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Fuzzy Prime And Fuzzy Prime Combination Labeling Of Graphs

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Abstract

A simple graph G with n junctions admits fuzzy prime if there exists a junction labeling $\sigma : V(G) \to (0,1]$ such that for any two neighborhood junctions u and v, $\gcd(\sigma(u), \sigma(v)) = 1$. The labeling σ is called prime combination labeling if for each positive integer r such that $10^{r-1} < n \le 10^r$, the induced bridge labeling $\mu(xy)$ equals $\frac{1}{10^r}\binom{10^r\sigma(x)}{10^r\sigma(y)}$ or $\frac{1}{10^r}\binom{10^r\sigma(y)}{10^r\sigma(x)}$ according as $\sigma(x) > \sigma(y)$ or $\sigma(y) > \sigma(x)$ is injective onto the subset of $(0,1]$. In this paper, we study fuzzy prime and fuzzy combination labeling of some families of connected graphs.

Keywords: Prime labeling, Fuzzy labeling, Fuzzy prime labeling, Fuzzy prime combination labeling.

AMS Subject Classification: 05C78

1. Introduction

Graph theory has obvious utility in its applications in Science. However, when viewed apart from these applications, it yields beautiful mathematical gems, and can be used as a lens to study other area of Mathematics. We use graph theory and specifically graph labeling as a lens to study prime numbers. Fuzzy set is a newly emerging mathematical framework to exemplify the phenomenon of uncertainty in real life tribulations. This article is a further contribution on fuzzy labeling graphs.

Mathematically in graph hypothesis, a graph labeling is the allocation of labels (commonly represented by an integer) to the graph junctions or graph bridges or both. There are several types of labeling. Among the various types of labeling, our focus is on prime labeling.

Motivated by the concepts of fuzzy labeling and prime labeling in this paper, we define fuzzy prime and fuzzy prime combination labeling. In this paper we investigate the fuzzy prime labeling behavior of several graphs like grid, ladder, generalized Petersen graph and duplication of helm, Gear, Crown and star graphs. We also show a few classes of graphs are fuzzy prime combination labeling graphs.

2. Review of literature

The notion of graph labeling was introduced by Rosa [26] in 1967. The various types of labeling are investigated by Gallian [13]. In 1965, Zadeh [38] introduced the concept of fuzzy set as a generalization of crisp sets. In 1975, Rosenfeld [27] discussed the concept of fuzzy graph whose basic idea was introduced by Kauffman [20] in 1973. Yeh and Bang [37] have introduced various connectedness concepts in fuzzy graph. Fuzzy end nodes and fuzzy cut nodes were studied by Bhattacharya [6], Bhutani et al. [7] and Bhutani and Rosenfeld [8]. Bhutani and Rosenfeld [9] developed the strong arcs in fuzzy graph. The concept of fuzzy labeling was introduced by Nagoor gani and Rajalaxmi [23] in 2012. For recent result of fuzzy labeling we refer to Shanmugapriya et al. [31], Tabraiz et al. [34], Jayasimman et al. [16] and Sujatha et al. [32].

The notion of the prime labeling originated with Entringer [12] and was introduced by Tout et al. [35] in 1982. Acharya and Gill [2] defined grid graph structures and Sundaram et al. [33] discussed prime labeling concept of planar grid graphs. Prime labeling concept of ladder graphs is discuss by Berliner et al. [5]. Prime labeling concepts of prism graphs is given by Prajapati and Gajjar [24]. Meena and Naveen [22] have investigated prime labeling for the duplication of the graphs. Watkins [36] defined generalized Petersen graphs and Holton and Sheehan [15], Prajapati and Gajjar [25] and Saražin et al. [29] developed the prime labeling of generalized Petersen graphs. For recent result of fuzzy labeling we refer to Kansagara and Patel [19], Schroeder [30], Donovan and Wiglesworth [11], Arockiamary and Vijayalakshmi [4] and Abughazaleb and Abughneim [1].

Hedge and Shetty [14] have introduced combination labeling and they established many interesting properties on them. Annadurai and Megala [3] combined the ideas of prime and combination labeling into prime combination labeling in which they investigated several results on this concept.

3. Preliminaries

A graph is an ordered pair $G = (V, E)$, where V is the set of all junctions of G, which is non empty and E is the set of all bridges of G. Two junctions x, y in a graph G are said to be neighborhood in G if xy is an bridge of G. A simple graph is a graph without loops and multiple bridges. A graph is a connected graph if, for each pair of junctions, there exists at least one single path which joins them. A finite graph is a graph in which the junction set and the bridge set are finite sets. An undirected graph is graph, i.e., a set of objects (called junctions or bridges) that are connected together, where all the bridges are bidirectional. The degree of a junction u is the number of bridges incident to u.

Next, we define a few important families of graphs. For $n \geq 2$, the path is a connected graph consisting of two junctions with degree 1 and $n - 2$ junctions of degree 2 and is denoted by P_n. For $n \geq 3$, the cycle is a connected graph consisting of n junctions, each of degree 2 and is denoted by C_n. Note that both P_n and C_n have n junctions while P_n has n − 1 bridges and C_n has n bridges.

The following definitions and theorems are used in our study.

Definition 3.1 (Rosa [26]) If the junctions or bridges or both of the graph are assigned valued subject to certain conditions it is known as *graph labeling*.

Definition 3.2 (Rosenfeld [27]) Let V be a set. A *fuzzy graph* $G=(\sigma,\mu)$ is a pair of functions $\sigma : V \to [0,1]$ and $\mu : V \times V \to [0,1]$, where for all $u,v \in V$, we have $\mu(u,v) \leq \sigma(u) \wedge \sigma(v)$.

Definition 3.3 (Nagoor gani and Rajalaxmi [23]) Let $G=(\sigma,\mu)$ be a fuzzy graph. If σ and μ are bijective functions. Then G is said to be a *fuzzy labeling graph* if the membership value if bridges and junctions are distinct and $\mu(u,v) < \sigma(u) \wedge \sigma(v)$ for all $u,v \in V$.

Definition 3.4 (Tout et al. [35]) Let $G = (V, E)$ be a graph. If $f : V \to \{1,2,\ldots,|V|\}$ is an one-to-one correspondence function. Then f is said to be a *prime labeling* if for each $e = uv \in E$, we have $\gcd(f(u), f(v)) = 1$. The graph that admits a prime labeling is called a *prime graph*.

Definition 3.5 (Annadurai and Megala [3]) Let $G=(V,E)$ be a graph with p junctions and q bridges. An one-to-one correspondence $f : V(G) \to \{1,2,\ldots,p\}$ is said to be a *prime combination labeling* if

(i) for each pair of neighborhood junctions u and v, $\gcd\left(f(u), f(v)\right)=1$, and

(ii) the induced bridge labeling $P_f(uv)$ equals $\binom{f(u)}{f(v)}$ or $\binom{f(v)}{f(u)}$ according as $f(u) > f(v)$ or $f(v) > f(u)$ is injective onto the set of natural numbers.

A graph with a prime combination labeling is called a prime combination graph.

Definition 3.6 Let $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ be any two graphs. The *Cartesian product* $G_1 \times G_2$ of graphs G_1 and G_2 is a graph such that

(i) The junction set of $G_1 \times G_2$ is the Cartesian product $V(G_1)\times V(G_2)$; and

(ii) The junctions (u_1,v_1) and (u_2,v_2) are neighborhood in $G_1 \times G_2$ if and only if either

(a) $u_1 = u_2$ and v_1 is neighborhood to v_2 in G_2, or

(b) $v_1 = v_2$ and u_1 is neighborhood to u_2 in G_1.

Definition 3.7 (Acharya and Gill [2]) The Cartesian product $P_n \times P_m$, where $m \le n$, is called a *grid graph*, where P_n is a path of order n.

Theorem 3.1 (Sundaram et al. [33]) The planer grid $P_m \times P_n$ has a prime labeling.

Theorem 3.2 (Kanetkar [18]) Let n is an odd prime. Then,

(i) If $n = 5$ *or* $n \equiv 3$ *or* $9 \pmod{10}$ and $(n+1)^2+1$ is a prime, then $P_{n+1} \times P_{n+1}$ has a prime labeling.

(ii) If n is not congruent to $2 \pmod{7}$, then $P_n \times P_{n+2}$ has a prime labeling.

Definition 3.8 The Cartesian product $P_n \times P_2$ is called a *ladder*, where P_n is the path on n junctions.

Theorem 3.3 (Berliner et al. [5]) The ladder graphs are prime.

Definition 3.9 A cycle of order n is denoted by C_n. The Cartesian product $C_n \times P_2$ is called *prism* graph.

Theorem 3.4 (Prajapati and Gajjar [24]) The prism graphs are prime.

Definition 3.10 The set of junctions neighborhood to a junction u of G is denoted by $N(u)$. Duplication of a junction v_k of a graph G produces a new graphs G_k by adding a junction v'_k with $N(v'_k) = N(v_k)$. The graph obtained by duplicating all the junctions of a graph G is called *duplication* of G.

Definition 3.11 (Watkins [36]) Let $n \ge 3$ and $k \in Z_n - \{0\}$ be the integers. The *generalized Petersen graph* $P(n,k)$ is defined on the set $\{v_i, u_i : i \in Z_n\}$ of $2n$ junctions, with the neighborhoods given by $v_i v_{i+1}, v_i u_i, u_i u_{i+k}$ for $1 \le i \le n$, subscripts modulo n.

4. Fuzzy prime labeling of graphs

Now, we define fuzzy prime labeling as follows:

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Definition 4.1. Let G be a graph with n junctions. Let $\sigma: V(G) \to (0,1]$ be an injective map. Then σ is called a fuzzy prime labeling of G if $\gcd(\sigma(x), \sigma(y)) = 1$ where x and y are neighborhood. A graph with fuzzy prime labeling is called a fuzzy prime graph.

Fuzzy prime labeling of grid, ladder and prism graphs are proved as follows.

Theorem 4.1. Let n be an odd prime. Then the grid $P_n \times P_{n+2}$ has a fuzzy prime labeling for $n \not\equiv 2 (\bmod\ 7)$.

Proof. Let n be an odd prime such that $n \not\equiv 2 (\bmod\ 7)$. Let u_{ij} be the junctions of the grid $P_n \times P_{n+2}$ where $1 \le i \le n+2$ and $1 \le j \le n$. Let $u_{ij}u_{i,j+1}$, $1 \le i \le n+2$, $1 \le j \le n-1$ and $u_{ij}u_{i+1,j}$, $1 \le i \le n+1$, $1 \le j \le n$ be the bridges of the grid $p_n \times p_{n+2}$.

For $r = 1,2,3,\ldots$ such that $10^{r-1} < n \le 10^r$, we define the junction membership function $\sigma: V(P_n \times P_{n+2}) \to (0,1]$ as $\sigma(u_{11}) = (n^2+2n)10^{-r}$, $\sigma(u_{ij}) = 10^{-r}((j-1)n+i)$ $(1 \le i \le n-1, 2 \le j \le n-1)$, $\sigma(u_{i1}) = 10^{-r} ni$ $(i = 3,5,\ldots,n-2)$, $\sigma(u_{i1}) = 10^{-r} i$ $(i = 2,4,6,\ldots,n-1)$, $\sigma(u_{nj}) = 10^{-r} j$ $(j = 1,3,5,\ldots,n-2)$, $\sigma(u_{nj}) = 10^{-r} nj$ $(j = 2,4,6,\ldots,n-1)$, $\sigma(u_{in}) = 10^{-r}((n-1)n+i)$ $(1 \le i \le n, i \ne n-2, i \ne n)$, $\sigma(u_{nn}) = 10^{-r}(n^2-2)$, $\sigma(u_{n-2,n}) = n^2 10^{-r}$, $\sigma(u_{n+1,j}) = (n^2+j)10^{-r}$ $(1 \le j \le n)$, $\sigma(u_{n+2,j}) = (n^2+n+j)10^{-r}$ $(2 \le j \le n-1)$, $\sigma(u_{n+2,1}) = n10^{-r}$ and $\sigma(u_{n+2,n}) = (n^2+n+j)10^{-r}$.

Then σ is an injection. Let $e = xy$ be an arbitrary bridge. We claim that the above junction labeling gives a fuzzy prime labeling.

Case 1: If $e = u_{ij}u_{i,j+1}$, then, for $1 \le i \le n-1$ and $2 \le j \le n-2$,

$$\begin{aligned} gcd(\sigma(u_{ij}), \sigma(u_{i,j+1})) &= gcd(((j-1)n+i)10^{-r}, ((j+1-1)n+i)10^{-r}) \\ &= gcd((jn-n+i)10^{-r}, (jn+i)10^{-r}) \\ &= \gcd(jn-n+i, jn+i) \end{aligned}$$

Now, we claim that $\gcd(jn-n+i, jn+i) = 1$. Suppose that $d | (jn-n+i)$ and $d | (jn+i)$. Then $d | ((jn-n+i)-(jn+i)) \Rightarrow d | -n \Rightarrow d | n \Rightarrow d = 1\ or\ n$ (Since n is a prime). If $d = n$, then $n | (jn+i)$. Since $n | n$, $n | jn$. Now, $n | (jn+i)$ and $n | jn \Rightarrow n | (jn+i) - jn \Rightarrow n | i$ 5 i.e., $\Rightarrow\Leftarrow$ since $i < n$. Thus, $d = 1$. Therefore, $gcd(\sigma(u_{ij}), \sigma(u_{i,j+1})) = 1$.

