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An Analysis of Generalized N-Projective Curvature Tensor of Lorentzian β-Kenmotsu Manifolds Admitting Zamkovoy Connection

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Abstract:- This paper investigates Lorentzian β-Kenmotsu Manifolds with Zamkovoy connections. We introduce a new (0, 2) type symmetric tensor Z˚, derived from the N-projective curvature tensor, termed the generalized N-projective curvature tensor. We prove that when these manifolds exhibit generalized N-projectively semi-symmetric properties, they become Einstein manifolds. Additionally, we show that the condition of generalized N-projective ϕ-symmetry on Lorentzian β-Kenmotsu Manifold with Zamkovoy connection implies that the manifold is again an Einstein manifold.

Mathematics Subject Classification- *53C15, 53C25, 35C05, 53C20, 53D10.*

Keywords*: Zamkovoy connections, Lorentzian β-Kenmotsu Manifolds, N-projective curvature tensor, Einstein manifolds.*

1. Introduction

Lorentzian manifolds are smooth manifolds equipped with a Lorentzian metric, which generalizes the notion of distance and angle from Euclidean spaces to spaces with a non-degenerate, indefinite quadratic form. These manifolds often arise in the study of general relativity, where spacetime is modeled as a Lorentzian manifold.

This paper contains the term N-projective curvature tensor , which was first introduced by G.P. Pokhariyal and R.S. Mishra [3]. This curvature tensor has further been studied by R. H. Ojha[4] and many other researchers [5, 6, 7, 8, 9, 1, 2]. For more details, we refer to [21, 22, 23, 24, 25, 26, 27, 28, 29] and the references therein.

The type $(0,3)$ N-projective curvature tensor N^* is given by

$$N^*(U,V)W = R(U,V)W - \frac{1}{2(n-1)}[S(V,W)U - S(U,W)V + g(V,W)QU - g(U,W)QV] \tag{1.1}$$

for all vector fields U, V and $W \in \chi(\ M)$.

The symbol $R(U,V)W$ refers to the Riemannian curvature tensor of type (0, 3) and S denotes the Ricci tensor, i.e, $S(L,M) = g(QL,M)$, where Q being the Ricci operator of type (1,1). The type $(0,4)$ N-projective curvature tensor field N^* is given by

$$'N^*(U,V,W,L) = {'R}(U,V,W,L) - \frac{1}{2(n-1)}[S(V,W)g(U,L) - S(U,W)g(V,L) + g(V,W)S(U,L) - g(U,W)S(V,L)] \tag{1.2}$$

where, $'N^*(U,V,W,L) = g(N^*(U,V)W,L)$ and $'R(U,V,W,L) = g(R(U,V)W,L)$

for arbitrary vector fields $U,V,W,L \in \chi(M)$. C.A. Mantica and Y.J. Suh [11] considered a new symmetric tensor Z of type $(0,2)$, given by

$$\boxed{Z(U,V) = S(U,V) + \omega g(U,V)} \tag{1.3}$$

with ω as an arbitrary scalar function. This tensor Z has been used by [12, 13] to obtain a new tensor field out of a given tensor field. We use it to generalise the N^*-projective curvature tensor. Using equation (3) in the equation (2), we get

$$'N^*(U,V,W,L) = {'R}(U,V,W,L) - \frac{1}{2(n-1)}[Z(V,W)g(U,L) - Z(U,W)g(V,L) + g(V,W)Z(U,L) - g(U,W)Z(V,L) \; - \frac{\omega}{(n-1)}[g(U,W)Z(V,L) - g(V,W)Z(U,L))] \tag{1.4}$$

If we denote the first five terms on the right hand side of the above equation by $'N^{**}(U,V,W,L)$, i.e.,

$$'N^{**}(U,V,W,L) = {'R}(U,V,W,L) - \frac{1}{2(n-1)}[Z(V,W)g(U,L) - Z(U,W)g(V,L) + g(V,W)Z(U,L) - g(U,W)Z(V,L)] \tag{1.5}$$

then the equation (4) can be rewritten as

$$'N^{**}(U,V,W,L) = {'N^*}(U,V,W,L) + \frac{\omega}{(n-1)}[g(U,W)Z(V,L) - g(V,W)Z(U,L))] \tag{1.6}$$

The new tensor field $'N^{**}$ defined by the equation (1.6) is termed as generalized N-projective curvature tensor.

