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# Prime Division Labeling for Some Standard Graph Families

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**Abstract:** We have defined the labeling pattern, named Prime Division Labeling. We have proved some results of Prime Division Labeling for various graph families like Path graphs, Star graphs, Circuit graphs.

**Keywords**: Prime Division Labeling, Prime Division Graphs,

## 1. Introduction

Graph labelling is one of the remarkable research area in the field of Graph theory which consists of several open problems available in the literature for various graph families. In 1960's A.Rosa [9] introduced graph labeling while in 1980 prime labeling technique was primarily explored by E.Roger for circuit less connected graphs. He proposed a conjecture that, to this day, is still unmet, all circuit less connected graphs permit prime labeling. Tout et al. [12] introduced the method of prime graphs. We have considered a simple connected graph in this work. Throughout this paper we have considered vertex prime division labeling for various graph families where V=V(G) considered to be the set of nodes or vertices E = E(G) considered to be the set of connections or edges of the graph G.

A labeled graph G can be defined as if either its vertices or edges or both vertices and edges are labeled with some numbers (generally by non-negative integers), symbols and alphabets. Let G = (V,E) be a graph with n vertices. A bijective mapping *f: V*→{1,2,…,n} is called a prime labeling if for each edge, labels of its end vertices are relatively prime with each other. A graph is said to be prime graph if it admits prime labeling [6]. Some circuit related graphs and its prime labeling are investigated by Vaidya and Kanani [13]. Prime labeling for special graph families are explored by Weeraratna et al. [11] and also prime labeling of Jahangir graph was studied by Lakshmi et al. [1].

**Main Results:**

**Definition:1.1** If vertices or edges or both vertices and edges of a graph are labeled with some numbers (Generally by non-negative integers), symbols and alphabets then the graph is considered as labeled graph.

**Definition:1.2** Let $G = (V, E)$ be a graph. A vertex labeling injective function $f: V \to N$ is said to be prime division labeling if for each edge $e = xy$, either $\frac{f(x)}{f(y)}$ or $\frac{f(y)}{f(x)}$ is least possible prime number. A graph that admits prime division labeling is called prime division graph.

**Illustration 1.1** Prime Division Labeling for some standard graphs is shown in figure given below.

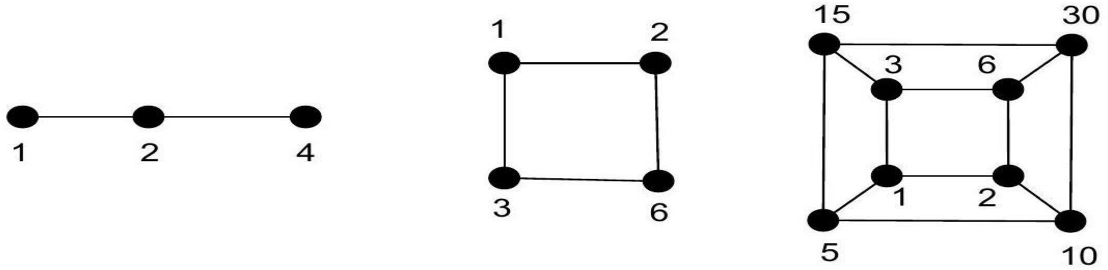


**Figure 1: Prime Division Labeling for some standard Graphs.**

**Theorem-1.1** [Sufficient condition for Prime Division Graph]

For a $k$-regular simple connected graph $G$ with $2^k$ vertices, if all the vertices are labeled with all possible factors of $p_1 \times \ldots \times p_k$ where $p_k$ is $k^{th}$ prime number, then the graph $G$ admits prime division labelling.

**Proof:** We will prove this result using theory of mathematical induction on $k$.

**Case-1:** For $k = 1$

Consider 1-regular simple connected graph with $2^k = 2^1 = 2$ vertices which gives us path $P_2$ with two vertices. To define prime division labeling consider one-one mapping $f: V(G) \to N$ such that $f(v_i) = i$, Where $i = 1,2$. Thus $\frac{f(v_2)}{f(v_1)} = \frac{2}{1} = 2$. Hence $G$ is a prime division graph. Thus result is true for $k = 1$.

**Case-2:** For $k = j$

Assume that result is true for $k = j$, Thus the result is true for $j$-regular, simple connected graph with $2^j$ vertices where all vertices are labeled with each possible factor of $p_1 \times \ldots . \times p_j$.