Case 2: If $e = u_{ij}u_{i+1,j}$, then

$$\begin{aligned} gcd\left(\sigma\left(u_{ij}\right),\sigma\left(u_{i+1,j}\right)\right) &= gcd\left(\left((j-1)n+i\right)10^{-r},\left((j-1)n+i+1\right)10^{-r}\right) \\ &= gcd\left((jn-n+i)10^{-r},(jn-n+i+1)10^{-r}\right) \\ &= \gcd(jn-n+i, jn-n+i+1) \\ &= 1 \end{aligned}$$

for $1 \le i \le n-2$ and $2 \le j \le n-1$ (Since $jn-n+i$ and $jn-n+i+1$ are consecutive integer).

Case 3: If $e = u_{11}u_{21}$. Suppose that $d\left|\left(n^2+2n\right)\right.$ and $d\left|2\right.$. Then $d = 1$ *or* 2. Now, $d\left|2\right. \Rightarrow d\left|2n\right. \Rightarrow d\left|\left(n^2+2n-2n\right)\right. \Rightarrow d\left|n^2\right.$. If $d=2$, $2\left|n^2\right.$ which is not possible. Thus, $d=1$. Therefore, $\gcd\left(\left(n^2+2n\right)10^{-r}, 2\times 10^{-r}\right)=1$. Hence $gcd\left(\sigma\left(u_{11}\right),\sigma\left(u_{21}\right)\right)=1$.

Case 4: If $e = u_{11}u_{12}$. If $d\left|\left(n^2+2n\right)\right.$ and $d\left|\left(n+1\right)\right.$, then $d\left|n(n+1)\right. \Rightarrow d\left|\left\{\left(n^2+2n\right)-n(n+1)\right\}\right. \Rightarrow d\left|\left(n^2+2n-n^2-n\right)\right. \Rightarrow d\left|n\right.$. If $d=n$, then $n\left|(n+1)\right.$ which is impossible. Thus, $d=1$. Therefore, $\gcd\left(n^2+2n, n+1\right)=1 \Rightarrow \gcd\left(\left(n^2+2n\right)10^{-r},(n+1)10^{-r}\right)=1 \Rightarrow gcd\left(\sigma\left(u_{11}\right),\sigma\left(u_{12}\right)\right)=1$.

Case 5: If $e = u_{i1}u_{i+1,1}$ ($i = 2,3,\ldots,n-1$), then we need to verity that $gcd\left(\sigma\left(u_{i1}\right),\sigma\left(u_{i-1,1}\right)\right) = gcd\left(\sigma\left(u_{i1}\right),\sigma\left(u_{i+1,1}\right)\right) = gcd\left(\sigma\left(u_{n-1,1}\right),\sigma\left(u_{n1}\right)\right)=1$ for $i = 3,5,\ldots,n-2$. Now, we have,

(i) If $d\left|ni\right.$ and $d\left|(i-1)\right.$, then $d\left|n(i-1)\right. \Rightarrow d\left|\left(ni-n(i-1)\right)\right. \Rightarrow d\left|n\right.$. If $d=n$, $n\left|(i-1)\right.$ which cannot happen, since $i-1<n$. Thus, $d=1$. $\therefore$ $\gcd\left(ni10^{-r},(i-1)10^{-r}\right)=1$. Hence, $gcd\left(\sigma\left(u_{i1}\right),\sigma\left(u_{i-1,1}\right)\right)=1$.

(ii) If $d\left|ni\right.$ and $d\left|(i+1)\right.$, then $d\left|n(i+1)\right. \Rightarrow d\left|\left(n(i+1)-ni\right)\right. \Rightarrow d\left|n\right.$. If $d=n$, $n\left|(i+1)\right.$ which again is not possible. That is, $d=1$. $\therefore$ $\gcd\left(ni10^{-r},(i+1)10^{-r}\right)=1$. Hence, $gcd\left(\sigma\left(u_{i1}\right),\sigma\left(u_{i+1,1}\right)\right)=1$.

(iii) Clearly, $\gcd\left((n-1)10^{-r},10^{-r}\right)=1$. Thus, $gcd\left(\sigma\left(u_{n-1,1}\right),\sigma\left(u_{n1}\right)\right)=1$.

Case 6: If $e = u_{i1}u_{i2}$, then we need to verity that $gcd\left(\sigma\left(u_{i1}\right),\sigma\left(u_{i2}\right)\right)=1$ for $i = 2,3,\ldots,n$. Now, we have,

(i) Suppose $d\left|ni\right.$ and $d\left|(n+i)\right.$, then $d\left|n(n+i)\right. \Rightarrow d\left|\left(n^2+ni-ni\right)\right. \Rightarrow d\left|n^2\right. \Rightarrow d = n^2$ *or* n *or* 1. If $d=n^2$, $n^2\left|ni\right. \Rightarrow n\left|i\right.$, not possible. If $d=n$, $n\left|n+i\right. \Rightarrow n\left|(n+i-n)\right.$ (Since $n\left|n\right.$) $\Rightarrow n\left|i\right.$, i.e., $\Rightarrow\Leftarrow$ (Since $i<n$). Thus, $d=1 \Rightarrow \gcd\left(ni10^{-r},(n+i)10^{-r}\right)=1$, for $i=3,5,\ldots,n-2 \Rightarrow gcd\left(\sigma\left(u_{i1}\right),\sigma\left(u_{i2}\right)\right)=1$, for $i=3,5,\ldots,n-2$.

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(ii) If $d|i$ and $d|(i+1)$, then $d|(n+i-i) \Rightarrow d|n$. If $d=n$, $n|i$ (Which is not possible). That is, $d=1 \Rightarrow \gcd(i10^{-r},(n+i)10^{-r})=1$, for $i=2,4,\ldots,n-1 \Rightarrow gcd(\sigma(u_{i1}),\sigma(u_{i2}))=1$, for $i=2,4,\ldots,n-1$.

(iii) Clearly, $\gcd(2n10^{-r},10^{-r})=1$. Hence, $gcd(\sigma(u_{n1}),\sigma(u_{n2}))=1$.

Case 7: If $e=u_{ni}u_{n(i-1)}$. We now show that $gcd(\sigma(u_{ni}),\sigma(u_{n(i-1)}))=gcd(\sigma(u_{ni}),\sigma(u_{n(i+1)}))=gcd(\sigma(u_{n(n-1)}),\sigma(u_{nn}))=1$ for $i=3,5,\ldots,n-2$. That is to verify that

(i) $\gcd(i10^{-r},n(i-1)10^{-r})=1$.

(ii) $\gcd(i10^{-r},n(i+1)10^{-r})=1$.

(iii) $\gcd(n(n-1)10^{-r},(n^2-2)10^{-r})=1$.

In this cases, if d is a common factor, then $d|n$. But $d=n$ yields $n|i$ (First two cases) and $n|n-2$ (Last case), which should not happen.

Case 8: If $e=u_{i,n-1}u_{in}$, then, for $i=1,2,\ldots,n-1, i\neq n-2$,

$$\begin{aligned} gcd(\sigma(u_{i,n-1}),\sigma(u_{in})) &= gcd(((n-2)n+i)10^{-r},((n-1)n+i)10^{-r}) \\ &= gcd((n^2-2n+i)10^{-r},(n^2-n+i)10^{-r}) \\ &= \gcd(n^2-2n+i,n^2-n+i) \end{aligned}$$

Let $d|(n^2-2n+i)$ and $d|(n^2-n+i)$. Then $d|(n^2-2n+i-n^2+n-i) \Rightarrow d|-n \Rightarrow d|n \Rightarrow d=1$ or n (Since n is a prime). If $d=n$, then $n|(n^2-2n+i) \Rightarrow n|(n^2-2n+i-n^2+2n) \Rightarrow n|i$, not possible. Thus, $d=1$. Therefore, $gcd(\sigma(u_{i,n-1}),\sigma(u_{in}))=1$.

Case 9: If $e=u_{n-2,n-1}u_{n-2,n}$, then

$$\begin{aligned} gcd(\sigma(u_{n-2,n-1}),\sigma(u_{n-2,n})) &= gcd(((n-1-1)n+n-2)10^{-r},n^2 10^{-r}) \\ &= gcd(((n-2)n+n-2)10^{-r},n^2 10^{-r}) \\ &= \gcd((n^2-2n+n-2)10^{-r},n^2 10^{-r}) \\ &= \gcd((n^2-n-2)10^{-r},n^2 10^{-r}) \\ &= \gcd(n^2-n-2,n^2) \end{aligned}$$

Let $d \mid (n^2-n-2)$ and $d \mid n^2$. Then $d = 1\ or\ n\ or\ n^2$ (Since n is a prime). If $d = n^2$, we have $n^2 \mid (n^2-n-2) \Rightarrow n^2 \mid (n^2-n-2-n^2) \Rightarrow n \mid -n-2 \Rightarrow n \mid n+2$ which happens only when $n = 0\ or\ 1\ or\ 2$. If $d = n$, $n \mid (n^2-n-2) \Rightarrow n \mid (n^2-n-2-n^2+n) \Rightarrow n \mid -2 \Rightarrow n \mid 2$, which is impossibility. Thus, $d = 1$. Hence, $gcd(\sigma(u_{n-2,n-1}),\sigma(u_{n-2,n})) = 1$.

Case 10: If $e = u_{n+1,j}u_{n+1,j+1}$, then

$$\begin{aligned} gcd(\sigma(u_{n+1,j}),\sigma(u_{n+1,j+1})) &= gcd((n^2+j)10^{-r},(n^2+j+1)10^{-r}) \\ &= \gcd(n^2+j, n^2+j+1) \\ &= 1 \end{aligned}$$

for $1 \le j \le n-1$ (Since $n^2 + j$ and $n^2 + j + 1$ are consecutive integer).

Case 11: If $e = u_{n+2,j}u_{n+2,j+1}$, then

$$\begin{aligned} gcd(\sigma(u_{n+2,j}),\sigma(u_{n+2,j+1})) &= gcd((n^2+n+j)10^{-r},(n^2+n+j+1)10^{-r}) \\ &= \gcd(n^2+n+j, n^2+n+j+1) \\ &= 1 \end{aligned}$$

for $1 \le j \le n-1$ (Since n^2+n+j and $n^2+n+j+1$ are consecutive integer).

Case 12: If $e = u_{n+2,1}u_{n+2,2}$, then we have to verify that $\gcd(n10^{-r},(n^2+n+2)10^{-r}) = 1$. Suppose that $d \mid n$ and $d \mid (n^2+n+2)$. Then $d \mid n \Rightarrow d = 1\ or\ n$. Now, if $d = n$, we get $n \mid (n^2+n+2) \Rightarrow n \mid (n^2+n+2-n^2-n) \Rightarrow n \mid 2$ which is not possible. Thus, $d = 1$. Therefore, $\gcd(n10^{-r},(n^2+n+2)10^{-r}) = 1$. Hence $gcd(\sigma(u_{n+2,1}),\sigma(u_{n+2,2})) = 1$.

Case 13: If $e = u_{n+2,n-1}u_{n+2,n}$, then we have to prove that $\gcd((n^2+2n-1)10^{-r},(n^2+n+1)10^{-r}) = 1$. Let $d \mid (n^2+2n-1)$ and $d \mid (n^2+n+1)$, then $d \mid (n^2+2n-1-n^2-n-1) \Rightarrow d \mid n-2 \Rightarrow d \mid n(n-2) \Rightarrow d \mid (n^2-2n) \Rightarrow d \mid (n^2-2n-n^2-n-1) \Rightarrow d \mid (-3n-1) \Rightarrow d \mid (3n+1)$. But $d \mid (n-2) \Rightarrow d \mid 3(n-2) \Rightarrow d \mid (3n-6)$. But $d \mid (3n+1)$ and $d \mid (3n-6) \Rightarrow d \mid (3n+1-3n+6) \Rightarrow d \mid 7 \Rightarrow d = 1\ or\ 7$. If $d = 7$, then $7 \mid (n-2)$ which should not happen because $n \not\equiv 2 \pmod 7$. $\therefore\ d = 1$. Therefore, $\gcd((n^2+2n-1)10^{-r},(n^2+n+1)10^{-r}) = 1$. Hence $gcd(\sigma(u_{n+2,n-1}),\sigma(u_{n+2,n})) = 1$.