The concept of Zamkovoy cannonical connection or in short Zamkovoy connection was first introduced by S. Zamkovoy [10] on a para-contact manifold. After this introduction many authors have developed and studied Zamkovoy connection on many different manifolds such as generalized pseudo-Ricci symmetric Sasakian manifolds [14], almost pseudo-symmetric Sasakian manifolds [15], para-Kenmotsu manifold [16], Sasakian manifolds [17] and LP-Sasakian manifolds [18].

For an n-dimensional almost contact metric manifold M equipped with metric structure (ϕ,ξ,η,g) consisting of a (1,1) tensor field ϕ , a vector field ξ , a 1-form η and a Riemannian metric g, the relation between Zamkovoy connection $\breve{\nabla}$ and Levi-civita connection ∇ is given by

$$\boxed{\breve{\nabla}_U V = \nabla_U V + (\nabla_U \eta)(V)\xi - \eta(V)\nabla_U \xi + \eta(U)\phi V} \tag{1.7}$$

for all $U,V \in \chi(M)$.

This paper delves into the study of the generalized N-projective curvature tensor of Lorentzian β-Kenmotsu Manifold with respect to the Zamkovoy connection, exploring various properties. The paper is divided into six parts:

- Section 2 gives preliminaries on the Lorentzian β-Kenmotsu Manifold .
- Section 3 describes about the generalized N-projective curvature tensor in Lorentzian β-Kenmotsu Manifold with the Zamkovoy connection.
- Section 4 provides proof that the generalized N-projectively semi-symmetric Lorentzian β-Kenmotsu Manifold with Zamkovoy connection is an Einstein manifold.
- Section 5 gives the result that generalized N-projectively Lorentzian β-Kenmotsu Manifold is either an Einstein manifold or $\omega = \beta^2 \frac{(1-n)}{2}$.
- Finally, in the last section, we provide proof that Lorentzian β-Kenmotsu Manifold satisfying $\phi^2((\nabla_L \breve{N}^{**})(U,V)W) = 0$ is an Einstein manifold.

2. Preliminaries

Let M be a differentiable manifold of dimension n. We call M as Lorentzian β-Kenmotsu manifold if it admits a (1, 1)-tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and Lorentzian metric g which satisfy[19]

$$\eta(\xi) = -1, \tag{2.1}$$

$$\phi\xi = 0, \tag{2.2}$$

$$\eta(\phi U) = 0, \tag{2.3}$$

$$\phi^2 U = U + \eta(U)\xi, \tag{2.4}$$

$$g(U,\xi) = \eta(U), \tag{2.5}$$

$$g\big(\phi(U),\phi(V)\big) = g(U,V) + \eta(U)\eta(V), \tag{2.6}$$

for all $U,V \in \chi(M)$.

Also, an Lorentzian β-Kenmotsu manifold M is satisfying

$$\nabla_U \xi = -\beta[U + \eta(U)\xi], \tag{2.7}$$

$$\nabla_U \eta(V) = \beta[g(U,V) - \eta(U)\eta(V)], \tag{2.8}$$

$$(\nabla_U \phi)V = \beta[g(\phi U,V) + \eta(V)\phi U], \tag{2.9}$$

where '∇' denotes the operator of covariant differentiation with respect to the Lorentzian metric g.

Further, on an Lorentzian β-Kenmotsu manifold M the following relations hold [19]

$$\eta(R(U,V)W) = \beta^2[g(U,W)\eta(V) - g(V,W)\eta(U)], \tag{2.10}$$

$$R(\xi,U)V = \beta^2[\eta(V)U - g(U,V)\xi], \tag{2.11}$$

$$R(U,V)\xi = \beta^2[\eta(U)V - \eta(V)U], \tag{2.12}$$

$$S(U,\xi) = -(n-1)\beta^2\eta(U), \tag{2.13}$$

$$Q\xi = -(n-1)\beta^2\xi, \tag{2.14}$$

$$S(\xi,\xi) = (n-1)\beta^2, \tag{2.15}$$

$$g(\xi,\xi) = \eta(\xi) = -1, \tag{2.16}$$

Definition 2.1

Let M be a Lorentzian β-Kenmotsu manifold. We call M as generalized η-Einstein manifold if its Ricci tensor S is of the form [20]

$$S(U,V) = \alpha_1 g(U,V) + \alpha_2 \eta(U)\eta(V),$$

where, α_1 and α_2 are smooth functions on M.

