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# Computation Method for a Differential-Difference Equation with Boundary Layer in Neuronal Variability Modelling using a Mixed Nonpolynomial Spline

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***Abstract:-*** This article introduces a computational difference scheme designed to solve singularly perturbed differential-difference equations (SPDDE) exhibiting boundary layer behaviour, utilizing a mixed nonpolynomial spline. To effectively manage boundary layer oscillations, the SPDDE is transformed into an equivalent two-point boundary layer problem. The adjustment of the fitting factor within the difference scheme is crucial for controlling these oscillations. The Thomas algorithm is employed to illustrate the discrete system of the difference scheme. This computational strategy demonstrates a second-order convergence rate, with a brief discussion on convergence analysis. The effectiveness of this approach is validated through numerical examples, with comprehensive comparisons confirming its reliability and consistency. Graphical representations of layer profiles are provided for various delay and advance parameter values.

**Keywords**: Singularly perturbed differential - difference equation, Mixed non-polynomial spline, Boundary layer

## 1. Introduction

The use of singularly perturbed differential difference equations (SPDDEs) is widespread in mathematical modelling across many scientific and engineering fields to accurately depict real-world scenarios. Generally, these equations require a small positive parameter that multiplies the largest derivative in the equation, while also including a delay/advance parameter. The significance of such differential equations is paramount in the fields of biosciences, control theory, economics, and engineering. Over the past few decades, several approaches have been developed to tackle the difficulties presented by singularly perturbed differential-difference problems. Control analysis and design studies [10], hybrid optical systems [2], population dynamics study [11], physiological control system investigations [16], predator-prey modelling [17], and competitive tumour growth models [27] frequently confront these problems. For a more comprehensive exploration of the mathematical models in this field, we decided to organize our equations by referring to the publications of Doolan and Miller [3] and Kokotovic et al. [10].

Over recent years, several investigative techniques have been devised to tackle problems associated with singularly perturbed convection delay problems. The pioneering research conducted by Lange & Miura [13–14] has been significant in exploring boundary value algorithms for solving SPDDEs. A particular focus of these authors is the analysis of singularly perturbed linear second-order differential-difference equations that include small shifts. SPDDEs are frequently derived from mathematical approaches that employ first exit time theory to estimate the time until the initial spike occurs at a certain initiating point in diffusive processes [19].

$$0.5\,\sigma^2\emptyset''(\mathcal{S}) + \left(\mu_D - \frac{\mathcal{S}}{\tau}\right)\emptyset'(\mathcal{S}) + \pi_{\mathcal{S}}\emptyset(\mathcal{S} + a_{\mathcal{S}}) + \vartheta_{\mathcal{S}}\emptyset(\mathcal{S} + i_{\mathcal{S}}) + (\pi_{\mathcal{S}} + \vartheta_{\mathcal{S}})\emptyset(\mathcal{S}) = -1$$

where the initial characteristic term is derived from the logarithmic decrease observed in two successive jumps in response to the supply operations. The diffusion moments $\mu_D$ and $\sigma$ quantify the impact of neural connections on cell stimulation in the Weiner manipulation. In this manipulation, the barrier potential decreases in a linear manner to reach a stationary level, with a barrier time constant $\tau$. Given that the reaction components are the combination of excitation and inhibitory inputs, that they follow a Poisson distribution [23].

The authors in [25] proposed an exponentially fitted spline approach to solve singularly perturbed differential-difference equations (SPDDEs). Swamy et al. [24] employed an exponentially fitted Galerkin method to address SPDDEs with delay and advance parameters, incorporating a fitting factor to manage abrupt changes within the boundary layer. Adilaxmi et al. [1] introduced a numerical approach involving two fitting factors at the convective and diffusion coefficients to solve SPDDEs exhibiting boundary layer behaviour. This two-parameter fitted technique is designed to achieve accurate results. To solve SPDDEs with mixed shifts, where solutions display boundary layer behaviour at the left end of the interval, Lakshmi Sirisha et al. [12] proposed a mixed finite difference method. This method divides the domain into inner and outer regions and introduces a terminal boundary point to the domain.

The numerical methods for solving SPDDEs utilizing finite difference schemes were proposed by Kadalbajoo and Sharma [6-9]. Kadalbajoo and Kumar [5] tackled SPDDEs by utilizing the B-Spline collocation method with a fitted mesh configuration on a mesh that is uniformly distributed. To gain deeper understanding of SPDDEs, it is recommended that readers refer to the published publications [16, 17, 18, 20] and the corresponding references.

## 2. Description of the Problem

We have considered a boundary layer problem which includes a small delay and advanced terms of the form:

$$\varepsilon\emptyset''(\mathcal{S}) + \mathcal{P}(\mathcal{S})\emptyset'(\mathcal{S}) + q(\mathcal{S})\emptyset(\mathcal{S} - \delta) + r(\mathcal{S})\emptyset(\mathcal{S}) + \omega(\mathcal{S})\emptyset(\mathcal{S} + \eta) = f(\mathcal{S}) \tag{1}$$

on (0, 1), under the boundary conditions

$$\emptyset(\mathcal{S}) = \varphi(\mathcal{S}) \text{ on } -\delta \le \mathcal{S} \le 0,\ \ \emptyset(\mathcal{S}) = \vartheta(\mathcal{S}) \text{ on } 1 \le \mathcal{S} \le 1 + \eta \tag{2}$$

where $0 < \varepsilon << 1$ is small perturbation parameter, $\mathcal{P}(\mathcal{S}), q(\mathcal{S}), r(\mathcal{S}), \omega(\mathcal{S}), f(\mathcal{S}),\ \varphi(\mathcal{S})\ and\ \vartheta(\mathcal{S})$ are sufficiently smooth enough and $0 < \delta = o(\varepsilon),\ 0 < \eta = o(\varepsilon)$ are the delay and the advance terms respectively. The layer at the left end of the domain is characterized when $\mathcal{P}(\mathcal{S}) - \delta q(\mathcal{S}) + \eta\omega(\mathcal{S}) > 0$ and layer at the right-end of the domain is characterized when $\mathcal{P}(\mathcal{S}) - \delta q(\mathcal{S}) + \eta\omega(\mathcal{S}) < 0$. when $p(\mathcal{S}) = 0$, the problem exhibits oscillatory solution or two layers depending upon the result of the expression whether $q(\mathcal{S}) + r(\mathcal{S}) + \omega(\mathcal{S})$ yields positive or negative value. The terms $\emptyset(\mathcal{S} - \delta)$ and $\emptyset(\mathcal{S} - \eta)$ can be expanded by employing Taylor series, since the solution $\emptyset(s)$ of the problem Eq. (1) is sufficiently differentiable and therefore,

$$\emptyset(\mathcal{S} - \delta) \approx \emptyset(\mathcal{S}) - \delta\emptyset'(\mathcal{S}) \tag{3a}$$

$$\emptyset(\mathcal{S} + \eta) \approx \emptyset(\mathcal{S}) + \eta\emptyset'(\mathcal{S}) \tag{3b}$$