Case 14: If $e = u_{n+1,j}u_{n+2,j}$, then we claim that $\gcd\left(\left(n^2+j\right)10^{-r},\left(n^2+n+j\right)10^{-r}\right)=1$, $j=2,3,\ldots,n-1$. Now, $d\left|\left(n^2+j\right)\right.$ and $d\left|\left(n^2+n+j\right)\right.$. Then $d\left|\left(\left(n^2+n+j\right)-\left(n^2+j\right)\right)\right. \Rightarrow d\left|n\right. \Rightarrow$ $d=1$ or n. Suppose $d=n$, then $n\left|\left(n^2+j\right)\right. \Rightarrow n\left|\left(\left(n^2+j\right)-n^2\right)\right. \Rightarrow n\left|j\right.$ which cannot happen, since $j<n$.

$\therefore\ d=1$

$\Rightarrow \gcd\left(\left(n^2+j\right)10^{-r},\left(n^2+n+j\right)10^{-r}\right)=1,\ j=2,3,\ldots,n-1$

$\Rightarrow\ gcd\left(\sigma\left(u_{n+1,j}\right),\sigma\left(u_{n+2,j}\right)\right)=1,\ j=2,3,\ldots,n-1.$

Case 15: If $e=u_{n+1,1}u_{n+2,1}$, then we have to show that $\gcd\left(\left(n^2+1\right)10^{-r},n10^{-r}\right)=1$. If $d\left|n\right.$ and $d\left|\left(n^2+1\right)\right.$, we have $d\left|n^2\right.$ and $d\left|\left(n^2+1\right)\right. \Rightarrow d\left|\left(n^2+1-n^2\right)\right. \Rightarrow d\left|1\right. \Rightarrow d=1$. Therefore, $\gcd\left(\left(n^2+1\right)10^{-r},n10^{-r}\right)=1$. Hence $gcd\left(\sigma\left(u_{n+1,1}\right),\sigma\left(u_{n+2,1}\right)\right)=1$.

Case 16: If $e=u_{n+1,n}u_{n+2,n}$, then $gcd\left(\sigma\left(u_{n+1,n}\right),\sigma\left(u_{n+2,n}\right)\right)=gcd\left(\left(n^2+n\right)10^{-r},\left(n^2+n+1\right)10^{-r}\right)=gcd\left(n^2+n,n^2+n+1\right)=1$.

That is, there are odd primes n for which $n\not\equiv 2\left(\bmod 7\right)$, yet the prescribed labeling is successful.

Theorem 4.2. Let n be an odd prime, $n\equiv 3\ or\ 9\left(\bmod 10\right)$ and $\left(n+1\right)^2+1$ also a prime, then the grid $P_{n+1}\times P_{n+1}$ has a fuzzy prime labeling.

Proof. Let n be an odd prime such that $n\equiv 3\ or\ 9\left(\bmod 10\right)$ and $\left(n+1\right)^2+1$ are prime, Let u_{ij} be the junctions of the grid $P_{n+1}\times P_{n+1}$ where $1\le i\le n+1$ and $0\le j\le n$. Let $u_{ij}u_{i,j+1}$, $1\le i\le n+1$, $0\le j\le n-1$ and $u_{ij}u_{i+1,j}$, $1\le i\le n, 0\le j\le n$ be the bridges of the grid $p_{n+1}\times p_{n+1}$.

For $r=1,2,3,\ldots$ such that $10^{r-1}<n\le 10^r$, we define the vertex membership function $\sigma:V\left(P_{n+1}\times P_{n+1}\right)\to\left(0,1\right]$ by $\sigma\left(u_{ij}\right)=\left(\left(j-1\right)n+i\right)10^{-r}$ $\left(1\le i\le n-1, 2\le j\le n-1\right)$, $\sigma\left(u_{i1}\right)=ni10^{-r}$ $\left(i=1,3,5,\ldots,n-2\right)$, $\sigma\left(u_{i1}\right)=i10^{-r}$ $\left(i=2,4,\ldots,n-1\right)$, $\sigma\left(u_{nj}\right)=j10^{-r}$ $\left(j=1,3,5,\ldots,n-2\right)$, $\sigma\left(u_{nj}\right)=nj10^{-r}$ $\left(j=2,4,\ldots,n-1\right)$, $\sigma\left(u_{in}\right)=\left(\left(n-1\right)n+i\right)10^{-r}$ $\left(1\le i\le n, i\ne n-2, i\ne n\right)$, $\sigma\left(u_{n1}\right)=10^{-r}$, $\sigma\left(u_{nn}\right)=\left(n^2-2\right)10^{-r}$, $\sigma\left(u_{n-2,n}\right)=n^2 10^{-r}$, $\sigma\left(u_{i0}\right)=\left(\left(n+1\right)^2+1-i\right)10^{-r}$ $\left(i=1,2,\ldots,n+1\right)$, and $\sigma\left(u_{n+1,j}\right)=\left(n^2+j\right)10^{-r}$ $\left(j=1,2,\ldots,n\right)$.

Then this map σ is an injective map. In order to show that fuzzy prime labeling is fuzzy prime, we must verify that the junction labels of the newly added column 0 by u_{i0} where $1\le i\le n+1$ and the junctions

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of the row $n+1$ by $u_{n+1,j}$ where $1\le j\le n$ arising from the endpoints of the new bridges are mutually prime. Let $e=xy$ be an arbitrary bridge.

Case 1: If $e=u_{i0}u_{i1}$, $i=1,3,...,n-2$. Let $d\left|\left((n+1)^2+1-i\right)\right.$ and $d\left|ni\right.$, this implies that $d\left|\left(n^2+2n+2-i\right)\right.$ and $d\left|ni\right.$. Suppose that $n\left|\left(n^2+2n+2-i\right)\right.$, then $n\left|\left(n^2+2n+2-i-n^2-2n\right)\right.$ $\Rightarrow n\left|(2-i)\right.$ which is not possible. That is, $d\neq n$. Now, $d\left|ni\right.\Rightarrow d\left|i\right.\Rightarrow d\left|\left(i+n^2+2n+2-i\right)\right.\Rightarrow$ $d\left|\left(n^2+2n+2\right)\right.\Rightarrow d\left|\left((n+1)^2+1\right)\right.\Rightarrow d=1$ or $(n+1)^2+1$. If $d=(n+1)^2+1$. then $(n+1)^2+1|i$ which is impossible, since $i<(n+1)^2+1$. Thus, $d=1$and $gcd\left(\sigma(u_{i0}),\sigma(u_{i1})\right)=1$, $i=1,3,...,n-2$. $\therefore$ $d=1$. Thus, $\gcd\left(\left((n+1)^2+1-i\right)10^{-r},ni10^{-r}\right)=1$. Hence $gcd\left(\sigma(u_{i0}),\sigma(u_{i1})\right)=1$.

Case 2: If $e=u_{i0}u_{i1}$ and let $d\left|\left((n+1)^2+1-i\right)\right.$ and $d\left|i\right.$, then $d\left|\left(\left((n+1)^2+1-i\right)+i\right)\right.$ $\Rightarrow$ $d\left|\left((n+1)^2+1\right)\right.\Rightarrow d=1$ or $(n+1)^2+1$. Suppose that $d=(n+1)^2+1$. Then $(n+1)^2+1|i$ which cannot happen, since $i<(n+1)^2+1$. $\therefore$ $d=1$. Thus, $\gcd\left(\left((n+1)^2+1-i\right)10^{-r},i10^{-r}\right)=1$, $i=2,4,...,n-1$. Hence $gcd\left(\sigma(u_{i0}),\sigma(u_{i1})\right)=1$, $i=2,4,...,n-1$.

Case 3: If $e=u_{n,0}u_{n,1}$, then $gcd\left(\sigma(u_{n,0}),\sigma(u_{n,1})\right)=gcd\left(\left((n+1)^2+1-n\right)10^{-r},10^{-r}\right)=gcd\left((n+1)^2+1-n,1\right)=1$.

Case 4: If $e=u_{n+1,0}u_{n+1,1}$. Let $d\left|\left((n+1)^2+1-(n+1)\right)\right.$ and $d\left|\left(n^2+1\right)\right.$, then we have $d\left|\left(\left((n+1)^2+1-(n+1)\right)-\left(n^2+1\right)\right)\right.$ $\Rightarrow$ $d\left|\left(\left((n+1)^2-n\right)-\left(n^2+1\right)\right)\right.$ $\Rightarrow$ $d\left|\left(\left(n^2+2n+1-n\right)-n^2-1\right)\right.$ $\Rightarrow$ $d\left|\left(n^2+n+1-n^2-1\right)\right.$ $\Rightarrow$ $d\left|n\right.$ $\Rightarrow$ $d\left|n^2\right.$. But $d\left|\left(n^2+1\right)\right.$ $\Rightarrow$ $d\left|\left(\left(n^2+1\right)-n^2\right)\right.\Rightarrow d\left|1\right.\Rightarrow d=1$. Therefore, $\gcd\left(\left((n+1)^2+1-(n+1)\right)10^{-r},\left(n^2+1\right)10^{-r}\right)=1$. Hence $gcd\left(\sigma(u_{n+1,0}),\sigma(u_{n+1,1})\right)=1$.

Case 5: If $e=u_{n+1,j}u_{n,j}$, $j=1,3,...,n-2$. Assume that $d\left|\left(n^2+j\right)\right.$ and $d\left|j\right.$, then $d\left|\left(\left(n^2+j\right)-j\right)\right.\Rightarrow$ $d\left|n^2\right.\Rightarrow d=n^2$ or n or 1. If $d\left|n^2\right.$, then $n^2\left|j\right.$, which is not possible, since $j<n^2$. If $d=n$, we get $n\left|j\right.$, which is an impossibility (Since $j<n$). Thus, $d=1$. Therefore, $\gcd\left(\left(n^2+j\right)10^{-r},j10^{-r}\right)=1\Rightarrow$ $gcd\left(\sigma(u_{n+1,j}),\sigma(u_{n,j})\right)=1$, $j=1,3,...,n-2$.

Case 6: If $e = u_{n+1,j}u_{n,j}$, $j = 2,4,...,n-1$. Assume that $d\left|\left(n^2 + j\right)\right.$ and $d\left|nj\right.$, then $d\left|n\left(n^2 + j\right)\right. \Rightarrow$ $d\left|\left(n^3 + nj\right)\right. \Rightarrow d\left|\left(\left(n^3 + nj\right) - nj\right)\right. \Rightarrow d\left|n^3\right. \Rightarrow d = n^3 \ or \ n^2 \ or \ n \ or \ 1$. If $d = n \ or \ n^2$, then we get $n\left|j\right.$, and if $d = n^3$, then we get $n^2\left|j\right.$. But $j < n^2$ and $j < n$. Therefore $d \neq n^3, n^2 \,\&\, n$. Thus, $d = 1$. Therefore, $\gcd\left(\left(n^2 + j\right)10^{-r}, nj10^{-r}\right) = 1 \Rightarrow gcd\left(\sigma\left(u_{n+1,j}\right), \sigma\left(u_{n,j}\right)\right) = 1$, $j = 2,4,...,n-1$.

Case 7: If $e = u_{n+1,n}u_{n,n}$. If $d\left|\left(n^2 + n\right)\right.$ and $d\left|\left(n^2 - 2\right)\right.$, then $d\left|\left(\left(n^2 + n\right) - n^2 + 2\right)\right. \Rightarrow d\left|\left(n+2\right)\right. \Rightarrow$ $d\left|n\left(n+2\right)\right. \Rightarrow d\left|\left(n^2 + 2n\right)\right. \Rightarrow d\left|\left(\left(n^2 + 2n\right) - \left(n^2 - 2\right)\right)\right. \Rightarrow d\left|\left(2n+2\right)\right.$. But $d\left|\left(n+2\right)\right. \Rightarrow$ $d\left|2\left(n+2\right)\right. \Rightarrow d\left|\left(2n+4\right)\right. \Rightarrow d\left|\left(\left(2n+4\right) - \left(2n+2\right)\right)\right. \Rightarrow d\left|2\right. \Rightarrow d\left|\left(2 - \left(n+2\right)\right)\right. \Rightarrow d\left|-n\right. \Rightarrow$ $d\left|n\right.$. If $d = n$, then $n\left|2\right.$, an impossibility (Since $2 < n$). Thus, $d = 1$. Therefore, $\gcd\left(\left(n^2 + n\right)10^{-r}, \left(n^2 - 2\right)10^{-r}\right) = 1 \Rightarrow gcd\left(\sigma\left(u_{n+1,n}\right), \sigma\left(u_{n,n}\right)\right) = 1$.