For the case, when $\alpha_3 = 0$ and $\alpha_2 = \alpha_3 = 0$, then the manifold is said to be an η-Einstein and Einstein, respectively.

3. Generalized N-Projective Curvature Tensor in Lorentzian β-Kenmotsu Manifold

The focus of this section is on examining the N-Projective Curvature Tensor of Lorentzian β-Kenmotsu Manifold with respect to the Zamkovoy connection. We then generalize its properties by introducing the tensor Z.

Adopting a similar format as in equation (1.1), we define the N-projective curvature tensor $\breve{N}$ with respect to the Zamkovoy connection $\breve{\nabla}$ by the following relation

$$\breve{N}^*(U,V)W = \breve{R}(U,V)W - \frac{1}{2(n-1)}[\breve{S}(V,W)U - \breve{S}(U,W)V + g(V,W)\breve{Q}U - g(U,W)\breve{Q}V] \quad (3.1)$$

Upon taking the inner product of $\breve{N}^*(U,V)W$ with the metric tensor g, we the type (0,4) tensor field $'\breve{N}^*$ shown below

$$'\breve{N}^*(U,V,W,X) = '\breve{R}(U,V,W,X) - \frac{1}{2(n-1)}[\breve{S}(V,W)g(U,X) - \breve{S}(U,W)g(V,X) + g(V,W)\breve{S}(U,X) - g(U,W)\breve{S}(V,X)] \quad (3.2)$$

where

$$'\breve{N}^*(U,V,W,X) = g(\breve{N}^*(U,V,W),X)$$

and

$$'\breve{R}(U,V,W,X) = g(\breve{R}(U,V,W),X)$$

where U,V,W and X are vector fields. Moreover, performing covariant differentiation of equation (3.1) with respect to L results in

$$(\nabla_L\breve{N}^*)(U,V,W) = (\nabla_L\breve{R})(U,V,W) - \frac{1}{2(n-1)}[(\nabla_L\breve{S})(V,W)U - (\nabla_L\breve{S})(U,W)V + g(V,W)(\nabla_L\breve{Q})U - g(U,W)(\nabla_L\breve{Q})V] \quad (3.3)$$

The expression (3.1) is rewritten using relation (1.3) in the following manner

$$'\breve{N}^*(U,V,W,X) = '\breve{R}(U,V,W,X) - \frac{1}{2(n-1)}[Z(V,W)g(U,X) - Z(U,W)g(V,X) + g(V,W)Z(U,X) - g(U,W)Z(V,X)] - \frac{\omega}{(n-1)}[g(U,W)g(V,X) - g(U,X)g(V,W)] \quad (3.4)$$

To construct a new tensor field from the expression provided above, we pick the first five terms on the right-hand side and write

$$'\breve{N}^{**}(U,V,W,X) = '\breve{R}(U,V,W,X) - \frac{1}{2(n-1)}[Z(V,W)g(U,X) - Z(U,W)g(V,X) + g(V,W)Z(U,X) - g(U,W)Z(V,X)] \quad (3.5)$$

We denote the tensor $'\breve{N}^{**}$, derived from the equation provided, as the generalized N-projective curvature tensor for Lorentzian β-Kenmotsu manifolds with respect to the Zamkovoy connection.

Considering equation (3.5), equation (3.4) is rewritten as

$$'\breve{N}^{**}(U,V,W,X) = '\breve{N}^*(U,V,W,X) + \frac{\omega}{(n-1)}[g(U,W)g(V,X) - g(U,X)g(V,W)] \quad (3.6)$$

Clearly, by setting $\omega = 0$, it follows from equation (3.6) that

$$'\breve{N}^{**}(U,V,W,X) = '\breve{N}^*(U,V,W,X), \quad (3.7)$$

Hence, when the scalar function ω becomes zero, it indicates that the two tensor fields, namely the N-projective and generalized N-projective curvature tensor fields, are identical.