Using Eq. (3) in Eq. (1), we get

$$\varepsilon y\emptyset''(\mathcal{S}) + u(\mathcal{S})\emptyset'(s) + v(\mathcal{S})\emptyset(\mathcal{S}) = f(\mathcal{S}) \tag{4}$$

with boundary conditions $\emptyset(0) = \varphi(0),\ \emptyset(1) = \vartheta(1)$ (5)

Here $u(\mathcal{S}) = \mathcal{P}(\mathcal{S}) - \delta q(\mathcal{S}) + \eta\omega(\mathcal{S}), v(\mathcal{S}) = q(\mathcal{S}) + r(\mathcal{S}) + \omega(\mathcal{S})$. Eq. (4) is a second order singular perturbation problem with perturbation parameter $0 < \varepsilon << 1$. The functions $u(\mathcal{S}), v(\mathcal{S}), f(\mathcal{S})$ and are assumed to be sufficiently smooth functions in $[a, b]$, and $\alpha,\beta$ are finite constants. The layer exists in the neighbourhood of $\mathcal{S} = 0$ , if $u(\mathcal{S}) \geq L > 0$ across the entire domain [0, 1], where $L$ is positive constant. The layer exists in the neighbourhood of $\mathcal{S} = 1$, if $u(\mathcal{S}) \leq \overline{L} < 0$ over the domain [0, 1], where $\overline{L}$.is a negative constant $\overline{L}$.

## 3. Mixed Non-Polynomial Cubic Spline

A uniform mesh $\Delta$ with nodal points $\mathcal{S}_i$ on $[0,1]$ has been considered, such that $\Delta: 0 = \mathcal{S}_0 < \mathcal{S}_1 < \mathcal{S}_2 <\ldots< \mathcal{S}_{n-1} < \mathcal{S}_n = 1$ where $\mathcal{S}_i = i\hbar, i = 0,1,\ldots,n$ and $\hbar = \frac{1}{n}$. A mixed non polynomial spline function $\psi_\Delta(\mathcal{S})$ of class $C^2[0,1]$is utilized to interpolate $\emptyset(\mathcal{S})$ at mesh points $\mathcal{S}_i$, $i = 0$ to $n$ depends on a parameter $k$, where $\psi_\Delta(\mathcal{S})$ in $[0,1]$ reduces to ordinary cubic spline as $k \to 0$. The nonpolynomial spline $\psi_\Delta(\mathcal{S})$ is considered for each subinterval $[\mathcal{S}_i, \mathcal{S}_{i+1}], i = 0,1,\ldots,n-1$ as below,

$$\psi_\Delta(\mathcal{S}) = a_i \sin\mathcal{K}(\mathcal{S}-\mathcal{S}_i) + b_i \cos\mathcal{K}(\mathcal{S}-\mathcal{S}_i) + c_i e^{\mathcal{K}(\mathcal{S}-\mathcal{S}_i)} + d_i e^{-\mathcal{K}(\mathcal{S}-\mathcal{S}_i)}, i = 0,1,2,\ldots,n \tag{6}$$

where $a_i, b_i, c_i$ and $d_i$ represents unknown coefficients in Eq. (6) and a free variable $k$ is utilised to improve the accuracy of the approach. Let the exact solution be $\emptyset(\mathcal{S})$ and an approximation to $\emptyset(\mathcal{S}_i)$ be $\emptyset_i$ which is obtained by the spline function $\psi_i(\mathcal{S})$ for each segment and passing through the points $(\mathcal{S}_i, y_i)$ and $(\mathcal{S}_{i+1}, y_{i+1})$. The coefficients of Eq. (6) are determined in terms of $\emptyset_i, \emptyset_{i+1}, M_i, M_{i+1}$ which are defined as

$$\psi_\Delta(\mathcal{S}_i) = \emptyset_i, \psi_\Delta(\mathcal{S}_{i+1}) = \emptyset_{i+1}$$
$$\psi_\Delta''(\mathcal{S}_i) = M_i, \psi_\Delta''(\mathcal{S}_{i+1}) = M_{i+1} \tag{7}$$

Using these conditions, we have $a_i = \frac{(k^2\emptyset_{i+1}-M_{i+1})-\cos\tau(k^2\emptyset_i-M_i)}{2k^2\sin\tau}$, $b_i = \frac{(k^2\emptyset_i-M_i)}{2k^2}$, $c_i = \frac{e^\tau(k^2\emptyset_{i+1}+M_{i+1})-(k^2\emptyset_i+M_i)}{2k^2(e^{2\tau}-1)}$

$d_i = \dfrac{e^{2\tau}(k^2\emptyset_i + M_i) - e^\tau(k^2\emptyset_{i+1} + M_{i+1})}{2k^2(e^{2\tau}-1)}, \tau = kh$ and $i = 0$ to $n-1$

Employing the continuity of the first derivative at the point $\mathcal{S} = \mathcal{S}_i$, the following tridiagonal system is obtained for $i = 1,2,\ldots,n-1$

$$\emptyset_{i-1} + \gamma\emptyset_i + \emptyset_{i+1} = h^2(\alpha M_{i-1} + \beta M_i + \alpha M_{i+1}) \tag{8}$$

In Eq. (8) $\alpha = \frac{(e^{2\tau}-2e^\tau\sin\tau-1)}{\emptyset^2(e^{2\tau}+2e^\tau\sin\tau-1)}$, $\beta = 2\frac{[e^{2\tau}(\sin\tau-\cos\tau)+(\sin\tau+\cos\tau)]}{\emptyset^2(e^{2\tau}+2e^\tau\sin\tau-1)}$, $\gamma = -2\frac{[e^{2\tau}(\sin\tau+\cos\tau)+(\sin\tau-\cos\tau)]}{(e^{2\tau}+2e^\tau\sin\tau-1)}$

When $(\alpha,\beta,\gamma) \to (1/6, 4/6, -2)$ as $\tau \to 0$ and then spline as defined by Eq. (8) has been transformed into ordinary cubic spline relation

$$(\emptyset_{i-1} - 2\emptyset_i + \emptyset_{i+1}) = \frac{h^2}{6}(M_{i-1} + 4M_i + M_{i+1}) \tag{9}$$

The Eq. (8) determines a relationship that results in $(n-1)$ linear algebraic equations involving $(n+1)$ unknowns denoted as $\mathcal{S}_i$ where $i$ ranges from 1 to $n-1$.