Case 8: If $e = u_{i0}u_{i+1,0}$, then for $1 \le i \le n$,

$$\begin{aligned} gcd\left(\sigma\left(u_{i0}\right), \sigma\left(u_{i+1,0}\right)\right) &= gcd\left(\left(\left(n+1\right)^2 + 1 - i\right)10^{-r}, \left(\left(n+1\right)^2 + 1 - \left(i+1\right)\right)10^{-r}\right) \\ &= gcd\left(\left(n^2 + 2n + 1 + 1 - i\right)10^{-r}, \left(n^2 + 2n + 1 + 1 - i - 1\right)10^{-r}\right) \\ &= gcd\left(\left(n^2 + 2n + 2 - i\right)10^{-r}, \left(n^2 + 2n + 1 - i\right)10^{-r}\right) \\ &= \gcd\left(n^2 + 2n + 2 - i, n^2 + 2n + 1 - i\right) \\ &= 1 \ (Since \ n^2 + 2n + 2 - i \ and \ n^2 + 2n + 1 - i \ are \ consecutive \ integer). \end{aligned}$$

Case 9: If $e = u_{n+1,j}u_{n+1,j+1}$, then $gcd\left(\sigma\left(u_{n+1,j}\right), \sigma\left(u_{n+1,j+1}\right)\right) = gcd\left(\left(n^2 + j\right)10^{-r}, \left(n^2 + j + 1\right)10^{-r}\right) = gcd\left(n^2 + j, n^2 + j + 1\right) = 1$ for $1 \le i \le n-1$ (Since $n^2 + j$ and $n^2 + j + 1$ are consecutive integer).

That is, there are odd primes n such that $n \equiv 3 \ or \ 9 \pmod{10}$ for which $\left(n+1\right)^2 + 1$ is a prime, yet the prescribed labeling is successful.

Theorem 4.3. If $n+1$ is a prime, then the ladder $P_n \times P_2$ has a fuzzy prime labeling.

Proof. Let $n+1$ be a prime. Let $P_n \times P_2$ be the ladder with junctions $v_1, v_2, ..., v_n$ and $u_1, u_2, ..., u_n$, where v_i is neighborhood to u_i for $1 \le i \le n$, v_i is neighborhood to v_{i+1} for $1 \le i \le n-1$, and u_i is neighborhood to u_{i+1} for $1 \le i \le n-1$.

For $r = 1,2,3,...$ such that $10^{r-1} < n \le 10^r$, we define the junction membership function $\sigma : V(P_n \times P_2) \to (0,1]$ by $\sigma(v_i) = i10^{-r}$ $(1 \le i \le n)$, $\sigma(u_i) = (n+i+1)10^{-r}$ $(1 \le i \le n-1)$ and $\sigma(u_n) = (n+1)10^{-r}$.

Then σ is an injection. Let $e = xy$ be an arbitrary bridge of $P_n \times P_2$. To prove $\gcd(\sigma(x), \sigma(y)) = 1$ we have the following cases:

Case 1: If $e = v_i v_{i+1}$, then $gcd(\sigma(v_i), \sigma(v_{i+1})) = gcd(i10^{-r}, (i+1)10^{-r}) = gcd(i, i+1) = 1$, for $1 \le i \le n-1$ (Since i and $i+1$ are consecutive integer).

Case 2: If $e = u_i u_{i+1}$, then $gcd(\sigma(u_i), \sigma(u_{i+1})) = gcd((n+i+1)10^{-r}, (n+i+2)10^{-r}) = gcd(n+i+1, n+i+2) = 1$, for $1 \le i \le n-2$ (Since $n+i+1$ and $n+i+2$ are consecutive integer).

Case 3: If $e = u_{n-1}u_n$, then if $d|2n$ and $d|(n+1)$. This implies that $d|2(n+1) \Rightarrow d|(2n+2) \Rightarrow d|(2n+2-2n) \Rightarrow d|2 \Rightarrow d = 1$ or 2. If $d = 2$, then $2|(n+1)$ which is impossible because $n+1$ is a prime. That is, $gcd(2n10^{-r}, (n+1)10^{-r}) = 1$. Hence, $gcd(\sigma(u_{n-1}), \sigma(u_n)) = 1$.

Case 4: If $e = v_i u_i, 1 \le i \le n-1$. If $d|i$ and $d|((n+1)+i)$. This implies that $d|(((n+1)+i)-i) \Rightarrow d|(n+1)$. Since $n+1$ is a prime, $n+1$ does not divide i. That is, $gcd(n+1, i) = 1 \Rightarrow gcd(i10^{-r}, (n+1+i)10^{-r}) = 1$. Hence, $gcd(\sigma(v_i), \sigma(u_i)) = 1$.

Case 4: If $e = v_n u_n$. Then $gcd(\sigma(v_n), \sigma(u_n)) = gcd(n10^{-r}, (n+1)10^{-r}) = gcd(n, n+1) = 1$.

Hence the ladder $P_n \times P_2$ has a fuzzy prime labeling if $n+1$ is a prime.

Theorem 4.4. If $p \ge 3$ is a prime number, then the prism graph $C_{p-1} \times P_2$ is fuzzy prime labeling.

Proof. Let $p \ge 3$ be a prime number. Let $v_{11}, v_{12}, v_{13}, ..., v_{1,p-1}$ be the junctions of one cycle and $v_{21}, v_{22}, v_{23}, ..., v_{2,p-1}$ be the junctions of other cycle and a junction v_{1i} is joined with v_{2i} by an bridge for $i = 1,2,..., p-1$.

For $r = 1,2,3,...$ such that $10^{r-1} < n \le 10^r$, we define the junction membership function $\sigma : V(C_{p-1} \times P_2) \to (0,1]$ by $\sigma(v_{ij}) = j10^{-r}$ $(i = 1, j = 1,2,..., p-1)$, $\sigma(v_{ij}) = (p+j)10^{-r}$ $(i = 2, j = 1,2,..., p-2)$ and $\sigma(v_{ij}) = p10^{-r}$ $(i = 2, j = p-1)$.

Then clearly σ is an injection. Let e be an any bridge of $C_{p-1} \times P_2$. To claim σ is a fuzzy prime labeling of $C_{p-1} \times P_2$ we have the following cases:

Case 1: If $e = v_{1j}v_{1(j+1)}$, then $gcd\left(\sigma\left(v_{1j}\right),\sigma\left(v_{1(j+1)}\right)\right) = gcd\left(j10^{-r},(j+1)10^{-r}\right) = gcd\left(j,j+1\right) = 1$, for $1 \le j \le p-2$ (Since j and $j+1$ are consecutive integer).

Case 2: If $e = v_{2j}v_{2(j+1)}$, then $gcd\left(\sigma\left(v_{2j}\right),\sigma\left(v_{2(j+1)}\right)\right) = gcd\left((p+j)10^{-r},(p+j+1)10^{-r}\right) = gcd\left(p+j,p+j+1\right) = 1$, for $1 \le j \le p-3$ (Since $p+j$ and $p+j+1$ are consecutive integer).

Case 3: If $e = v_{1(p-1)}v_{11}$, then $gcd\left(\sigma\left(v_{1(p-1)}\right),\sigma\left(v_{11}\right)\right) = gcd\left((p-1)10^{-r},10^{-r}\right) = gcd\left(p-1,1\right) = 1$.

Case 4: If $e = v_{2(p-2)}v_{2(p-1)}$, then $gcd\left(\sigma\left(v_{2(p-2)}\right),\sigma\left(v_{2(p-1)}\right)\right) = gcd\left((2p-2)10^{-r},p10^{-r}\right) = gcd\left(2(p-1),p\right) = 1$.

Case 5: If $e = v_{2(p-1)}v_{21}$, then $gcd\left(\sigma\left(v_{2(p-1)}\right),\sigma\left(v_{21}\right)\right) = gcd\left(p10^{-r},(p+1)10^{-r}\right) = gcd\left(p,p+1\right) = 1$.

Case 6: If $e = v_{1j}v_{2j}$, then for $1 \le j \le p-2$, $gcd\left(\sigma\left(v_{1j}\right),\sigma\left(v_{2j}\right)\right) = gcd\left(j10^{-r},(p+j)10^{-r}\right) = gcd\left(j,p+j\right) = gcd\left(j,p\right) = 1$.

Case 7: If $e = v_{1(p-1)}v_{2(p-1)}$, then $gcd\left(\sigma\left(v_{1(p-1)}\right),\sigma\left(v_{2(p-1)}\right)\right) = gcd\left((p-1)10^{-r},p10^{-r}\right) = gcd\left(p-1,p\right) = 1$.

Thus, $C_{p-1} \times P_2$ admits a fuzzy prime labeling. Hence the theorem.

Next we have a fuzzy prime labeling of generalized Petersen graphs are proved as follows.

Theorem 4.5. If $2^t + 1$ is a prime for $t \ge 2$, then generalized Petersen graph $P(n,k)$ is a fuzzy prime graph, where $n = 2^t + 2$ and $k = 2^{t-1} + 1$.

Proof. Let $v_1, v_2, \ldots, v_n$ be the outer junctions and $u_1, u_2, \ldots, u_n$ be the inner junctions of $P(n,k)$. For $r = 1,2,3,\ldots$ such that $10^{r-1} < n \le 10^r$, we define the junction membership function $\sigma : V(P(n,k)) \to (0,1]$ by $\sigma(v_i) = i10^{-r}$ $(1 \le i \le 2^t + 2)$, $\sigma(u_1) = (2^{t+1} + 4)10^{-r}$, $\sigma(u_{2i-1}) = (2^t + 4i - 2)10^{-r}$ $(i = 2,3,\ldots,2^{t-2} + 1)$, $\sigma\left(u_{2^{t-1}+2i+1}\right) = (2^t + 4i)10^{-r}$ $(i = 1,2,3,\ldots,2^{t-2})$, $\sigma(u_{2i}) = (2^t + 4i + 1)10^{-r}$ $(i = 1,2,3,\ldots,2^{t-2})$ and $\sigma\left(u_{2^{t-1}+2i}\right) = (2^t + 4i - 1)10^{-r}$ $(i = 1,2,3,\ldots,2^{t-2} + 1)$.

Then clearly σ is an injective function. Let e be an arbitrary bridge of $P(n,k)$. To prove σ is a fuzzy prime labeling of $P(n,k)$ we have the following cases:

Case 1: If $e = v_i v_{i+1}$, then $gcd(\sigma(v_i), \sigma(v_{i+1})) = gcd(i10^{-r}, (i+1)10^{-r}) = gcd(i, i+1) = 1$, for $i = 1, 2, \ldots, 2^t + 1$ (Since i and $i+1$ are consecutive integer).

Case 2: If $e = v_1 v_{2^t+2}$, then $gcd(\sigma(v_1), \sigma(v_{2^t+2})) = gcd(10^{-r}, (2^t+2)10^{-r}) = gcd(1, 2^t+2) = 1.$

Case 3: If $e = v_1 u_1$, then $gcd(\sigma(v_1), \sigma(u_1)) = gcd(10^{-r}, (2^{t+1}+4)10^{-r}) = gcd(1, 2^{t+1}+4) = 1.$

Case 4: If $e = v_{2i-1} u_{2i-1}$, then $gcd(\sigma(v_{2i-1}), \sigma(u_{2i-1})) = gcd((2i-1)10^{-r}, (2^t+4i-2)10^{-r}) = gcd(2i-1, 2^t+4i-2) = gcd(2i-1, 2^t) = 1$ for $i = 2, 3, \ldots, 2^{t-2} + 1$ (Since $2i-1$ is not multiple of 2) .

Case 5: If $e = v_{2^{t-1}+2i+1} u_{2^{t-1}+2i+1}$, then, for $i = 2, 3, \ldots, 2^{t-2}$,

$$gcd(\sigma(v_{2^{t-1}+2i+1}), \sigma(u_{2^{t-1}+2i+1})) = gcd((2^{t-1}+2i+1)10^{-r}, (2^t+4i)10^{-r}) = gcd(2^{t-1}+2i+1, 2^t+4i)$$
$$= gcd(2^{t-1}+2i+1, -1) = 1.$$

Case 6: If $e = v_{2i} u_{2i}$, then, $gcd(\sigma(v_{2i}), \sigma(u_{2i})) = gcd(2i10^{-r}, (2^t+4i+1)10^{-r}) = gcd(2i, 2^t+4i+1) = gcd(2i, 2^t+1) = 1$, as $2t+1$ is prime, for $i = 1, 2, \ldots, 2^{t-2}$ (Since $2i$ is not prime).

Case 7: If $e = v_{2^{t-1}+2i} u_{2^{t-1}+2i}$, then, for $i = 1, 2, \ldots, 2^{t-2} + 1$,

$$gcd(\sigma(v_{2^{t-1}+2i}), \sigma(u_{2^{t-1}+2i})) = gcd((2^{t-1}+2i)10^{-r}, (2^t+4i-1)10^{-r}) = gcd(2^{t-1}+2i, 2^t+4i-1)$$
$$= gcd(2^{t-1}+2i, -1) = 1.$$

Case 8: If $e = u_1 u_{2^{t-1}+2}$, then $gcd(\sigma(u_1),\ \sigma(u_{2^{t-1}+2})) = gcd(2^{t+1}+4, 2^t+3) = gcd(-2, 2^t+3) = 1$ (Since 2^t+3 is odd).