Remark 3.1 *It is very easy to note that the N-projective curvature tensor field of a Lorentzian β-Kenmotsu manifold relative to the Zamkovoy connection $\breve{\nabla}$ satisfies the following properties:*

1. $'\breve{N}^*(U,V,W,X) + '\breve{N}^*(V,U,W,X) = 0,$
2. $'\breve{N}^*(U,V,W,X) + '\breve{N}^*(U,V,X,W) = 0$
3. $'\breve{N}^*(U,V,W,X) = '\breve{N}^*(W,X,U,V)$

Theorem 3.1 *Generalized N-projective curvature tensor N^{**} of a Lorentzian β-Kenmotsu manifold relative to the Zamkovoy connection $\breve{\nabla}$ is satisfies the following properties:*

1. $'\breve{N}^{**}(U,V,W,X) + '\breve{N}^{**}(V,U,W,X) = 0,$
2. $'\breve{N}^{**}(U,V,W,X) + '\breve{N}^{**}(U,V,X,W) = 0$
3. $'\breve{N}^{**}(U,V,W,X) = '\breve{N}^{**}(W,X,U,V)$

Proof:

1) Interchanging the vector fields in first two slots in the equation (3.6) to obtain

$$'\breve{N}^{**}(V,U,W,X) = '\breve{N}^*(V,U,W,X) + \frac{\omega}{(n-1)}[g(V,W)g(U,X) - g(V,X)g(U,W)] \tag{3.8}$$

Next, we combine equations (3.6) and (3.8), applying property (I) from remark (3.1) to obtain

$$'\breve{N}^{**}(U,V,W,X) = -'\breve{N}^{**}(V,U,W,X)$$

Which provides evidence of the skew symmetry of the generalized N-projective curvature tensor $'\breve{N}^{**}$ in the first two slots.

2) We now exchange the vector fields in the last two slots of equation (3.6) to yield

$$'\breve{N}^{**}(U,V,X,W) = '\breve{N}^*(U,V,X,W) + \frac{\omega}{(n-1)}[g(U,X)g(V,W) - g(U,W)g(V,X)] \tag{3.9}$$

Next, we combine equations (3.6) and (3.9), applying property (II) from remark (3.1) to obtain

$$'\breve{N}^{**}(U,V,W,X) = -'\breve{N}^{**}(U,V,X,W)$$

This validates the skew symmetry of the tensor field $'\breve{N}^{**}$ in the last two slots. **3)** Proceeding further, we interchange U with W and V with L in equation (3.6) to obtain

$$'\breve{N}^{**}(W,X,U,V) = '\breve{N}^*(U,V,X,W) + \frac{\omega}{(n-1)}[g(U,X)g(V,W) - g(U,W)g(V,X)] \tag{3.10}$$

Considering property (III) from remark (3.1), the combination of equations (3.6) and (3.10) results in

$$'\breve{N}^*(U,V,W,X) = '\breve{N}^*(W,X,U,V)$$

which verifies the symmetry of the tensor field $'\breve{N}^*$ in the pair of slots.

Theorem 3.2 *The generalized N-projective curvature tensor field $\breve{N}^{**}$ of Lorentzian β-Kenmotsu manifold relative to the Zamkovoy connection satisfies the Bianchi's first identity*

$$\breve{N}^{**}(U,V)W + \breve{N}^{**}(V,W)U + \breve{N}^{**}(W,U)V = 0.$$

Proof:

From the equation (3.6), we have

$$\breve{N}^{**}(U,V)W = \breve{N}^*(U,V)W + \frac{\omega}{(n-1)}[g(U,W)V - g(V,W)U], \tag{3.11}$$

Rearranging U, V, and W cyclically in the equation above, we formulate the following two equations

$$\breve{N}^{**}(V,W)U = \breve{N}^{*}(V,W)U + \frac{\omega}{(n-1)}[g(V,U)W - g(W,U)V] \tag{3.12}$$

and

$$\breve{N}^{**}(W,U)V = \breve{N}^{*}(W,U)V + \frac{\omega}{(n-1)}[g(W,V)U - g(U,V)W] \tag{3.13}$$

Summing up equations (3.11), (3.12), and (3.13), and utilizing the fact that

$$\breve{N}^{*}(U,V)W + \breve{N}^{*}(V,W)U + \breve{N}^{*}(W,U)V = 0 \tag{3.14}$$

we obtain

$$\breve{N}^{**}(U,V)W + \breve{N}^{**}(V,W)U + \breve{N}^{**}(W,U)V = 0 \tag{3.15}$$

Hence, the theorem holds.