## 4. Derivation of the Difference Scheme

The proposed boundary value problem Eq. (1) can be discretized at the grid point $\mathcal{S}_i$, using the following approach:

$$\varepsilon\emptyset''(\mathcal{S}_i) + u(\mathcal{S}_i)\emptyset'(\mathcal{S}_i) + v(\mathcal{S}_i)\emptyset(\mathcal{S}_i) = f(\mathcal{S}_i) \tag{10}$$

By utilizing the second derivative of a spline, we establish

$$M_i = \frac{f_i - u_i\emptyset_i' - v_i\emptyset_i}{\varepsilon}, M_{i-1} = \frac{f_{i-1} - u_{i-1}\emptyset_{i-1}' - v_{i-1}\emptyset_{i-1}}{\varepsilon}, M_{i+1} = \frac{f_{i+1} - u_{i+1}\emptyset_{i+1}' - v_{i+1}\emptyset_{i+1}}{\varepsilon}$$

where $\emptyset_i' \approx \frac{\emptyset_{i+1}-\emptyset_{i-1}}{2h}, \emptyset_{i-1}' \approx \frac{-\emptyset_{i+1}+4\emptyset_i-3\emptyset_{i-1}}{2h}, \emptyset_{i+1}' \approx \frac{3\emptyset_{i+1}-4\emptyset_i+\emptyset_{i+1}}{2h}$, $u_i = u(\mathcal{S}_i), v_i = v(\mathcal{S}_i), f_i = f(\mathcal{S}_i)$.

Replacing the entries of $M_j$ for $j = i, i \pm 1$ in Eq. (8), we acquire

$$\left[\varepsilon - \frac{3}{2}\alpha u_{i-1}h + \alpha v_{i-1}h^2 - \frac{u_i}{2}\beta h + \frac{\alpha}{2}u_{i+1}h\right]\emptyset_{i-1} + [\varepsilon\gamma + 2\alpha u_{i-1}h + v_i\beta h^2 - 2\alpha u_{i+1}h]\emptyset_i +$$

$$\left[\varepsilon+\frac{3}{2}\alpha u_{i+1}h+\alpha v_{i+1}h^2+\frac{u_i}{2}\beta h-\frac{\alpha}{2}u_{i-1}h\right]\emptyset_{i+1}\ =h^2\left[\alpha f_{i-1}+\beta f_i+\ \alpha f_{i+1}\right] \tag{11}$$

Now introduce a fitting factor $\sigma_i(\rho)$ in Eq. (11) to manage the boundary layer behaviour, we have

$$\left[\sigma_i(\rho)\varepsilon-\frac{3}{2}\alpha u_{i-1}h+\alpha v_{i-1}h^2-\frac{u_i}{2}\beta h+\frac{\alpha}{2}u_{i+1}h\right]\emptyset_{i-1}+\left[\sigma_i(\rho)\varepsilon\gamma+2\alpha u_{i-1}h+v_i\beta h^2-2\alpha u_{i+1}h\right]\emptyset_i+$$
$$\left[\sigma_i(\rho)\varepsilon+\frac{3}{2}\alpha u_{i+1}h+\alpha v_{i+1}h^2+\frac{u_i}{2}\beta h-\frac{\alpha}{2}u_{i-1}h\right]\emptyset_{i+1}\ =h^2\left[\alpha f_{i-1}+\beta f_i+\ \alpha f_{i+1}\right] \tag{12}$$

The value of $\sigma_i(\rho)$ is acquired by the procedure given by Doolan et al. [3] and is given by

$$\sigma_i(\rho)=a_i\rho\left(\alpha+\frac{\beta}{2}\right)coth\left(\frac{a_i\rho}{2}\right).$$

Using Eq. (11), we get the following system of equations:

$$L_i\emptyset_{i-1}+C_i\emptyset_i+U_i\emptyset_{i+1}=H_i \quad \text{for } \ i=1,2,\dots,n-1 \tag{13}$$

Here $L_i=\sigma_i(\rho)\varepsilon-\frac{3}{2}\alpha u_{i-1}h+\alpha v_{i-1}h^2-\frac{u_i}{2}\beta h+\frac{\alpha}{2}u_{i+1}h,\quad C_i=\sigma_i(\rho)\varepsilon\gamma+2\alpha u_{i-1}h+v_i\beta h^2-2\alpha u_{i+1}h$

$$U_i=\sigma_i(\rho)\varepsilon+\frac{3}{2}\alpha u_{i+1}h+\alpha v_{i+1}h^2+\frac{u_i}{2}\beta h-\frac{\alpha}{2}u_{i-1}h,\ H_i=h^2\left[\alpha f_{i-1}+\beta f_i+\alpha f_{i+1}\right]$$

The tridiagonal system Eq. (13) is solved for $i=1,2,\dots,n-1$ to obtain the approximations $\emptyset_1,\emptyset_2,\dots,\emptyset_{n-1}$ of the solution $\emptyset(\mathcal{S})$ at $\mathcal{S}_1,\mathcal{S}_2,\dots\dots\mathcal{S}_{n-1}$ with the boundary conditions Eq. (5). The local truncation error of the developed scheme in Eq. (13) is determined as follows:

$$T_i(h)=\varepsilon\sigma_i(2+\gamma)+[1-(2\alpha+\beta)]\varepsilon\sigma_i\emptyset_i{}''h^2+\left[\varepsilon\sigma_i\emptyset_i{}^{iv}\left(\frac{1}{12}-\alpha\right)+u_i\emptyset_i{}'''\left(\frac{\beta}{6}-\frac{2\alpha}{3}\right)\right]h^4+\dots$$

Thus, truncation error indicates different orders for various values of $\alpha$ and $\beta$.

