffusion problem with two small parameters, Comput Methods Appl Math., 3, pp. 443–458.

[25] Sapna Pandit, Manoj Kumar (2014), Haar Wavelet Approach for Numerical Solution of Two Parameters Singularly Perturbation Boundary Value Problems, Appl. Math. Inf. Sci., 8(6), pp. 2965-2974.

[26] G.I. Shishkin, V.A. Titov (1976), A difference scheme for a differential equation with two small parameters multiplying the derivatives, Numer Method Contin Medium Mech., 7, pp. 145–155.

[27] M. Stojanovik (1994), On the optimally convergent splines difference scheme for one-dimensional reaction-diffesion problem, Appl Math Modell., 18, pp. 461–466.

[28] S. Valarmathi, N. Ramanujam (2003), Computational methods for solving two-parameter singularly perturbed boundary value problems for second-order ordinary differential equations, Appl Math Comp., 136, pp. 415–441

[29] A.B. Vasileva (1963), Asymptotic methods in the theory of ordinary differential equations containing small parameters in front of the highest derivatives, USSR Comput Math Math Phys., 3, pp. 823–863.

[30] R. Vulanovic (2001), A high order scheme for quasilinear boundary value problems with two small parameters, Computing, 67, pp. 287–303

[31] W.K. Zahra, Ashraf M. El Mhlawy (2013), Numerical solution of two-parameter singularly perturbed boundary value problems via exponential spline, Journal of King Saud University – Science., 25, pp. 201-208.