Case 9: If $e = u_{2i-1} u_{2^{t-1}+2i}$, then $gcd(\sigma(u_{2i-1}), \sigma(u_{2^{t-1}+2i})) = gcd((2^t+4i-2)10^{-r}, (2^t+4i-1)10^{-r}) = gcd(2^t+4i-2, 2^t+4i-1) = 1$, for $i = 2, 3, \ldots, 2^{t-2} + 1$ (Since 2^t+4i-2 and 2^t+4i-1 are consecutive integer).

Case 10: If $e = u_{2i}u_{2^{t-1}+2i+1}$, then $gcd\left(\sigma\left(u_{2i}\right),\sigma\left(u_{2^{t-1}+2i+1}\right)\right) = gcd\left(\left(2^t+4i+1\right)10^{-r},\left(2^t+4i\right)10^{-r}\right) = gcd\left(2^t+4i+1,2^t+4i\right)=1$, for $i=1,2,...,2^{t-2}$ (Since 2^t+4i+1 and 2^t+4i are consecutive integer).

Thus, $P(n,k)$ admits a fuzzy prime labeling. Hence the theorem.

Theorem 4.6. If $8n+5$ and $8n+9$ are prime, then generalized Petersen graph $P(4n+2,2n+1)$ is a fuzzy prime graph.

Proof. Let $v_1,v_2,v_3,...,v_{4n+2}$ be the outer junctions and $u_1,u_2,u_3,...,u_{4n+2}$ be the inner junctions of $P(4n+2,2n+1)$.

For $r=1,2,3,...$ such that $10^{r-1}<n\le 10^r$, we define the junction membership function $\sigma : V\left(P(4n+2,2n+1)\right)\to(0,1]$ by $\sigma(v_{2i+1})=(4i+1)10^{-r}\ (0\le i\le n)$, $\sigma(v_{2i+2})=(8n-4i+4)10^{-r}\ (0\le i\le n-1)$, $\sigma(v_{2n+2i+2})=(4i+4)10^{-r}\ (0\le i\le n)$, $\sigma(v_{2n+2i+3})=(8n-4i+1)10^{-r}\ (0\le i\le n-1)$, $\sigma(u_{2i+1})=(4i+2)10^{-r}\ (0\le i\le n)$, $\sigma(u_{2i+2})=(8n-4i+3)10^{-r}\ (0\le i\le n-1)$, $\sigma(u_{2n+2i+2})=(4i+3)10^{-r}\ (0\le i\le n)$ and $\sigma(u_{2n+2i+3})=(8n-4i+2)10^{-r}\ (0\le i\le n-1)$.

Then σ is an injection. Let $e=xy$ be an arbitrary bridge of $P(4n+2,2n+1)$. We now show that the labeling given above is fuzzy prime for the generalized Petersen graph $P(4n+2,2n+1)$ if $8n+5$ and $8n+9$ are prime.

Case 1: If $e=v_{2i+1}v_{2i+2}$, then $gcd\left(\sigma(v_{2i+1}),\ \sigma(v_{2i+2})\right)=gcd\left((4i+1)10^{-r},(8n-4i+4)10^{-r}\right)=gcd\left(4i+1,8n-4i+4\right)=gcd(4i+1,8n+5)=1$ for $0\le i\le n-1$ as $8n+5$ is prime (Since $4i+1<8n+5$).

Case 2: If $e=v_{2i+2}v_{2i+3}$, then $gcd\left(\sigma(v_{2i+2}),\sigma(v_{2i+3})\right)=gcd\left(\sigma(v_{2i+2}),\sigma\left(v_{2(i+1)+1}\right)\right)=gcd\left(8n-4i+4,4i+5\right)=gcd\left(8n+9,4i+5\right)=1$, for $0\le i\le n-1$ as $8n+9$ is prime (Since $8n+9<4i+5$).

Case 3: If $e=v_{2n+1}v_{2n+2}$, then $gcd\left(\sigma(v_{2n+1}),\sigma(v_{2n+2})\right)=gcd\left((4n+1)10^{-r},4\times 10^{-r}\right)=gcd\left(4n+1,4\right)=1$ (Since $4n+1$ is not multiple of 2).

Case 4: If $e=v_{2n+2i+2}v_{2n+2i+3}$, then, for $0\le i\le n-1$,

$$gcd\left(\sigma(v_{2n+2i+2}),\ \sigma(v_{2n+2i+3})\right)=gcd\left((4i+4)10^{-r},(8n-4i+1)10^{-r}\right)=gcd(4i+4,8n-4i+1)$$
$$=gcd\left(4i+4,8n+5\right)=1$$

(Since $4i+4<8n+5$ and $8n+5$ is prime).

Case 5: If $e = v_{2n+2i+3}v_{2n+2i+4}$, then for $0 \le i \le n-1$,
$$gcd\left(\sigma\left(v_{2n+2i+3}\right),\sigma\left(v_{2n+2i+4}\right)\right) = gcd\left(\sigma\left(v_{2n+2i+3}\right),\sigma\left(v_{2n+2(i+1)+2}\right)\right) = gcd\left((8n-4i+1)10^{-r},(4i+8)10^{-r}\right)$$
$$= gcd\left(8n-4i+1,4i+8\right) = gcd\left(8n+9,4i+8\right) = 1$$

(Since $4i+8 < 8n+9$ and $8n+9$ is prime).

Case 6: If $e = v_{4n+2}v_1$, then $gcd\left(\sigma\left(v_{4n+2}\right),\sigma\left(v_1\right)\right) = gcd\left((4n+4)10^{-r},10^{-r}\right) = gcd\left(4n+4,1\right) = 1$.

Case 7: If $e = v_{2i+1}u_{2i+1}$, then $gcd\left(\sigma\left(v_{2i+1}\right),\sigma\left(u_{2i+1}\right)\right) = gcd\left((4i+1)10^{-r},(4i+2)10^{-r}\right) = gcd\left(4i+1,4i+2\right) = 1$ for $0 \le i \le n$ (Since $4i+1$ and $4i+2$ are consecutive integer).

Case 8: If $e = v_{2i+2}u_{2i+2}$, then $gcd\left(\sigma\left(v_{2i+2}\right),\sigma\left(u_{2i+2}\right)\right) = gcd\left((8n-4i+4)10^{-r},(8n-4i+3)10^{-r}\right) = gcd\left(8n-4i+4,8n-4i+3\right) = 1$,
for $0 \le i \le n-1$ (Since $8n-4i+4$ and $8n-4i+3$ are consecutive integer).

Case 9: If $e = v_{2n+2i+2}u_{2n+2i+2}$, then $gcd\left(\sigma\left(v_{2n+2i+2}\right),\sigma\left(u_{2n+2i+2}\right)\right) = gcd\left((4i+4)10^{-r},(4i+3)10^{-r}\right) = gcd\left(4i+4,4i+3\right) = 1$, for $0 \le i \le n$ (Since $4i+3$ and $4i+4$ are consecutive integer).

Case 10: If $e = v_{2n+2i+3}u_{2n+2i+3}$, then $gcd\left(\sigma\left(v_{2n+2i+3}\right),\sigma\left(u_{2n+2i+3}\right)\right) = gcd\left((8n-4i+1)10^{-r},(8n-4i+2)10^{-r}\right) = gcd\left(8n-4i+1,8n-4i+2\right) = 1$, for $0 \le i \le n-1$ (Since $8n-4i+1$ and $8n-4i+2$ are consecutive integer).

Case 11: If $e = u_{2i+1}u_{2n+2i+2}$, then $gcd\left(\sigma\left(u_{2i+1}\right),\sigma\left(u_{2n+2i+2}\right)\right) = gcd\left((4i+2)10^{-r},(4i+3)10^{-r}\right) = gcd\left(4i+2,4i+3\right) = 1$, for $0 \le i \le n$ (Since $4i+2$ and $4i+3$ are consecutive integer).

Case 12: If $e = u_{2i+2}u_{2n+2i+3}$, then
$$gcd\left(\sigma\left(u_{2i+2}\right),\sigma\left(u_{2n+2i+3}\right)\right) = gcd\left((8n-4i+3)10^{-r},(8n-4i+2)10^{-r}\right) = gcd\left(8n-4i+3,8n-4i+2\right)$$
$$= gcd\left(8n-4i+3,8n-4i+2\right) = 1,$$

for $0 \le i \le n-1$ (Since $8n-4i+3$ and $8n-4i+2$ are consecutive integer).

Thus, $P\left(4n+2,2n+1\right)$ admits a fuzzy prime labeling.

Theorem 4.7. If $2n+1$, $4n+3$ and $6n+5$ are prime, then generalized Petersen graph $P\left(4n+4,2n+1\right)$ is a fuzzy prime graph.

Proof. Let $v_1, v_2, v_3, ..., v_{4n+4}$ be the outer junctions and $u_1, u_2, u_3, ..., u_{4n+4}$ be the inner junctions of $P(4n+4, 2n+1)$. For $r = 1,2,3,...$ such that $10^{r-1} < n \le 10^r$, we define the junction membership function $\sigma : V(P(4n+4, 2n+1)) \to (0,1]$ by $\sigma(v_i) = i10^{-r}$ $(1 \le i \le 4n+4)$, $\sigma(u_{2i}) = (4n+2i+3)10^{-r}$ $(1 \le i \le 2n+2)$, $\sigma(u_{2n+2i+3}) = (4n+2i+4)10^{-r}$ $(1 \le i \le n)$ and $\sigma(u_{2i-2n-1}) = (4n+2i+4)10^{-r}$ $(n+1 \le i \le 2n+2)$.

Then clearly σ is an injective function. Let e be an arbitrary bridge of $P(4n+4, 2n+1)$. To prove σ is a fuzzy prime labeling of $P(4n+4, 2n+1)$ we have the following cases:

Case 1: If $e = v_i v_{i+1}$, then $gcd(\sigma(v_i), \sigma(v_{i+1})) = gcd(i10^{-r}, (i+1)10^{-r}) = gcd(i, i+1) = 1$, for $1 \le i \le 4n+3$(Since i and $i+1$ are consecutive integer).

Case 2: If $e = v_{4n+4} v_1$, then $gcd(\sigma(v_{4n+4}), \sigma(v_1)) = gcd((4n+4)10^{-r}, 10^{-r}) = gcd(4n+4, 1) = 1.$

Case 3: If $e = u_{2i} u_{2n+3+2i}$, then $gcd(\sigma(u_{2i}), \sigma(u_{2n+3+2i})) = gcd((4n+3+2i)10^{-r}, (4n+4+2i)10^{-r}) = gcd(4n+3+2i, 4n+4+2i) = 1$ for $1 \le i \le n$ (Since $4n+3+2i$ and $4n+4+2i$ are consecutive integer).

Case 4: If $e = u_{2i} u_{2i-2n-1}$, then $gcd(\sigma(u_{2i}), \sigma(u_{2i-2n-1})) = gcd((4n+3+2i)10^{-r}, (4n+4+2i)10^{-r}) = gcd(4n+3+2i, 4n+4+2i) = 1$ for $n+1 \le i \le 2n+2$ (Since $4n+3+2i$ and $4n+4+2i$ are consecutive integer).

Case 5: If $e = u_{2n+3+2i} u_{2i+2}$, then $gcd(\sigma(u_{2n+3+2i}), \sigma(u_{2i+2})) = gcd((4n+4+2i)10^{-r}, (4n+5+2i)10^{-r}) = gcd(4n+4+2i, 4n+5+2i) = 1$, for $1 \le i \le n$ (Since $4n+4+2i$ and $4n+5+2i$ are consecutive integer).

Case 6: If $e = u_{2i-2n-1} u_{2i+2}$, then $gcd(\sigma(u_{2i-2n-1}), \sigma(u_{2i+2})) = gcd((4n+4+2i)10^{-r}, (4n+5+2i)10^{-r}) = gcd(4n+4+2i, 4n+5+2i) = 1$, for $n+1 \le i \le 2n+1$ (Since $4n+4+2i$ and $4n+5+2i$ are consecutive integer).

Case 7: If $e = u_{2n+3} u_2$, then $gcd(\sigma(u_{2n+3}), \sigma(u_2)) = gcd((8n+8)10^{-r}, (4n+5)10^{-r}) = gcd(8n+8, 4n+5) = gcd(-2, 4n+5) = 1$ (Since $4n+5$ is an odd).

Case 8: If $e = v_{2i} u_{2i}$, then $gcd(\sigma(v_{2i}), \sigma(u_{2i})) = gcd(2i10^{-r}, (4n+3+2i)10^{-r}) = gcd(2i, 4n+3+2i) = gcd(2i, 4n+3) = 1$, for $1 \le i \le 2n+2$ (Since $4n+3$is prime and $2i$ is odd).