(i) For $\alpha=\frac{1}{6},\beta=\frac{4}{6}$, and $\gamma=-2$, truncation error is fourth order.

(ii) For $\alpha=\frac{1}{12},\beta=\frac{5}{6}$, and $\gamma=-2$, truncation error is sixth order.

## 5. Convergence Analysis

The matrix representation of the system of equations Eq. (13) with the boundary conditions is

$$(D+F)W+G+T(h)=0 \tag{14}$$

where $D=[\varepsilon\sigma_i,\ \varepsilon\sigma_i\gamma,\varepsilon\sigma_i]=\begin{bmatrix}\varepsilon\sigma_i & \varepsilon & 0 & 0 & \dots\dots & 0\\ \varepsilon & \varepsilon\sigma_i\gamma & \varepsilon & 0 & \dots\dots & 0\\ 0 & \varepsilon & \varepsilon\sigma_i & \varepsilon & \dots\dots & 0\\ .. & .. & .. & .. & .. & ..\\ .. & .. & .. & .. & .. & ..\\ 0 & .. & .. & 0 & \varepsilon & \varepsilon\gamma\end{bmatrix}$,

$$F=[\tilde{z}_i,\tilde{r}_i,\widetilde{w}_i]=\begin{bmatrix}\tilde{r}_1 & \widetilde{w}_1 & 0 & 0 & \dots\dots & 0\\ \tilde{z}_2 & \tilde{r}_2 & \widetilde{w}_2 & 0 & \dots\dots & 0\\ 0 & \tilde{z}_3 & \tilde{r}_3 & \widetilde{w}_3 & \dots\dots & 0\\ .. & .. & .. & .. & .. & ..\\ .. & .. & .. & .. & .. & ..\\ 0 & .. & .. & 0 & \tilde{z}_{N-1} & \tilde{r}_{N-1}\end{bmatrix}$$

$\tilde{z}_i=-\frac{3}{2}\alpha u_{i-1}h+\alpha v_{i-1}h^2-\frac{u_i}{2}\beta h+\frac{\alpha}{2}u_{i+1}h,\ \tilde{r}_i=2\alpha u_{i-1}h+v_i\beta h^2-2\alpha u_{i+1}h$

$\widetilde{w}_i = \frac{3}{2}\alpha u_{i+1}h + \alpha v_{i+1}h^2 + \frac{u_i}{2}\beta h - \frac{\alpha}{2}u_{i-1}h$ for $i = 1,2,\ldots,n-1$ and $G = [q_1 - \tilde{z}_1\alpha, q_2,,\ldots,q_{n-1} - \widetilde{w}_{n-1}\beta]$ where $q_i = h^2[\alpha f_{i-1} + \beta f_i + \alpha f_{i+1}]$, for $i = 2,3,\ldots,N-1$, $T(h) = O(h^4)$ for $\alpha = \frac{1}{12}, \beta = \frac{5}{6}$, and $\gamma = -2$ and $W = [W_1, W_2, W_3, \ldots, W_{n-1}]^T$, $T(h) = [T_1, T_2, \ldots, T_{n-1}]^T, O = [0,0,\ldots,0]^T$ represents the associated vectors of Eq. (14). Let $w = [w_1, w_2, \ldots, w_{n-1}]^T \cong W$ which satisfies the equation

$$(D + F)w + G = O \tag{15}$$

Let the discretization error be $e_i = w_i - W_i$ where $i$ ranges from 1 to $n-1$ so that$E = [e_1, e_2, \ldots, e_{n-1}]^T = w - W$. The error equation is acquired by Subtracting Eq. (14) from Eq. (15)

$$(D + F)E = T(h) \tag{16}$$

Let $\xi_1, \xi_2$ be positive constants such that $|u(s)| \le \xi_1$ and $|v(s)| \le \xi_2$. If $(i,j)^{th}$element of $F$ be $F_{i,j}$, then

$$|F_{i,i+1}| = \sigma_i\varepsilon + \frac{3}{2}\alpha u_{i+1}h + \alpha v_{i+1}h^2 + \frac{u_i}{2}\beta h - \frac{\alpha}{2}u_{i-1}h$$

$$|F_{i,i-1}| = \sigma_i\varepsilon - \frac{3}{2}\alpha u_{i-1}h + \alpha v_{i-1}h^2 - \frac{u_i}{2}\beta h + \frac{\alpha}{2}u_{i+1}h$$

Therefore, when the value of $h$ sufficiently small, we conclude that

$$|F_{i,i+1}| < \varepsilon, i \text{ ranges from 1 to } n-2 \tag{17a}$$

$$|F_{i,i-1}| < \varepsilon, i \text{ ranges from 2 to } n-2 \tag{17b}$$

Hence $(D + F)$ is irreducible (Ref. [28]). Assume that $\bar{S}_i$ represents sum of the entries of the $i^{th}$ row of the matrix $(D + F)$, we acquire

$$\bar{S}_i = \sum_{j=1}^{N-1} M_{ij} = -\frac{\varepsilon\sigma_i}{h^2} + \frac{3\alpha}{2h}u_{i-1} - \frac{\alpha}{2h}u_{i+1} + \alpha v_{i+1} + \left(\frac{u_i}{2h} + v_i\right)\beta \quad \text{for } i = 1$$

$$\bar{S}_i = \sum_{j=1}^{N-1} M_{ij} = \alpha b_{i-1} + b_i\beta + \alpha b_{i+1} \text{ for } i = 2, 3, \ldots, n-2.$$

$$\bar{S}_{N-1} = \sum_{j=1}^{N-1} M_{N-1\,j} = \frac{-\varepsilon\sigma_i}{h^2} - \frac{\alpha}{2h}u_{i-1} - \frac{3\alpha}{2h}u_{i+1} + \alpha v_{i-1} - \left(\frac{u_i}{2h} - v_i\right) \quad \text{for } i = n-1$$

Let $\xi_{1^*} = \min_{1\le i\le N}|u(s_i)|$ and $\xi_1^* = \max_{1\le i\le N}|u(s_i)|$, $\xi_{2^*} = \min_{1\le i\le N}|v(s_i)|$ and $\xi_2^* = \max_{1\le i\le N}|v(s_i)|$. It is confirmed that the monotonicity of $(D + F)$ holds true [26,28] for sufficiently small $h$, since $0 < \varepsilon << 1$ and $\varepsilon \propto o(h)$.