**Case-3:** For $k = j + 1$

To Prove: For $j + 1$-regular, simple connected graph with $2^{j+1}$ vertices. Where all vertices are labeled with each possible factors of $p_1 \times \ldots \times p_j \times p_{j+1}$ is prime division graph.

Let $G = (V, E)$ be $j$-regular, simple connected graph with $2^j$ vertices where all vertices are labeled with each possible factor of $p_1 \times \ldots . \times p_j$. According to case- 2 such graph G is prime division graph. Now each possible factors of $p_1 \times . . \times p_j$ are $1, p_1, p_2, \ldots, p_j, p_1 \times p_2, p_1 \times p_3, \ldots, p_1 \times p_j, \ldots, p_1 \times p_2 \times p_3 \times \ldots \times p_j$ which are $2^j$ elements. As, $\sum_{r=0}^{j} \binom{j}{r} = \binom{j}{0} + \binom{j}{1} + \cdots + \binom{j}{j} = 2^j$.

Consider the copy of graph $G = (V, E)$ as $G' = (V', E')$ for which the vertex labeling function $g: V' \to N$ defined as $g(v_i') = f(v_i) \times p_{j+1}, 1 \leq i \leq 2^j$.

Now for every edge $e = uv$ of graph $G$, $\frac{f(u)}{f(v)}$ or $\frac{f(v)}{f(u)}$ is prime number as $G$ is a prime division graph. For every edge $e' = u'v'$ of graph $G'$. $\frac{g(u')}{g(v')} = \frac{f(u) \times p_{j+1}}{f(v) \times p_{j+1}} = \frac{f(u)}{f(v)}$ and $\frac{g(v')}{g(u')} = \frac{f(v) \times p_{j+1}}{f(u) \times p_{j+1}} = \frac{f(v)}{f(u)}$.

Hence $\frac{g(u')}{g(v')}$ or $\frac{g(v')}{g(u')}$ is prime number.

Therefore graph $G' = (V', E')$ is a prime division graph. Consider a graph $G^* = (V^*, E^*)$ generated by joining each vertex $v_i$ of graph $G$ with $v_i'$ of graph $G'$.Therefore $|V^*(G^*)| = |V(G)| + |V'(G')| = 2^j + 2^j = 2(2^j) = 2^{j+1}$. As $G$ is $j$-regular graph and $G'$ is also $j$-regular graph, $G^*$ will be $j + 1$-regular graph. Because of joining each vertex $v_i$ with vertex $v_i'$ will increase one degree of each vertex of graph $G^*$.

All labels of graph $G^*$ are $1, p_1, p_2, \ldots, p_j, p_{j+1}, p_1 \times p_{j+1}, p_2 \times p_{j+1}, \ldots, p_j \times p_{j+1}, \ldots, p_1 \times p_2 \times p_3 \times \ldots \times p_j \times p_{j+1}$ which are $2^{j+1}$ elements and also these are the all possible factors of $p_1 \times p_2 \times \ldots \times p_j \times p_{j+1}$. For every edge $e^* = uu'$ where $u \in G$ and $u' \in G'$. The ratio of labels of $u'$ and $u$ will be $\frac{g(u')}{f(u)} = \frac{f(u) \times p_{j+1}}{f(u)} = p_{j+1}$, which is a prime number. Hence $G^* = (V^*, E^*)$ is a prime division graph. Therefore $G^*$ is $j + 1$-regular graph with $2^{j+1}$ vertices which are labeled with all possible factors of $p_1 \times \ldots \times p_j \times p_{j+1}$ and it is a prime division graph.

Thus the result is true for $k = j + 1$. By Principle of Mathematical induction Theorem is true for all positive integers.

**Theorem-1.2** Every Path $P_n$, $n \geq 2$ is a Prime Division Graph.

**Proof:** Let $P_n$, $n \geq 2$ be the path with ordered set of vertices $\{v_1, v_2, \ldots, v_n\}$. To obtain a prime division labeling function for path $P_n$ we must have a ratio of labels of two adjacent vertices as least prime number. Define an injective function $f : V(P_n) \to N$ such that $f(v_i) = 2^{i-1}$, where $i = 1, 2, \ldots, n$. By assigning the labels to each vertex as $f(v_1) = 2^0 = 1$, $f(v_2) = 2^1 = 2, ..., f(v_n) = 2^{n-1}$. So we have $\frac{f(v_{i+1})}{f(v_i)} = \frac{2^{(i+1)-1}}{2^{i-1}} = 2$ , which is a prime number, where $v_i$ and $v_{i+1}$ for $1 \leq i < n$ are two adjacent vertices. Therefore, this labeling pattern give rise to prime division labeling. Hence $P_n$ is a prime division graph.