Theorem 3.3 *The generalized N-projective curvature tensor field $\breve{N}^{**}$ of Lorentzian β-Kenmotsu manifold relative to the Zamkovoy connection satisfies the following identities:*

a) $\breve{N}^{**}(U,V)W = [\frac{3\beta^2}{2} + \frac{\omega}{(n-1)}][g(U,W)V - g(V,W)U] + \frac{1}{2(n-1)}[S(U,W)V - S(V,W)U]$

b) $\breve{N}^{**}(\xi,V)W = -\breve{N}^{**}(V,\xi)W = [\frac{\beta^2}{2} + \frac{\omega}{(n-1)}][\eta(W)V - g(V,W)\xi] + \frac{1}{2(n-1)}[\eta(W)QV - S(V,W)U\xi]$

c) $\breve{N}^{**}(U,V)\xi = [\frac{\beta^2}{2} + \frac{\omega}{(n-1)}][\eta(U)V - \eta(V)U] + \frac{1}{2(n-1)}[\eta(U)QV - \eta(V)QU]$

Proof-

a) By performing the inner product operation on equation (3.11) using ξ, we arrive at

$$\eta(\breve{N}^{**}(U,V)W) = \eta(\breve{N}^{*}(U,V)W) + \frac{\omega}{(n-1)}[g(U,W)\eta(V) - g(V,W)\eta(U)]$$

Now, Applying equations (2.6), (2.11), (2.21), (2.23), and (3.1) as described above leads to the desired outcome.

b) Now, Substituting ξ in place of U within equation (3.11), the expression becomes

$$\breve{N}^{**}(\xi,V)W = \breve{N}^{*}(\xi,V)W + \frac{\omega}{(n-1)}[g(\xi,W)V - g(V,W)\xi]$$

By incorporating equations (2.6), (2.21), (2.23), and (3.1) into the equation provided, we obtain the result.

c) When W is replaced by ξ in equation (3.11), the resulting expression is

$$\breve{N}^{**}(U,V)\xi = \breve{N}^{*}(U,V)\xi + \frac{\omega}{(n-1)}[g(U,\xi)V - g(V,\xi)U]$$

which, in view of the equations (2.6), (2.21), (2.22) and (3.1), proves the result.

4. Generalized N-projectively Lorentzian β-Kenmotsu Manifold with Zamkovoy connection satisfying $(R(\xi,Y).\breve{N}^{**}).(U,V)W$

Within this section, we examine Lorentzian β-Kenmotsu Manifolds that are generalized N-projectively semi-symmetric, with respect to the Zamkovoy connection.

Definition 4.1. A para-Kenmotsu manifold is called as semi-symmetric manifold [30] if its curvature tensor satisfies

$$R(U,V).R = 0, \tag{4.1}$$

where the curvature operator $R(U,V)$ is the derivation of the tensor algebra at each point of the manifold.

Analogous to the definition (4.1), we propose the following definition:

Definition 4.2 A para-Kenmotsu manifold is generalized N-projectively semi-symmetric if it satisfies the condition of the form

$$R(U,V).\breve{N}^{**}=0, \tag{4.2}$$

where $\breve{N}^{**}$ is generalized N-projective curvature tensor of para-Kenmotsu manifold relative to the Zamkovoy connection.