Case 9: If $e = v_{2n+3+2i}u_{2n+3+2i}$, then

$$gcd\left(\sigma\left(v_{2n+3+2i}\right),\sigma\left(u_{2n+3+2i}\right)\right) = gcd\left((2n+3+2i)10^{-r},(4n+4+2i)10^{-r}\right) = gcd\left(2n+3+2i,4n+4+2i\right)$$
$$= gcd\left(2n+3+2i,2n+1\right) = gcd\left(2+2i,2n+1\right) = 1$$

for $1 \le i \le n$ (Since $2n+1$ is prime and $2+2i$ is even).

Case 10: If $e = v_{2i-2n-1}u_{2i-2n-1}$, then

$$gcd\left(\sigma\left(v_{2i-2n-1}\right),\sigma\left(u_{2i-2n-1}\right)\right) = gcd\left((2i-2n-1)10^{-r},(4n+4+2i)10^{-r}\right) = gcd\left(2i-2n-1,4n+4+2i\right)$$
$$= gcd\left(2i-2n-1,6n+5\right) = 1$$

for $n+1 \le i \le 2n+2$ (Since $6n+5$ is prime).

Thus, $P(4n+4,2n+1)$ admits a fuzzy prime labeling which complete the proof.

Now we have a fuzzy prime labeling of duplicate graphs are proved as follows.

Theorem 4.8. If n is even. Then the graph G obtained by duplicating all the junctions in the rim of helm H_n is a fuzzy prime graph.

Proof. Let $c,u_i,v_i\,(1 \le i \le n)$ be the junctions of H_n and let $cu_i,u_iv_i\,(1 \le i \le n)$, $u_iu_{i+1}\,(1 \le i \le n-1)$ and u_1u_n be the bridges of H_n. Let G be the graph obtained by duplicating all the rim junctions in H_n and let the new junctions be $u'_1,u'_2,\ldots,u'_n$. Then $c,u_i,v_i,u'_i\,(1 \le i \le n)$ are the junctions of G and $\{cu_i,cu'_i,u_iv_i,v_iu'_i\,(1 \le i \le n)\}$, $\{u_iu_{i+1},u_iu'_{i+1},u'_i\,u_{i+1}\,(1 \le i \le n-1)\}$ and $\{u_1u_n,u_nu'_1,u'_n\,u_1\}$ are the bridges of G.

For $r = 1,2,3,\ldots$ such that $10^{r-1} < n \le 10^r$, we define the junction membership function $\sigma : V(G) \to (0,1]$ by $\sigma(c) = 10^{-r}$, $\sigma(u_i) = (3i-1)10^{-r}\ (1 \le i \le n, i \ne 2(\bmod 5))$, $\sigma(v_i) = 3i10^{-r}\ (1 \le i \le n)$, $\sigma(u'_i) = (3i+1)10^{-r}\ (1 \le i \le n, i \ne 2(\bmod 5))$, $\sigma(u_i) = (3i+1)10^{-r}\ (1 \le i \le n, i \equiv 2(\bmod 5))$ and $\sigma(u'_i) = (3i-1)10^{-r}\ (1 \le i \le n, i \equiv 2(\bmod 5))$.

Then clearly the function σ is an injective. Let e be an arbitrary bridge of G. To show σ is a fuzzy prime labeling of G we have the following cases:

Case 1: If $e = cu_i$, then $gcd\left(\sigma(c),\sigma(u_i)\right) = gcd\left(10^{-r},(3i-1)10^{-r}\right) = gcd(1,3i-1) = 1$, for $i \ne 2(\bmod 5)$, $1 \le i \le n$. Similarly, $gcd\left(\sigma(c),\sigma(u_i)\right) = 1$ for $i \equiv 2(\bmod 5), 1 \le i \le n$.

Case 2: If $e = cu_i'$, then $gcd\left(\sigma(c),\sigma(u_i')\right) = gcd\left(10^{-r},(3i+1)10^{-r}\right) = gcd(1,3i+1) = 1$, for $i \ne 2(\bmod 5)$, $1 \le i \le n$. Similarly, $gcd\left(\sigma(c),\sigma(u_i')\right) = 1$ for $i \equiv 2(\bmod 5), 1 \le i \le n$.

Case 3: If $e = u_i v_i$, then $gcd\left(\sigma(u_i),\sigma(v_i)\right) = gcd\left((3i-1)10^{-r}, 3i10^{-r}\right) = gcd\left(3i-1, 3i\right) = 1$, for $i \neq 2(\bmod 5)$, $1 \le i \le n$ (Since $3i-1$ and $3i$ are consecutive integer). Similarly, $gcd\left(\sigma(u_i),\sigma(v_i)\right) = 1$ for $i \equiv 2(\bmod 5)$, $1 \le i \le n$.

Case 4: If $e = v_i u_i'$, then $gcd\left(\sigma(v_i),\sigma(u_i')\right) = gcd\left(3i10^{-r}, (3i+1)10^{-r}\right) = gcd\left(3i, 3i+1\right) = 1$, for $i \neq 2(\bmod 5)$, $1 \le i \le n$ (Since $3i+1$ and $3i$ are consecutive integer). Similarly, $gcd\left(\sigma(v_i),\sigma(u_i')\right) = 1$ for $i \equiv 2(\bmod 5)$, $1 \le i \le n$.

Case 5: If $e = u_i u_{i+1}$, then $gcd\left(\sigma(u_i),\sigma(u_{i+1})\right) = gcd\left((3i-1)10^{-r}, (3i+2)10^{-r}\right) = gcd\left(3i-1, 3i+2\right) = 1$, for $i \neq 2(\bmod 5)$, $1 \le i \le n-1$ (Since $3i-1$ and $3i+2$ are not multiples of 3 and differ by 3). Similarly, $gcd\left(\sigma(u_i),\sigma(u_{i+1})\right) = 1$ for $i \equiv 2(\bmod 5)$, $1 \le i \le n-1$.

Case 6: If $e = u_1 u_n$, then

$$\begin{aligned} gcd\left(\sigma(u_1),\sigma(u_n)\right) &= gcd\left(2\times 10^{-r}, (3n-1)10^{-r}\right) \ or \ gcd\left(2\times 10^{-r}, (3n+1)10^{-r}\right) \\ &= gcd\left(2, 3n-1\right) \ or \ gcd\left(2, 3n+1\right) \\ &= 1 \end{aligned}$$

(Since $3n-1$ and $3n+1$ are odd as n is even).

Case 7: If $e = u_i' u_{i+1}$, then $gcd\left(\sigma(u_i'),\sigma(u_{i+1})\right) = gcd\left((3i+1)10^{-r}, (3i+2)10^{-r}\right) = gcd\left(3i+1, 3i+2\right) = 1$, for $i \neq 2(\bmod 5)$, $1 \le i \le n-1$ (Since $3i+1$ and $3i+2$ are consecutive integer). Similarly, $gcd\left(\sigma(u_i'),\sigma(u_{i+1})\right) = 1$ for $i \equiv 2(\bmod 5)$, $1 \le i \le n-1$.

Case 8: If $e = u_i u_{i+1}'$, then $gcd\left(\sigma(u_i),\sigma(u_{i+1}')\right) = gcd\left((3i-1)10^{-r}, (3i+4)10^{-r}\right) = gcd\left(3i-1, 3i+4\right) = 1$, for $i \neq 2(\bmod 5)$, $1 \le i \le n-1$ (Since $3i-1$ and $3i+4$ are not multiples of 5 and differ by 5). Also, $gcd\left(\sigma(u_i),\sigma(u_{i+1}')\right) = gcd\left((3i+1)10^{-r}, (3i+2)10^{-r}\right) = gcd\left(3i+1, 3i+2\right) = 1$ for $i \equiv 2(\bmod 5)$, $1 \le i \le n-1$.

Case 9: If $e = u_1' u_n$, then $gcd\left(\sigma(u_1'),\sigma(u_n)\right) = gcd\left(4\times 10^{-r}, (3n-1)10^{-r}\right) = gcd\left(4, 3n-1\right) = 1$ (Since $3n-1$ is odd as n is even).

Case 10: If $e = u_n' u_1$, then $gcd\left(\sigma(u_n'),\sigma(u_1)\right) = gcd\left((3n+1)10^{-r}, 2\times 10^{-r}\right) = gcd\left(3n+1, 2\right) = 1$ (Since $3n+1$ is odd as n is even).

Therefore, the graph G admits a fuzzy prime labeling.

Theorem 4.9. The graph G obtained by duplicating all the junctions of the helm H_n, except the apex junction, is a fuzzy prime graph.

Proof. Let $c, u_i, v_i\,(1 \le i \le n)$ be the junctions of H_n and let $cu_i, u_iv_i\,(1 \le i \le n)$, $u_iu_{i+1}\,(1 \le i \le n-1)$ and u_1u_n be the bridges of H_n. Let G be the graph obtained by duplicating all the junctions in H_n, except the apex junction c. Let $u'_1, u'_2, ..., u'_n$ and $v'_1, v'_2, ..., v'_n$ be the new junctions of G by duplicating $u_1, u_2, ..., u_n$ and $v_1, v_2, ..., v_n$. Then $c, u_i, v_i, u'_i, v'_i\,(1 \le i \le n)$ are the junctions of G and $\{cu_i, cu'_i, u_iv_i, v_iu'_i, u_iv'_i\,(1 \le i \le n)\}$, $\{u_iu_{i+1}\,(1 \le i \le n-1)\}$, $\{u_iu'_{i+1}\,(2 \le i \le n-1)\}$, $\{u'_{i-1}u_i\,(2 \le i \le n)\}$ and $\{u_1u_n, u_nu'_1, u'_nu_1, u_1u'_2\}$ are the bridges of G.

For $r = 1,2,3,...$ such that $10^{r-1} < n \le 10^r$, we define the junction membership function $\sigma : V(G) \to (0,1]$ by $\sigma(c) = 10^{-r}$, $\sigma(u_1) = 4 \times 10^{-r}$, $\sigma(u'_1) = 2 \times 10^{-r}$, $\sigma(v_1) = 3 \times 10^{-r}$, $\sigma(v'_1) = (4n+1)10^{-r}$, $\sigma(u_i) = (4i-1)10^{-r}\,(2 \le i \le n, i \ne 1(\text{mod } 3))$, $\sigma(v_i) = (4i-2)10^{-r}\,(2 \le i \le n)$, $\sigma(u'_i) = (4i-3)10^{-r}\,(2 \le i \le n, i \ne 1(\text{mod } 3))$, $\sigma(v'_i) = 4i10^{-r}\,(2 \le i \le n)$, $\sigma(u_i) = (4i-3)10^{-r}\,(2 \le i \le n, i \equiv 1(\text{mod } 3))$ and $\sigma(u'_i) = (4i-1)10^{-r}\,(2 \le i \le n, i \equiv 1(\text{mod } 3))$.

Then σ is an injective. Let e be an any bridge of G. To claim σ is a fuzzy prime labeling of G we have the following cases:

Case 1: If $e = cu_i$, then $gcd(\sigma(c), \sigma(u_i)) = 1$ and if $e = cu_i'$, then $gcd(\sigma(c), \sigma(u_i')) = 1$ for $1 \le i \le n$ (Since $\sigma(c) = 1$).

Case 2: If $e = u_iv_i$, then clearly, $gcd(\sigma(u_i), \sigma(v_i)) = 1$, for $1 \le i \le n$.

Case 3: If $e = u_i'v_i$, then clearly, $gcd(\sigma(u_i'), \sigma(v_i)) = 1$, for $1 \le i \le n$.

Case 4: If $e = u_iu_{i-1}$, then, for $i = 3,4,...,n$,

$$\begin{aligned} gcd(\sigma(u_i), \sigma(u_{i-1})) &= gcd((4i-1)10^{-r}, (4i-5)10^{-r}) \quad or \quad gcd((4i-3)10^{-r}, (4i-7)10^{-r}) \\ &= gcd(4i-1, 4i-5) \quad or \quad gcd(4i-3, 4i-7) \\ &= 1 \end{aligned}$$

(Since $4i-1$ and $4i-5$ are odd integers that are differ by 4 and also, $4i-3$ and $4i-7$ are odd integers that are differ by 4).

Case 5: If $e = u_1u_n$, then

$$\begin{aligned} gcd(\sigma(u_1), \sigma(u_n)) &= gcd(4 \times 10^{-r}, (4n-1)10^{-r}) \quad or \quad gcd(4 \times 10^{-r}, (4n-3)10^{-r}) \\ &= gcd(4, 4n-1) \quad or \quad gcd(4, 4n-3) \\ &= 1 \end{aligned}$$

(Since $4n-1$ and $4n-3$ are odd).

Case 6: If $e=u_1u_2$, then obviously, $gcd\left(\sigma(u_1),\sigma(u_2)\right)=1$.