Therefore $(D + F)^{-1}$ exists and $(D + F)^{-1} \ge 0$. Thus, using Eq. (16), we get

$$\|E\| \le \|(D + F)^{-1}\| \quad \|T\| \tag{18}$$

Assume that $(i,k)^{th}$ element of $(D + F)^{-1}$ be $(D + F)^{-1}_{i,k}$. Now define $\|(D + F)^{-1}\| = \max_{1\le i\le N-1} \sum_{k=1}^{n-1}(D + F)^{-1}_{i,k}$ and $\|T(h)\| = \max_{1\le i\le N-1}|T(h)|$. Since $(D + F)^{-1}_{i,k} \ge 0$ and $\sum_{k=1}^{n-1}(D + F)^{-1}_{i,k}\bar{S}_k = 1$ where $i$ ranges from 1 to $n-1$. Hence

$$(D + F)^{-1}_{i,1} \le \frac{1}{\bar{S}_1} < \frac{1}{h^2\xi} \tag{19a}$$

$$(D + F)^{-1}_{i,n-1} \le \frac{1}{\bar{S}_{n-1}} < \frac{1}{h^2\xi} \tag{19b}$$

Furthermore, $\sum_{k=2}^{n-2}(D + F)^{-1}_{i,k} \le \frac{1}{\min_{2\le k\le n-2}\bar{S}_k} < \frac{1}{h^2\xi}, i = 2,3,\ldots,n-2$. (19c)

Utilizing Eq. (18) with the help of Eqs. (19a) - (19c), we get

$$\|E\| \le O(h^2). \tag{20}$$

Hence, the method given in Eq. (13) is second order convergent for $\alpha = \frac{1}{12}, \beta = \frac{5}{6}, \gamma = -2$.

## 6. Numerical Illustrations

The model problems of the types Eqs. (1) – (2) are taken into consideration (ref. [6]) to validate the applicability of the method. Here $\emptyset(s) = c_1 e^{m_1 \mathcal{S}} + c_2 e^{m_2 \mathcal{S}} + \frac{f}{C}$ is the exact solution of the problem where $C = b + c + d$, $c_1 = \frac{-f+\emptyset C+e^{m_2}(f-\phi C)}{(e^{m_1}-e^{m_2})C}$, $c_2 = \frac{[f-\emptyset C+e^{m_1}(-f+\phi C)]}{(e^{m_1}-e^{m_2})C}$, $m_1 = \frac{\left[-(p-q\delta+\omega\eta)+\sqrt{(p-q\delta+\omega\eta)^2-4\varepsilon C}\right]}{2\varepsilon}$ and $m_2 = \frac{\left[-(p-q\delta+\omega\eta)-\sqrt{(p-q\delta+\omega\eta)^2-4\varepsilon C}\right]}{2\varepsilon}$.

Illustration 1. $\varepsilon\emptyset''(\mathcal{S}) + \emptyset' + 2\emptyset(\mathcal{S}-\delta) - 3\mathcal{S} = 0$ with $\emptyset(\mathcal{S}) = 1,\ -\delta \le \mathcal{S} \le 0, \emptyset(\mathcal{S}) = 1,\ 1 \le \mathcal{S} \le 1+\eta$

Illustration 2. $\varepsilon\emptyset''(\mathcal{S}) + \emptyset' - 3\emptyset + 2\emptyset(\mathcal{S}+\eta) = 0$ with $\emptyset(\mathcal{S}) = 1, -\delta \le \mathcal{S} \le 0, \emptyset(\mathcal{S}) = 1, 1 \le \mathcal{S} \le 1+\eta$

Illustration 3. $\varepsilon\emptyset''(\mathcal{S}) + \emptyset' - 2\emptyset(\mathcal{S}-\delta) - 5\emptyset + \emptyset(\mathcal{S}+\eta) = 0$ with $\emptyset(\mathcal{S}) = 1, -\delta \le \mathcal{S} \le 0,\ \emptyset(\mathcal{S}) = 1, 1 \le \mathcal{S} \le 1+\eta$

Illustration 4. $\varepsilon\emptyset''(\mathcal{S}) - \emptyset' - 2\emptyset(\mathcal{S}-\delta) + \emptyset = 0$ with $\emptyset(\mathcal{S}) = 1,\ -\delta \le \mathcal{S} \le 0, \emptyset(\mathcal{S}) = -1,\ 1 \le \mathcal{S} \le 1+\eta$

Illustration 5. $\varepsilon\emptyset''(\mathcal{S}) - \emptyset' + \emptyset - 2\emptyset(\mathcal{S}+\eta) = 0$ with $\emptyset(\mathcal{S}) = 1, -\delta \le \mathcal{S} \le 0, \emptyset(\mathcal{S}) = -1, 1 \le \mathcal{S} \le 1+\eta$

Illustration 6. $\varepsilon\,\emptyset''(\mathcal{S}) - y' - 2\emptyset(\mathcal{S}-\delta) + \emptyset - 2\emptyset(\mathcal{S}+\eta) = 0$ with $\emptyset(s) = 1, -\delta \le \mathcal{S} \le 0, \emptyset(\mathcal{S}) = -1,\ 1 \le \mathcal{S} \le 1+\eta$.

## 7. Discussions and conclusion

A numerical technique employing a mixed nonpolynomial spline is proposed in the article for solving the SPDDE. A three-term relation is derived using the difference approach. A rapid analysis has been carried out on the convergence of the mechanism. The technique is supported by a range of computational demonstrations. The proposed approach has been tested and practically implemented to tackle various issues.