Theorem 4.1 *A generalized N-projectively semi-symmetric Lorentzian β-Kenmotsu manifold with respect to the Zamkovoy connection is an η-Einstein manifold.*

Proof:

Consider

$$R(X,Y).\breve{N}^{**}=0$$

We now assign X as ξ in the above expression to obtain

$$(R(\xi,Y).\breve{N}^{**})(U,V)W=0$$

for all $X,Y,U,V,W\in\chi(M)$, which gives

$$R(\xi,Y).(\breve{N}^{**}(U,V).W)-\breve{N}^{**}(R(\xi,Y).U,V).W-\breve{N}^{**}(U,R(\xi,Y).V).W-\breve{N}^{**}(U,V).R(\xi,Y).W=0 \tag{4.3}$$

As a result of the relation (2.11), the above equation reduces to

$$\eta(\breve{N}^{**}(U,V)W)Y-g(Y,\breve{N}^{**}(U,V)W)\xi-\eta(U)\breve{N}^{**}(Y,V)W-\eta(V)\breve{N}^{**}(U,Y)W-\eta(W)\breve{N}^{**}(U,V)Y$$
$$+g(Y,U)\breve{N}^{**}(\xi,V)W+g(Y,V)\breve{N}^{**}(U,\xi)W+g(Y,W)\breve{N}^{**}(U,V)\xi=0 \tag{4.4}$$

We then perform the inner product of the above expression with the vector field ξ, applying equations (2.4), (3.6), (3.14), (3.15), and (3.16) to find

$$'\breve{N}^{**}(U,V,W,Y)-[\frac{3\beta^2}{2}+\frac{\omega}{(n-1)}][g(U,Y)\eta(V)\eta(W)-g(V,Y)\eta(U)\eta(W)]$$
$$-\frac{1}{2(n-1)}[S(U,Y)\eta(V)\eta(W)-S(V,Y)\eta(U)\eta(W)]+\frac{\omega}{(n-1)}][g(U,Y)\eta(V)\eta(W)$$
$$-g(V,Y)\eta(U)\eta(W)]+\frac{\beta^2}{2}[g(V,W)g(Y,U)-g(U,W)g(Y,V)]$$
$$+\frac{1}{2(n-1)}[S(V,W)g(Y,U)-S(U,W)g(Y,V)]+\frac{\omega}{(n-1)}[g(V,W)g(Y,U)-g(U,W)g(Y,V)]=0 \tag{4.5}$$

With equations (2.22) and (3.2) taken into account, the equation above becomes

$$'\breve{R}(U,V,W,Y)=\frac{1}{2(n-1)}[\breve{S}(V,W)g(U,Y)-\breve{S}(U,W)g(V,Y)+g(V,W)\breve{S}(U,Y)$$
$$-g(U,W)\breve{S}(V,Y)]-[\frac{3\beta^2}{2}+\frac{\omega}{(n-1)}][g(U,Y)\eta(V)\eta(W)-g(V,Y)\eta(U)\eta(W)]$$
$$-\frac{1}{2(n-1)}[S(U,Y)\eta(V)\eta(W)-S(V,Y)\eta(U)\eta(W)]+\frac{\omega}{(n-1)}][g(U,Y)\eta(V)\eta(W)$$
$$-g(V,Y)\eta(U)\eta(W)]+\frac{\beta^2}{2}[g(V,W)g(Y,U)-g(U,W)g(Y,V)]$$
$$+\frac{1}{2(n-1)}[S(V,W)g(Y,U)-S(U,W)g(Y,V)]+\frac{\omega}{(n-1)}[g(V,W)g(Y,U)-g(U,W)g(Y,V)]=0 \tag{4.6}$$

Considering $e_i : i = 1,2,\ldots,n$ as an orthonormal basis, we substitute $U = Y = e_i$ into the equation above. By summing over i, we arrive at

$$\omega g(V,W) = (n-1)\eta(V)\eta(W) \tag{4.7}$$

Now, putting $W = \xi$ in above equation, we get

$$\omega = (1-n)$$

which justifies the theorem.

5. Lorentzian β-Kenmotsu Manifold with Zamkovoy connection satisfying $\breve{N}^{**}(U,V).S = 0$

We proceed by considering a Lorentzian β-Kenmotsu Manifold which satisfies the following condition

$$\breve{N}^{**}(U,V).S = 0 \tag{5.1}$$

for all vector fields U and V .

where, $\breve{N}^{**}$ is called the generalized N-projective curvature tensor field relative to the Zamkovoy connection.