**Remark-1.1** For any path $P_n$, $1 \leq f(v_i) \leq 2^{n-1}$, where $v_i$ is any vertex of $P_n$.

**Illustration 1.2** Prime Division Labeling of $P_n$


**Figure-2: $P_n$ and its Prime Division Labeling**

**Theorem-1.3** Every star graph $K_{1,n}$ for $n \geq 1$ is a Prime Division Graph.

**Proof:** Consider the smallest star graph $K_{1,1}$ which is path with two vertices, according to Theorem-1.2, every path is a prime division graph. So $K_{1,1}$ is a prime division graph. Now consider $K_{1,2}$ which is isomorphic to path $P_3$, according to Theorem-1.2, every path is prime division graph. Consider $K_{1,n}$ with vertex set $\{v_0, v_1, ..., v_n\}$, where $v_0$ is an apex vertex and $\{v_1, ..., v_n\}$ are pendant vertices of the graph. Define an injective mapping.

$f : V(K_{1,n}) \to N$ such that

$$f(v_i) = \begin{matrix} p_i & 1 \leq i \leq n \\ 1 & i = 0 \end{matrix}$$

where $p_n$ is $n^{th}$ prime number. Therefore, for any pair of adjacent vertices $v_0$ and $v_n$,

$$\frac{f(v_n)}{f(v_0)} = \frac{p_n}{1} = p_n \,.$$

Therefore $K_{1,n}$ is a prime division graph.

**Illustration 1.3** $K_{1,n}$ and its Prime Division Labeling is shown in Figure-3.

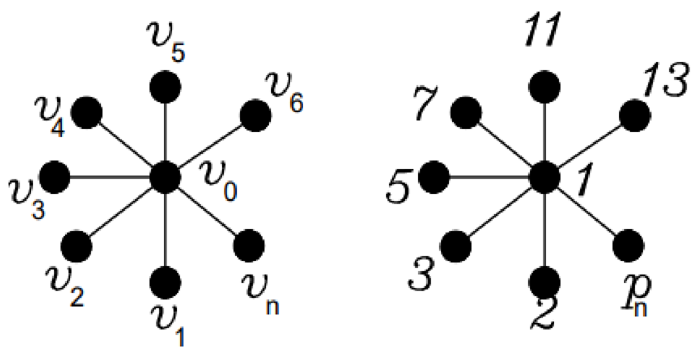


**Figure-3: $K_{1,n}$ and its Prime Division Labeling**

**Remark-1.2** For prime division Labeling of any star graph $K_{1,n}$, $1 \leq f(v_i) \leq p_n$ where $p_n$ is $n^{th}$ primenumber.

**Theorem-1.4** If graph $G = (V, E)$ contains cycle $C_3$ in it then the graph $G$ does not admit prime division labeling.

**Proof:** Consider cycle $C_3$. Let $f: V \to N$ be the vertex labeling function. Suppose $f(v_1) = 1$. By the definition of prime division graph for any adjacent pair of two vertices $v_i$ and $v_j$ either $\frac{f(v_i)}{f(v_j)}$ or $\frac{f(v_j)}{f(v_i)}$ must be prime. We have to consider $f(v_2) = p_i$, where $p_i$ is prime and $f(v_3) = p_j$, where $p_j$ is prime. Now $\frac{f(v_2)}{f(v_3)} = \frac{p_i}{p_j}$, which is not a prime number or $\frac{f(v_3)}{f(v_2)} = \frac{p_j}{p_i}$, which is also not a prime number. Note that the division of two prime numbers is always a non-integer number. So the ratio of two prime labels is not a prime number. Hence cycle $C_3$ does not admit prime division labeling. Therefore, cycle $C_3$ is not a prime division graph. Hence all super graphs of $C_3$ does not admit prime division labeling. $C_3$ is shown in Figure-4.

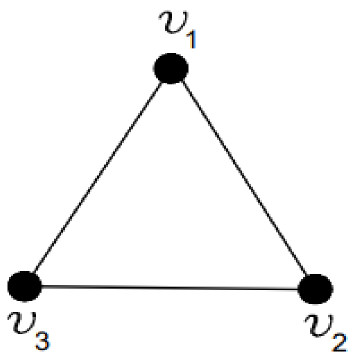


**Figure-4: $C_3$**

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