For illustrations 1-6, the MAEs in the solutions are tabulated in Tables 1-6. In pursuance the contemplated strategy has provided precisely to acquire accurate results relative to the approach produced in [6] in Comparison with computed errors which were produced in [6]. The solutions of the illustrations for various values of $\delta, \eta$ are demonstrated at Figures 1-8. Based on the analysis of Figures 1-4, it has been observed that an increase in δ leads to a reduction in the width of the left end boundary layer. Conversely, an increase in η results in an enlargement of the boundary layer. Based on Figures 5-8, it has been observed that as the value of δ increases, the width of the right end boundary layer enlarged, while it found shrunk as η increases

**Table 1. MAE in Illustration 1 with $\varepsilon = 0.1$**

| $N \rightarrow$ | 8 | 32 | 128 | 512 |
|---|---|---|---|---|
| $\delta \downarrow$ | | Proposed method | | |
| 0.00 | 2.8681(03) | 1.9231(04) | 1.2064(05) | 7.5418(07) |
| 0.05 | 2.8400(03) | 1.8703(04) | 1.1730(05) | 7.3326(07) |
| 0.09 | 2.7729(03) | 1.8094(04) | 1.1351(05) | 7.0958(07) |
| | | Results in [6] | | |
| 0.00 | 9.9078(02) | 3.7007(02) | 9.5467(03) | 2.1450(03) |
| 0.05 | 9.6596(02) | 3.6405(02) | 9.2466(03) | 2.0299(03) |
| 0.09 | 9.2774(02) | 3.5566(02) | 8.9517(03) | 1.9248(03) |

$\boldsymbol{Table\ 2.\ MAE\ in\ Illustration\ 2\ with\ \varepsilon = 0.1}$

| $N \rightarrow$ | 8 | 32 | 128 | 512 |
|---|---|---|---|---|
| $\eta \downarrow$ | | Proposed method | | |
| 0.00 | 2.8681(03) | 1.9231(04) | 1.2064(05) | 7.5418(07) |
| 0.05 | 2.9150(03) | 1.9563(04) | 1.2274(05) | 7.6734(07) |
| 0.09 | 2.9436(03) | 1.9702(04) | 1.2371(05) | 7.7341(07) |
| | | Results in [6] | | |
| 0.00 | 9.9078(02) | 3.7007(02) | 9.5467(03) | 2.1450(03) |
| 0.05 | 9.9775(02) | 3.7270(02) | 9.7965(03) | 2.2447(03) |
| 0.09 | 1.0031(01) | 3.7238(02) | 9.9628(03) | 4.5869(03) |

**Table 3. MAE for Illustration 3 with $\boldsymbol{\varepsilon = 0.1}$**

| $N \rightarrow$ | 8 | 32 | 128 | 512 |
|---|---|---|---|---|
| $\delta \downarrow$ $\eta = 0.05$ | | Proposed method | | |
| 0.00 | 1.4743(02) | 1.0010(03) | 6.2934(05) | 3.9348(06) |
| 0.05 | 1.5547(02) | 1.0631(03) | 6.6867(05) | 4.1808(06) |
| 0.09 | 1.6086(02) | 1.1077(03) | 6.9761(05) | 4.3619(06) |
| $\eta \downarrow$ $\delta = 0.05$ | | | | |
| 0.00 | 1.5164(02) | 1.0332(03) | 6.4947(05) | 4.0606(06) |
| 0.05 | 1.5547(02) | 1.0631(03) | 6.6867(05) | 4.1808(06) |
| 0.09 | 1.5828(02) | 1.0855(03) | 6.8335(05) | 4.2731(06) |
| $\delta \downarrow \eta = 0.05$ | | Results in [6] | | |
| 0.00 | 9.1902(02) | 3.4534(02) | 1.1643(02) | 3.0046(03) |
| 0.05 | 1.0233(01) | 3.8231(02) | 1.2958(02) | 3.3513(03) |
| 0.09 | 1.1018(01) | 4.1108(02) | 1.4001(02) | 3.6292(03) |
| $\eta \downarrow$ $\delta = 0.05$ | | | | |
| 0.00 | 9.7200(02) | 3.6404(02) | 1.2294(02) | 3.1778(03) |
| 0.05 | 1.0233(01) | 3.8231(02) | 1.2958(02) | 3.3513(03) |
| 0.09 | 1.0632(01) | 3.9658(02) | 1.3483(02) | 3.4905(03) |

***Table 4. MAE for Illustration 4 for with $\varepsilon = 0.1$***

| $N \rightarrow$ | 8 | 32 | 128 | 512 |
|---|---|---|---|---|
| $\delta \downarrow$ | Proposed method | | | |
| 0.00 | 9.0740(03) | 5.2137(04) | 3.2532(05) | 2.0332(06) |
| 0.05 | 8.6107(03) | 5.0104(04) | 3.1181(05) | 1.9483(06) |
| 0.09 | 8.1459(03) | 4.7837(04) | 2.9870(05) | 1.8670(06) |
| | Results in [6] | | | |
| 0.00 | 7.8474(02) | 4.6789(02) | 1.7279(02) | 4.4308(03) |
| 0.05 | 9.2225(02) | 3.8283(02) | 1.4877(02) | 3.8067(03) |
| 0.09 | 1.0509(01) | 3.1492(02) | 1.2993(02) | 3.3193(03) |

**Table 5. MAE in Illustration 5 with $\varepsilon = 0.1$**

| $N \rightarrow$ | 8 | 32 | 128 | 512 |
|---|---|---|---|---|
| $\eta \downarrow$ | Proposed method | | | |
| 0.00 | 9.0740(03) | 5.2137(04) | 3.2532(05) | 2.0332(06) |
| 0.05 | 9.4199(03) | 5.3953(04) | 3.3648(05) | 2.1027(06) |
| 0.09 | 9.6224(03) | 5.5419(04) | 3.4383(05) | 2.1493(06) |
| | Results in [6] | | | |
| 0.00 | 7.8474(02) | 4.6789(02) | 1.7279(02) | 4.4308(03) |
| 0.05 | 6.8345(02) | 5.5164(02) | 1.9725(02) | 5.0676(03) |
| 0.09 | 8.3282(02) | 6.1682(02) | 2.1696(02) | 5.5845(03) |