Theorem 5.1 *A Lorentzian β-Kenmotsu manifold conceding Zamkovoy connection and satisfying the condition $\breve{N}^{**}(U,V).S = 0$ is either an Einstein manifold or $\omega = \beta^2 \frac{(1-n)}{2}$ on it.*

Proof:

Suppose the Lorentzian β-Kenmotsu manifold satisfies the condition

$$(\breve{N}^{**}(\xi,X).S)(U,V) = 0 \tag{5.2}$$

This implies

$$S(\breve{N}^{**}(\xi,X)U,V) + S(U,\breve{N}^{**}(\xi,X)V) = 0 \tag{5.3}$$

By utilizing the equations (2.14), (2.15) and (3.14) in the above equation, we get

$$A[S(X,V)\eta(U) + S(U,X)\eta(V)] + A(n-1)\beta^2[g(X,V)\eta(U) + g(X,U)\eta(V)] = 0 \tag{5.4}$$

where, $A = [\frac{\beta^2}{2} + \frac{\omega}{(n-1)}]$

Upon replacing U by ξ in the above equation and making use of equation (2.4), (2.6) and (2.14), we arrive at

$$A[S(X,V) + (n-1)\beta^2 g(X,V)] = 0 \tag{5.5}$$

This leads us to conclude that either

$$\omega = \beta^2 \frac{(1-n)}{2}$$

or

$$S(X,V) = -(n-1)\beta^2 g(X,V)$$

This justifies the theorem.

6. A Lorentzian β-Kenmotsu Manifold with Zamkovoy connection satisfying $\phi^2((\nabla_L \breve{N}^{**})(U,V)W) = 0$

Our focus in this section turns to locally N-projectively ϕ-symmetric manifold , considering them in the context of the Zamkovoy connection. The notion of local ϕ-symmetry for Sasakian manifolds was introduced by Takahashi [31].

Definition 6.1. A Riemannian manifold is known to be locally ϕ-symmetric if

$$\phi^2((\nabla_L \breve{R})(U,V)W) = 0 \tag{6.1}$$

for vector fields U, V and W and L orthogonal to ξ.

Analogous to the conditions (6.1), we consider a Lorentzian β-Kenmotsu manifold satisfying

$$\phi^2((\nabla_L \breve{N}^{**})(U,V)W) = 0 \tag{6.2}$$

for arbitary vector fields U, V, W, L and call it as a N-projectively ϕ-symmetric manifold.

Theorem 6.1 *A Lorentzian β-Kenmotsu manifold conceding Zamkovoy connection satifying $\phi^2((\nabla_L \breve{N}^{**})(U,V)W) = 0$ is an Einstein manifold.*

Proof:

Taking covariant derivative of equation (3.11) with respect to L gives us

$$(\nabla_L \breve{N}^{**})(U,V)W = (\nabla_L \breve{N}^{*})(U,V)W + \frac{dr(\omega)}{(n-1)}[g(U,W)V - g(V,W)U] \tag{6.3}$$

Now, substituting equation (3.3) in the above equation, we arrive at

$$(\nabla_L \breve{N}^{**})(U,V)W = (\nabla_L \breve{R})(U,V)W + \frac{dr(\omega)}{(n-1)}[g(U,W)V - g(V,W)U]$$

$$-\frac{1}{(2n-1)}[(\nabla_L \breve{S})(V,W)U - (\nabla_L \breve{S})(U,W)V]$$

$$+[g(V,W)(\nabla_L \breve{Q})U - g(U,W)(\nabla_L \breve{Q})V] \tag{6.4}$$

As we know,

$$\phi^2((\nabla_L \breve{N}^{**})(U,V,W)) = 0 \tag{6.5}$$

Using equation (2.1) in above equation (6.5), we get

$$(\nabla_L \breve{N}^{**})(U,V)W = \eta((\nabla_L \breve{N}^{**})(U,V)W)\xi \tag{6.6}$$