Case 7: If $e=u_iu_{i+1}'$, then for $2\le i\le n-1$,

$$\begin{aligned} gcd\left(\sigma(u_i),\sigma(u_{i+1}')\right) &= gcd\left((4i-1)10^{-r},(4i+1)10^{-r}\right) \ \ or\ \ gcd\left((4i-3)10^{-r},(4i+3)10^{-r}\right) \\ &= gcd\left(4i-1,4i+1\right) \ \ or\ \ gcd\left(4i-3,4i+3\right) \\ &= 1 \end{aligned}$$

(Since $4i-1$ and $4i+1$ are odd integers that are differ by 2 and also, $4i-3$ and $4i+3$ are odd integers that are not multiples of 3 and differ by 6).

Case 8: If $e=u_1u'_2$, then obviously, $gcd\left(\sigma(u_1),\sigma(u'_2)\right)=1$.

Case 9: If $e=u_nu'_1$, then clearly, $gcd\left(\sigma(u_n),\sigma(u'_1)\right)=1$.

Case 10: If $e=u_iu'_{i-1}$, then for $3\le i\le n-1$,

$$\begin{aligned} gcd\left(\sigma(u_i),\sigma(u'_{i-1})\right) &= gcd\left((4i-1)10^{-r},(4i-7)10^{-r}\right) \ \ or\ \ gcd\left((4i-3)10^{-r},(4i-5)10^{-r}\right) \\ &= gcd\left(4i-1,4i-7\right) \ \ or\ \ gcd\left(4i-3,4i-5\right) \\ &= 1 \end{aligned}$$

(Since $4i-1$ and $4i-7$ are odd integers that are not multiples of 3 and differ by 6 and also, $4i-3$ and $4i-5$ are odd consecutive integer).

Case 11: If $e=u'_nu_1$, then clearly, $gcd\left(\sigma(u'_n),\sigma(u_1)\right)=1$.

Case 12: If $e=u_2u'_1$, then obviously, $gcd\left(\sigma(u_2),\sigma(u'_1)\right)=1$.

Case 13: If $e=u_iv'_i$, then for $2\le i\le n$,

$$\begin{aligned} gcd\left(\sigma(u_i),\sigma(v'_i)\right) &= gcd\left((4i-1)10^{-r},4i10^{-r}\right) \ \ or\ \ gcd\left((4i-3)10^{-r},4i10^{-r}\right) \\ &= gcd\left(4i-1,4i\right) \ \ or\ \ gcd\left(4i-3,4i\right) \\ &= 1 \end{aligned}$$

(Since $4i-1$ and $4i$ are consecutive integer and also, $4i-3$ and $4i$ are not multiples of 3 and differ by 3).

Case 14: If $e=u_1v'_1$, then clearly, $gcd\left(\sigma(u_1),\sigma(v'_1)\right)=1$.

Therefore, the graph G is a fuzzy prime graph.

Theorem 4.10. The graph obtained by duplicating all the junctions in crown C_n^* is a fuzzy prime graph.

Proof. Let C_n^* be the crown with junctions $v_1,v_2,\ldots,v_n$ and $u_1,u_2,\ldots,u_n$, where v_i is neighborhood to u_i for $1\le i\le n$, u_i is neighborhood to u_{i+1} for $1\le i\le n-1$ and u_n is neighborhood to u_1. Let G be the graph obtained by duplicating all the junctions in C_n^*. Let $u'_1,u'_2,\ldots,u'_n$ and $v'_1,v'_2,\ldots,v'_n$ be the new junctions of G by duplicating $u_1, u_2, ..., u_n$ and $v_1, v_2, ..., v_n$ respectively. Then $u_i, v_i, u'_i, v'_i \left(1 \le i \le n\right)$ are the junctions of G and $\left\{u_i v_i, v_i u'_i, u_i v'_i \left(1 \le i \le n\right)\right\}$, $\left\{u_i u_{i+1}, u'_i u_{i+1}, u_i u'_{i+1} \left(1 \le i \le n-1\right)\right\}$ and $\left\{u_n u_1, u_n u'_1, u'_n u_1\right\}$ are the bridges of G.

For $r = 1, 2, 3, ...$ such that $10^{r-1} < n \le 10^r$, we define the junction membership function $\sigma : V(G) \to (0,1]$ by $\sigma(u_1) = 10^{-r}$, $\sigma(u'_1) = 2 \times 10^{-r}$, $\sigma(v_1) = 3 \times 10^{-r}$, $\sigma(u_i) = (4i-1)10^{-r} \left(2 \le i \le n, i \ne 1(\text{mod } 3)\right)$, $\sigma(v_i) = (4i-2)10^{-r} \left(2 \le i \le n\right)$, $\sigma(u'_i) = (4i-3)10^{-r} \left(2 \le i \le n, i \ne 1(\text{mod } 3)\right)$, $\sigma(v'_i) = 4i10^{-r} \left(1 \le i \le n\right)$, $\sigma(u_i) = (4i-3)10^{-r} \left(2 \le i \le n, i \equiv 1(\text{mod } 3)\right)$ and $\sigma(u'_i) = (4i-1)10^{-r} \left(2 \le i \le n, i \equiv 1(\text{mod } 3)\right)$.

Then the map σ is an injective. If e is an any bridge of G. We now σ is a fuzzy prime for the following cases:

Case 1: If $e = u_i u_{i+1}$, then $gcd\left(\sigma(u_i), \sigma(u_{i+1})\right) = gcd\left((4i-1)10^{-r}, (4i+3)10^{-r}\right) = gcd\left(4i-1, 4i+3\right) = 1$, for $i \ne 1(\text{mod}3)$, $2 \le i \le n-1$ (Since $4i-1$ and $4i+3$ are odd integer and differ by 4). Similarly, $gcd\left(\sigma(u_i), \sigma(u_{i+1})\right) = 1$ for $i \equiv 1(\text{mod}3)$, $2 \le i \le n-1$.

Case 2: If $e = u_n u_1$, then clearly, $gcd\left(\sigma(u_n), \sigma(u_1)\right) = 1$.

Case 3: If $e = u_2 u_1$, then obviously, $gcd\left(\sigma(u_2), \sigma(u_1)\right) = 1$.

Case 4: If $e = u_i v_i$, then $gcd\left(\sigma(u_i), \sigma(v_i)\right) = gcd\left((4i-1)10^{-r}, (4i-2)10^{-r}\right) = gcd\left(4i-1, 4i-2\right) = 1$, for $i \ne 1(\text{mod}3)$, $2 \le i \le n$, (Since $4i-1$ and $4i-2$ are consecutive integer). Similarly, $gcd\left(\sigma(u_i), \sigma(v_i)\right) = 1$ for $i \equiv 1(\text{mod}3)$, $2 \le i \le n$.

Case 5: If $e = u_1 v_1$, then clearly, $gcd\left(\sigma(u_1), \sigma(v_1)\right) = 1$.

Case 6: If $e = v_i u'_i$, then $gcd\left(\sigma(v_i), \sigma(u'_i)\right) = gcd\left((4i-2)10^{-r}, (4i-3)10^{-r}\right) = gcd\left(4i-2, 4i-3\right) = 1$, for $i \ne 1(\text{mod}3)$, $2 \le i \le n$, (Since $4i-2$ and $4i-3$ are consecutive integer). Similarly, $gcd\left(\sigma(v_i), \sigma(u'_i)\right) = 1$ for $i \equiv 1(\text{mod}3)$, $2 \le i \le n$.

Case 7: If $e = v_1 u'_1$, then clearly, $gcd\left(\sigma(v_1), \sigma(u'_1)\right) = 1$.

Case 6: If $e=u_iv'_i$, then $gcd\left(\sigma(u_i),\sigma(v'_i)\right)=gcd\left((4i-1)10^{-r},4i10^{-r}\right)=gcd\left(4i-1,4i\right)=1$, for $i\neq 1(\bmod 3)$, $2\leq i\leq n$, (Since $4i-1$ and $4i$ are consecutive integer).

Case 8: If $e=u_iv'_i$, then $gcd\left(\sigma(u_i),\sigma(v'_i)\right)=gcd\left((4i-3)10^{-r},4i10^{-r}\right)=gcd\left(4i-3,4i\right)=1$, for $i\equiv 1(\bmod 3)$, $2\leq i\leq n$, (Since $4i-3$ and $4i$ are not multiples of 3 and differ by 3).

Case 9: If $e=u_iu'_{i-1}$, then $gcd\left(\sigma(u_i),\sigma(u'_{i-1})\right)=gcd\left((4i-3)10^{-r},(4i-5)10^{-r}\right)=gcd\left(4i-3,4i-5\right)=1$, for $i\equiv 1(\bmod 3)$, $2\leq i\leq n$, (Since $4i-3$ and $4i-5$ are odd consecutive integer).

Case 10: If $e=u_iu'_{i-1}$, then $gcd\left(\sigma(u_i),\sigma(u'_{i-1})\right)=gcd\left((4i-1)10^{-r},(4i-7)10^{-r}\right)=gcd\left(4i-1,4i-7\right)=1$, for $i\neq 1(\bmod 3)$, $2\leq i\leq n$, (Since $4i-1$ and $4i-7$ are not multiples of 3 and differ by 6).

Case 11: If $e=u_1u'_n$, then clearly, $gcd\left(\sigma(u_1),\sigma(u'_n)\right)=1$.

Case 12: If $e=u_iu'_{i+1}$, then $gcd\left(\sigma(u_i),\sigma(u'_{i+1})\right)=gcd\left((4i-1)10^{-r},(4i+1)10^{-r}\right)=gcd\left(4i-1,4i+1\right)=1$, for $i\neq 1(\bmod 3)$, $2\leq i\leq n-1$, (Since $4i-1$ and $4i+1$ are odd consecutive integer).

Case 13 If $e=u_iu'_{i+1}$, then $gcd\left(\sigma(u_i),\sigma(u'_{i+1})\right)=gcd\left((4i-3)10^{-r},(4i+3)10^{-r}\right)=gcd\left(4i-3,4i+3\right)=1$, for $i\equiv 1(\bmod 3)$, $2\leq i\leq n-1$, (Since i is not multiple of 3 and that $4i-3$ and $4i+3$ are differ by 6).

Case 14: If $e=u_nu'_1$, then clearly, $gcd\left(\sigma(u_n),\sigma(u'_1)\right)=1$.

Case 15: If $e=u_1u'_2$, then obviously, $gcd\left(\sigma(u_1),\sigma(u'_2)\right)=1$.

Therefore, every two neighborhood junction membership values are distinct and mutually prime which completes the proof. Thus, the graph G is a fuzzy prime graph.

Theorem 4.11. The graph obtained by duplicating all the junctions of the star $K_{1,n}$ is a fuzzy prime graph.

Proof. Let $K_{1,n}$ be the crown with junctions $c,v_i\,(1\leq i\leq n)$, where c is neighborhood to v_i for $1\leq i\leq n$. Let G be the graph obtained by duplicating all the junctions of the star $K_{1,n}$. Let $v'_1,v'_2,...,v'_n$ and c' be the new junctions of G by duplicating $v_1,v_2,...,v_n$ and c respectively. Then $c,c',v_i,v'_i\,(1\leq i\leq n)$ are the junctions of G and $\{cv_i,c'v_i,cv'_i\,(1\leq i\leq n)\}$ are the bridges of G.

For $r=1,2,3,...$ such that $10^{r-1}<n\le 10^r$, we define the junction membership function $\sigma:V(G)\to(0,1]$ by $\sigma(c)=10^{-r}$, $\sigma(c')=2\times 10^{-r}$, $\sigma(v_i)=(2i+1)10^{-r}, 1\le i\le n$ and $\sigma(v'_i)=2i10^{-r}$.

Therefore, it can be easily verified that every two neighborhood junction membership values are distinct and mutually prime which complete the proof.

5. Fuzzy prime combination labeling of graphs

In this section, we modify another type of labeling of a graph G with n junctions called fuzzy prime combination labeling. A fuzzy prime labeling σ on a graph G on n junctions is called a fuzzy prime combination labeling if for each positive integer r such that $10^{r-1}<n\le 10^r$, the induced bridge labeling $\mu(xy)$ equals $\frac{1}{10^r}\binom{10^r\sigma(x)}{10^r\sigma(y)}$ or $\frac{1}{10^r}\binom{10^r\sigma(y)}{10^r\sigma(x)}$ according as $\sigma(x)>\sigma(y)$ or $\sigma(y)>\sigma(x)$ is injective onto the subset of $(0,1]$. A graph with a fuzzy prime combination labeling is called a fuzzy prime combination graph.

Now, the fuzzy prime combination labeling is defined below.

Definition 4.1. Let G be a graph with n junctions m bridges. For each positive integer r such that $10^{r-1}<n\le 10^r$, let $\sigma:V(G)\to(0,1]$ be an injective map such that $\gcd(\sigma(x),\sigma(y))=1$. Then σ is called a fuzzy prime combination labeling of G if the induced bridge labeling $\mu(xy)$ equals $\frac{1}{10^r}\binom{10^r\sigma(x)}{10^r\sigma(y)}$ or $\frac{1}{10^r}\binom{10^r\sigma(y)}{10^r\sigma(x)}$ according as $\sigma(x)>\sigma(y)$ or $\sigma(y)>\sigma(x)$ is injective onto the subset of $(0,1]$. A graph with a fuzzy prime combination labeling is called a fuzzy prime combination graph.