Table 6. MAE in Illustration 6 with $\varepsilon = 0.1$

| $N \rightarrow$ | 8 | 32 | 128 | 512 |
|---|---|---|---|---|

| | | | | |
|---|---|---|---|---|
| $\delta\downarrow$ | $\eta = 0.05$ | Proposed method | | |
| 0.00 | 1.8115(02) | 1.0328(03) | 6.4031(05) | 4.0020(06) |
| 0.05 | 1.6810(02) | 9.5187(04) | 5.9260(05) | 3.7026(06) |
| 0.09 | 1.5665(02) | 8.8187(04) | 5.5198(05) | 3.4484(06) |
| $\eta\downarrow$ | $\delta = 0.05$ | | | |
| 0.00 | 1.5365(02) | 8.6624(04) | 5.4148(05) | 3.3829(06) |
| 0.05 | 1.6810(02) | 9.5187(04) | 5.9260(05) | 3.7026(06) |
| 0.09 | 1.7866(02) | 1.0172(03) | 6.3105(05) | 3.9437(06) |
| $\delta\downarrow$ | $\eta = 0.05$ | Results in [6] | | |
| 0.00 | 9.9300(02) | 3.6850(02) | 1.3316(02) | 3.4288(03) |
| 0.05 | 9.9972(02) | 3.2184(02) | 1.1671(02) | 2.9957(03) |
| 0.09 | 1.0044(01) | 2.8503(02) | 1.0389(02) | 2.6637(03) |
| $\eta\downarrow$ | $\delta = 0.05$ | | | |
| 0.00 | 1.0055(01) | 2.7595(02) | 1.0078(02) | 2.5829(03) |
| 0.05 | 9.9972(02) | 3.2184(02) | 1.1671(02) | 2.9957(03) |
| 0.09 | 9.9440(02) | 3.5914(02) | 1.2973(02) | 3.3404(03) |

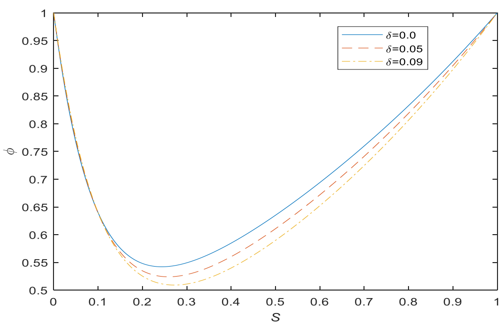


Fig. 1 Layer description in Illustration 1 with $\varepsilon = 0.1$

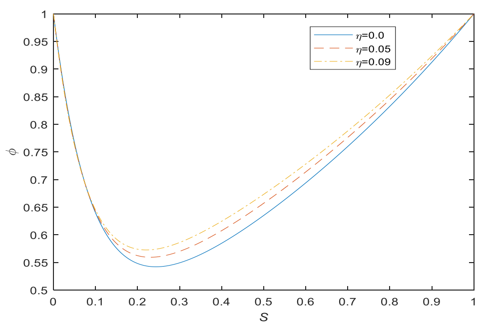


Fig. 2 Layer description in Illustration 2 with $\varepsilon = 0.1$

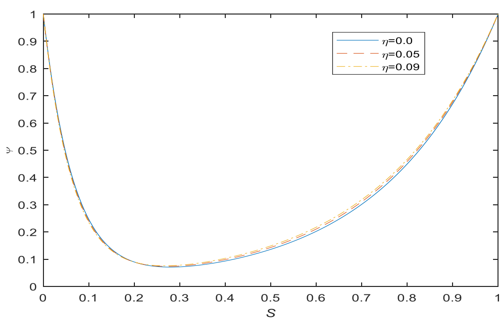


**Fig. 3** Layer description in Illustration 3 with

$\varepsilon = 0.1, \delta = 0.5\varepsilon$

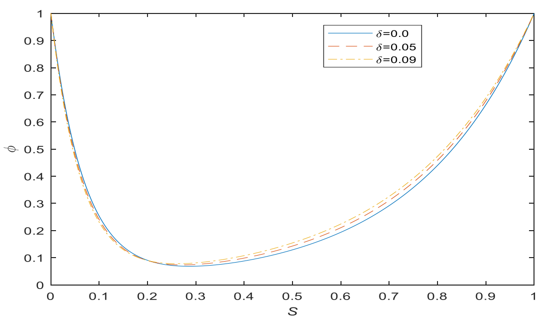


**Fig. 4** Layer description in Illustration 3 with

$\varepsilon = 0.1, \eta = 0.5\varepsilon$

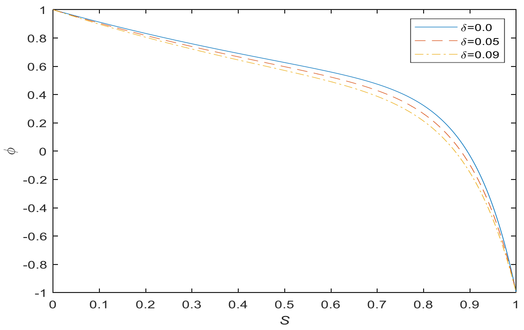


**Fig. 5** Layer description in Illustration 4 with $\varepsilon = 0.1$

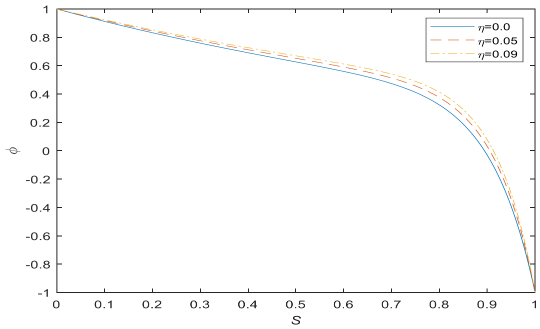


**Fig. 6** Layer description in Example 5 with $\varepsilon = 0.1$

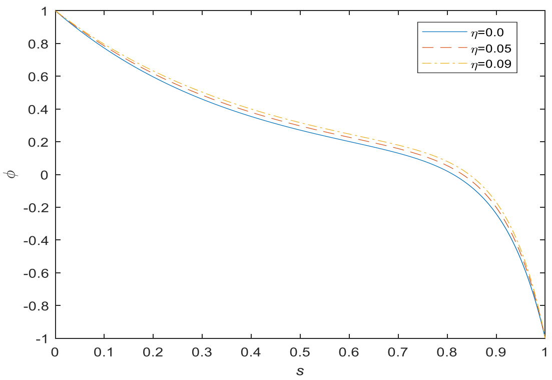


**Fig. 7** Layer description in Example 6 with

$\varepsilon = 0.1, \delta = 0.5\varepsilon$

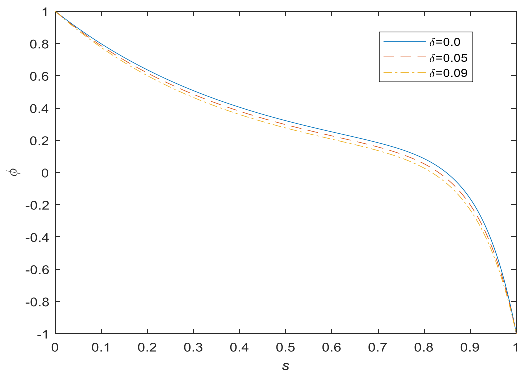


**Fig. 8** Layer description in Example 6 with

$\varepsilon = 0.1, \delta = 0.5\varepsilon$

**Conflicts of interest statement**- The authors declare that they have no conflict of interest.

**References**

[1] M. Adilaxmi, D. Bhargavi, K. Phaneendra (2019), Numerical Solution of Singularly Perturbed Differential-Difference Equations using Multiple Fitting Factors, Communications in Mathematics and Applications. 10(4), pp. 681–691.

[2] M. W. Derstine, H. M. Gibbs, , F. A. Hopf and D. L. Kaplan (1982), Bifurcation gap in a hybrid optical system, Phys. Rev. A., 26, pp. 3720-3722.

[3] E. P. Doolan, J. J. H. Miller and W. H. A. Schilders (1980), Uniform Numerical Methods for Problems with Initial and Boundary Layers, Boole Press, Dublin.

[4] D. Kumara Swamy, K. Phaneendra, Y. N. Reddy (2018), Accurate Numerical Method for Singularly Perturbed Differential Difference Equations with Mixed Shifts. Khayyam. J. Math., 4, pp. 110 –122.

[5] M. K. Kadalbajoo, D. Kumar (2008), Fitted mesh B-spline collocation method for singularly perturbed differential–difference equations with small delay, Appl. Math. Comput., 204 (1), pp. 90–98.

[6] M.K. Kadalbajoo, K. K. Sharma (2005), Numerical treatment of mathematical model arising from a model of neuronal variability. Journal of Mathematical Analysis and Applications. 307, 606-627.

[7] M.K. Kadalbajoo, K. K. Sharma (2003), An ε-uniform fitted operator method for solving boundary- value problems for singularly perturbed delay differential equations: layer behavior, Int. J. Comput. Math., 80 (10), pp. 1261–1276.

[8] M.K. Kadalbajoo, K. K. Sharma (2004), Numerical analysis of singularly perturbed delay differential equations with layer behavior, Appl. Math. Comput., 157(1), pp. 11–28.

[9] M.K. Kadalbajoo, K. K. Sharma (2004), ε-Uniform fitted mesh method for singularly perturbed differential–difference equations: Mixed type of shifts with layer behavior, Int. J. Comput. Math. , 81(1), pp. 49–62.

[10] P. V. Kokotovic, H. K. Khalil and J. O'Reilly (1986), Singular perturbation methods in control analysis and design. McGraw-Hill, Academic Press.

[11] Y. Kuang (1993), Delay Differential Equations with Applications in Population Dynamics. Mathematics in Science and Engineering, Academic Press Inc, Boston.

[12] Lakshmi Sirisha, K. Phaneendra and Y. N. Reddy (2018), Mixed finite difference method for singularly perturbed differential difference equations with mixed shifts via domain decomposition, Ain Shams Engineering Journal. 9(4), pp. 647-654

[13] C. G. Lange, R. M. Miura (1985), Singular perturbation analysis of boundary value problems for differential-difference equations III Turning point problems, SIAM. J. Appl. Math., 45(5), pp. 708—734.

[14] C. G. Lange, R. M. Miura (1994), Singular perturbation analysis of boundary-value problems for differential-difference equations V Small shifts with layer behaviour, SIAM. J. Appl. Math., 54(1), pp. 249-272.

[15] C. G. Lange, R. M. Miura (1994), Singular perturbation analysis of boundary-value problems for differential-difference equations VI small shifts with rapid oscillations, SIAM. J. Appl. Math., 54(1), pp. 273—283.

[16] M. C. Mackey, L. Glass (1977), Oscillations and chaos in physiological control systems, Science, 197, pp. 287-289.

[17] A. Martin, S. Raun (2001), Predetor-prey models with delay and prey harvesting, J. Math. Bio., 43, pp. 247-267.

[18] J.J.H. Miller, E. O'Riordan and G. I. Shishkin (1996), Fitted Numerical Methods for Singular Perturbation Problem, World Scientific, Singapore.

[19] P. Miroslav Musila (1991), Generalized Stein's model for anatomically complex neurons, Bio-Systems, 25, pp. 179–191.

[20] O'Malley Jr., R.E.: Two parameter singular perturbation problems for second order equations. J. Math. Mech. 16, 1143–1164 (1967)

[21] R. E. O'Malley (1974), Introduction to Singular Perturbations. Academic Press, New York.

[22] K. Phaneendra, D. Kumara Swamy and Y. N. Reddy (2018), Computational method for singularly perturbed delay differential equations with twin layers or oscillatory behaviour, Khayyam J. Math., 4(2), pp. 110 -122.

[23] G. Sampath, S. K. Srinivasan(1977), Stochastic models for spike trains of single neurons, Lecture Notes in Biomathematics, Springer, Berlin.

[24] D. K. Swamy, K. Phaneendra and Y. N. Reddy (2016), Solution of Singularly perturbed differential difference equations with mixed shifts using Galerkin method with exponential fitting, Chinese Journal of Mathematics, 2016.

[25] R. K. A. S. Venkata, M. M. K. Palli (2017), A numerical approach for solving singularly perturbed convection delay problems via exponentially fitted spline method, Calcolo, 54, pp. 943 – 967.

[26] R. S. Varga (1962), Matrix Iterative Analysis, Prentice-Hall, Englewood Cliffs, New Jersey.

[27] M. Villasana, A. Radunskaya (2003), A delay differential equation model for tumor growth, J. Math. Biol., 47(3), pp. 270-294.

[28] D. M. Young (1971), Iterative Solutions of Large Linear Systems, Academic press, New York.