We discuss the fuzzy prime combination labeling of some graphs.

Theorem 4.1. The path P_n admits a fuzzy prime combination labeling.

Proof. Let P_n be the path $v_1v_2...v_n$. For $r=1,2,3,...$ such that $10^{r-1}<n\le 10^r$, define $\sigma:V(G)\to(0,1]$ by $\sigma(v_i)=\frac{i}{10^r}\ \forall\ 1\le i\le n$. Then the neighborhood junction membership values are mutually prime and the bridges v_iv_{i+1} have distinct membership values $\frac{1}{10^r}\binom{i+1}{i}$ for $1\le i\le n-1$. Hence the graph P_n admits a fuzzy prime combination labeling.

Theorem 4.2. The cycle C_n admits a fuzzy prime combination labeling for $n>4$ and n is odd.

Proof. Let C_n be the cycle $v_1,v_2,...,v_n$. For $r=1,2,3,...$ such that $10^{r-1}<n\le 10^r$, define $\sigma:V(G)\to(0,1]$ by $\sigma(v_1)=\frac{2}{10^r}$, $\sigma(v_2)=\frac{1}{10^r}$ and $\sigma(v_i)=\frac{i}{10^r}\ \forall\ 3\le i\le n$. Then the neighborhood junction membership values are mutually prime. By the definition of μ, it is clear that

$\mu(u_1u_2)=\frac{1}{10^r}\binom{2}{1}$, $\mu(u_2u_3)=\frac{1}{10^r}\binom{3}{1}$, $\mu(u_iu_{i+1})=\frac{1}{10^r}\binom{i+1}{i}\ \forall\ 3\le i\le n-1$ and $\mu(u_nu_1)=\frac{1}{10^r}\binom{n}{2}$. Hence the required graph admits a fuzzy prime combination labeling.

Theorem 4.3. The star $K_{1,n}$ admits a fuzzy prime combination labeling.

Proof. Let v_0 be a center junction and $v_1,v_2,...,v_n$ be the other junctions of $K_{1,n}$. For $r=1,2,3,...$ such that $10^{r-1}<n\le 10^r$, define $\sigma:V(G)\to(0,1]$ by $\sigma(v_0)=\frac{1}{10^r}$ and $\sigma(v_i)=\frac{i+1}{10^r}\ \forall\ 1\le i\le n$. Then the neighborhood junction membership values are mutually prime and the bridges v_0v_i have distinct membership values $\frac{i+1}{10^r}$ for $1\le i\le n$. Hence the given graph admits a fuzzy prime combination labeling.

Theorem 4.4. Olive tree admits a fuzzy prime combination labeling.

Proof. Let v_{00} be the root of the given Olive tree G. Let $v_{11},v_{12},...,v_{1n}$ be the junctions in the first level such that there are n bridges. Let $v_{22},v_{23},...,v_{2n}$ be the junctions in the second level such that there are $n-1$ bridges. Let $v_{33},v_{34},...,v_{3n}$ be the junctions in the third level such that there are $n-2$ bridges.

Proceeding like this, Let v_{nn} be the unique junction in the n^{th} level and the corresponding lonely bridges be $v_{(n-1)n}v_{nn}$. We define the junction membership function $\sigma:V(G)\to(0,1]$ by $\sigma(v_{00})=\frac{1}{10^r}$; and $\sigma(v_{ij})=\sigma\left(v_{(j-1)(j-1)}\right)+\frac{i}{10^r}$, $1\le j\le n$, $1\le i\le j$, where $r=1,2,3,...$ such that $10^{r-1}<n\le 10^r$. Then we get

$$\gcd\left(\sigma(v_{00})\times 10^r,\sigma(v_{1j})\times 10^r\right)=1,\ 1\le j\le n$$

$$\Rightarrow\quad \gcd\left(\frac{\sigma(v_{00})\times 10^r}{10^r},\frac{\sigma(v_{1j})\times 10^r}{10^r}\right)=1,\ 1\le j\le n$$

$$\Rightarrow\quad \gcd\left(\sigma(v_{00}),\sigma(v_{1j})\right)=1,\ 1\le j\le n.$$

Also, we have

$$\gcd\left(\sigma(v_{ij})\times 10^r,\sigma(v_{(i+1)j})\times 10^r\right)=1,\ 2\le j\le n,\ 1\le i\le j-1$$

$$\Rightarrow\quad \gcd\left(\frac{\sigma(v_{ij})\times 10^r}{10^r},\frac{\sigma(v_{(i+1)j})\times 10^r}{10^r}\right)=1,\ 2\le j\le n,\ 1\le i\le j-1$$

$$\Rightarrow\quad \gcd\left(\sigma(v_{ij}),\sigma(v_{(i+1)j})\right)=1,\ 2\le j\le n,\ 1\le i\le j-1$$

Hence, every two neighborhood junction membership values are distinct and $\gcd\left(10^r\sigma(x),10^r\sigma(y)\right)=1$ where x and y are neighborhood. Also, the induced bridge membership values are

$$\mu\left(v_{00}v_{1j}\right)=\sigma\left(v_{(j-1)(j-1)}\right)+\frac{1}{10^r},\ 1\le j\le n;\text{ and}$$

$$\mu\left(v_{ij}v_{(i+1)j}\right)=\sigma\left(v_{(j-1)(j-1)}\right)+\frac{i+1}{10^r},\ 2\le j\le n,\ 1\le i\le j-1$$

Hence, the membership values of the bridges are all distinct and followed by the definition of bridge function μ , a map μ is injective. This, completes the proof.

Theorem 4.5. The graph $S_{m,n}$ admits a fuzzy prime combination labeling.

Proof. Let v_0 be the center junction of star and v_j^i be the junctions of path of length m where $1\le i\le n$ and $1\le j\le m$. We define the junction membership function $\sigma:V(G)\to(0,1]$ by $\sigma(v_0)=\frac{1}{10^r}$; and for $1\le i\le n$, $\sigma\left(v_j^i\right)=\frac{1}{10^r}\left((i-1)m+j+1\right)$, $1\le j\le m$, where $r=1,2,3,\ldots$ such that $10^{r-1}<n\le 10^r$. Then, we give

$$\gcd\left(\sigma(v_0)\times 10^r,\sigma\left(v_1^{\,i}\right)\times 10^r\right)=1,\ 1\le j\le n$$

$$\Rightarrow\qquad \gcd\left(\frac{\sigma(v_0)\times 10^r}{10^r},\frac{\sigma\left(v_1^{\,i}\right)\times 10^r}{10^r}\right)=1,\ 1\le j\le n$$

$$\Rightarrow\qquad \gcd\left(\sigma(v_0),\sigma\left(v_1^{\,i}\right)\right)=1,\ 1\le j\le n.$$

Also, we have

$$\gcd\left(\sigma\left(v_j^{\,i}\right)\times 10^r,\sigma\left(v_{j+1}^{\,i}\right)\times 10^r\right)=1,\ 1\le i\le n,\ 1\le j\le m-1$$

$$\Rightarrow\qquad \gcd\left(\frac{\sigma\left(v_j^{\,i}\right)\times 10^r}{10^r},\frac{\sigma\left(v_{j+1}^{\,i}\right)\times 10^r}{10^r}\right)=1,\ 1\le i\le n,\ 1\le j\le m-1$$

$$\Rightarrow\qquad \gcd\left(\sigma\left(v_j^{\,i}\right),\sigma\left(v_{j+1}^{\,i}\right)\right)=1,\ 1\le i\le n,\ 1\le j\le m-1$$

Hence, every two neighborhood junction membership values are distinct and mutually prime. Also, the induced bridge membership values are

$$\mu\left(v_0v_1^i\right)=\frac{1}{10^r}\left((i-1)m+2\right),\ 1\le i\le n,\text{ and}$$

$$\mu\left(v_j^iv_{j+1}^i\right)=\frac{1}{10^r}\left((i-1)m+j+2\right),\ 1\le i\le n,\ 1\le j\le m-1.$$

Thus, the membership values of the bridges are all distinct. Clearly, the bridge function μ is injective. Hence, the graph $S_{m,n}$ admits a fuzzy prime combination labeling.

Theorem 4.6. The graph $C_n \oplus P_m$ admits a fuzzy prime combination labeling when $m+n$ is odd.

Proof. Denote the graph $G = C_n \oplus P_m$ be the junctions of C_n by $u_1, u_2, ..., u_n$ where u_1 is neighborhood to u_n and u_i is neighborhood to u_{i+1} for $1 \le i \le n-1$ and be the junctions of P_m by $v_1 v_2 ... v_m$ joined with the junction u_n of C_n. We define the junction membership function $\sigma : V(G) \to (0,1]$ by $\sigma(u_n) = \frac{2}{10^r}$, $\sigma(u_1) = \frac{1}{10^r}$, $\sigma(v_i) = \frac{i+2}{10^r}$, $1 \le i \le m$ and $\sigma(u_j) = \frac{m+j+1}{10^r}$, $2 \le j \le n-1$, where $r = 1,2,3,...$ such that $10^{r-1} < n \le 10^r$. Then we can easily verify that neighborhood junction membership values are all distinct and mutually prime. Also, the induced bridge membership values are

$$\mu(u_n u_1) = \frac{2}{10^r},$$

$$\mu(u_n v_1) = \frac{3}{10^r},$$

$$\mu(v_i v_{i+1}) = \frac{i+3}{10^r}, 1 \le i \le m-1,$$

$$\mu(u_j u_{j+1}) = \frac{m+j+2}{10^r}, 1 \le j \le n-2,$$

$$\mu(u_{n-1} u_n) = \frac{1}{10^r}\left(\frac{m+n}{2}\right).$$

Hence, the membership values of the bridges are all distinct and followed by the definition of μ which completes the proof.

Theorem 4.7. The graph $C_n \oplus K_{1,m}$ admits a fuzzy prime combination labeling if $m+n \ge 5$.

Proof. Denote $G = C_n \oplus K_{1,m}$ be the junctions of C_n by $u_1, u_2, ..., u_n$ where u_1 is neighborhood to u_n and u_i is neighborhood to u_{i+1} for $1 \le i \le n-1$ and be the junctions of $K_{1,m}$ by $v_1, v_2, ..., v_m$ which is attached with the junction u_n of C_n. There are two cases:

Case 1. $m+n$ is odd.

We define the membership function $\sigma : V(G) \to (0,1]$ as follows:

$$\sigma(u_n) = \frac{1}{10^r},$$

$$\sigma(u_{n-1}) = \frac{2}{10^r},$$

$$\sigma\left(v_i\right)=\frac{i+2}{10^r},\ 1\le i\le m \text{ and}$$

$$\sigma\left(u_j\right)=\frac{m+j+2}{10^r},1\le j\le n-2,$$

where $r=1,2,3,...$ such that $10^{r-1}<n\le 10^r$.

Case 2. $m+n$ is even.

We define the membership function $\sigma:V\left(G\right)\to\left(0,1\right]$ as follows:

$$\sigma\left(u_n\right)=\frac{1}{10^r},$$

$$\sigma\left(u_{n-1}\right)=\frac{2}{10^r},$$

$$\sigma\left(v_i\right)=\frac{i+2}{10^r},\ 1\le i\le m-1$$

$$\sigma\left(v_m\right)=\frac{m+n}{10^r} \text{ and}$$

$$\sigma\left(u_j\right)=\frac{m+j+1}{10^r},1\le j\le n-2,$$

where $r=1,2,3,...$ such that $10^{r-1}<n\le 10^r$.

Clearly, the junction membership values are all distinct and mutually prime. By the definition of μ, the bridge function is injective. Hence, the graph $C_n\oplus K_{1,m}$ admits a fuzzy prime combination labeling for $m+n\ge 5$.

5.1. Applications

Concept of graph theory have applications in many areas of computer science, including data mining, image segmentation, clustering, image capturing, networking, etc. Klir and Bo Yuan [21] and Sahoo and Pal [28] discussed the applications in fuzzy graphs. Labeled graphs serve us useful models for broad range of applications such as coding theory, X-ray, radar, astronomy, circuit design and communication networks, etc. Fuzzy labeling models yield more precision, flexibility, and compatibility to the system compared to the classical and fuzzy models. They have many applications in physics, chemistry, computer science, and other branches of mathematics. Kalaiarasi and Mahalakshmi [17] and Devaraj and Chellamani [10] discussed the applications in fuzzy labeling.

Conclusion

In this paper, we have proved that grid, ladder, generalized Petersen graph, duplication of helm, gear, crown and star graphs are fuzzy prime and some class of graphs are fuzzy prime combination graphs. The study of the existence of fuzzy prime labeling for other families of graphs is an area for further investigation.

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