Now, use of the equation (6.4) in the above equation (6.6), provides

$$(\nabla_L \breve{R})(U,V)W + \frac{dr(\omega)}{(n-1)}[g(U,W)V - g(V,W)U] - \frac{1}{(2n-1)}[(\nabla_L \breve{S})(V,W)U - (\nabla_L \breve{S})(U,W)V]$$

$$+[g(V,W)(\nabla_L \breve{Q})U - g(U,W)(\nabla_L \breve{Q})V] = \eta((\nabla_L \breve{R})(U,V)W)\xi + \frac{dr(\omega)}{(n-1)}[g(U,W)\eta(V) - g(V,W)\eta(U)]\xi$$

$$-\frac{1}{(2n-1)}[(\nabla_L \breve{S})(V,W)\eta(U) - (\nabla_L \breve{S})(U,W)\eta(V)]\xi + [g(V,W)\eta((\nabla_L \breve{Q})U) - g(U,W)\eta((\nabla_L \breve{Q})V)]\xi$$

Taking inner product of the above equation with the vector field X, we get

$$g((\nabla_L \breve{R})(U,V)W, X) + \frac{dr(\omega)}{(n-1)}[g(U,W)g(V,X) - g(V,W)g(U,X)] \tag{6.7}$$

$$-\frac{1}{(2n-1)}[(\nabla_L \breve{S})(V,W)g(U,X) - (\nabla_L \breve{S})(U,W)g(V,X)]$$

$$+[g(V,W)g((\nabla_L \breve{Q})U,X) - g(U,W)g((\nabla_L \breve{Q})V,X)]$$

$$= \eta((\nabla_L \breve{R})(U,V)W)\eta(X) + \frac{dr(\omega)}{(n-1)}[g(U,W)\eta(V) - g(V,W)\eta(U)]\eta(X)$$

$$-\frac{1}{(2n-1)}[(\nabla_L \breve{S})(V,W)\eta(U) - (\nabla_L \breve{S})(U,W)\eta(V)]\eta(X)$$

$$+[g(V,W)\eta((\nabla_L \breve{Q})U) - g(U,W)\eta((\nabla_L \breve{Q})V)]\eta(X)$$

Upon replacing U and X by e_i in the equation above and summing over i from 1 to n, we obtain

$$-dr\omega g(V,W) - \frac{1}{2}(\nabla_L \breve{S})(V,W) - \frac{1}{(2n-1)}[g(\nabla_L \breve{Q})e_i, e_i)g(V,W) - \tag{6.8}$$

$$g((\nabla_L \breve{Q})Y, e_i)g(e_i, W)] = -\eta((\nabla_L \breve{R})(e_i, V, W))\eta(e_i) - \frac{dr\omega}{(n-1)}[\eta(W)\eta(V)$$

$$+g(V,W)] - \frac{1}{2(n-1)}[(\nabla_L \breve{S})(V,W) + (\nabla_L \breve{S})(e_i, W)\eta(W)\eta(e_i)$$

$$-g(V,W)\eta((\nabla_L \breve{Q})e_i)\eta(e_i) + \eta((\nabla_L \breve{Q})Y)\eta(W)$$

If we take $W = \xi$ in the preceding equation, we have

$$\eta(V)dr\omega + \frac{(n-2)}{2(n-1)}(\nabla_L \breve{S})(V,\xi) + \frac{1}{2(n-1)}[dr(\breve{L})\eta(V)] \tag{6.9}$$

$$-\eta((\nabla_L \breve{R})(e_i, V, \xi))\eta(e_i) + \frac{1}{2(n-1)}[(\nabla_L \breve{S})(e_i, \xi)\eta(e_i)$$

$$+\eta(V)\eta((\nabla_L \breve{Q})e_i)\eta(e_i) = 0$$

We know,

$$\eta((\nabla_L \breve{R})(e_i, V)\xi)\eta(e_i) = g((\nabla_L \breve{R})(e_i, V)\xi, \xi)g(e_i, \xi) \tag{6.10}$$

Also,

$$g((\nabla_L \breve{R})(e_i, V)\xi, \xi) = g(\nabla_L \breve{R}(e_i, V)\xi, \xi) - g(\breve{R}(\nabla_L e_i, V)\xi, \xi) \tag{6.11}$$

$$-g(\breve{R}(e_i, \nabla_L V)\xi, \xi) - g(\breve{R}(e_i, V)\nabla_L \xi, \xi)$$

Since e_i is an orthonormal basis. So,$\nabla_L e_i$=0.

Thus, from equation (2.11), we get

$$g(\breve{R}(e_i, \nabla_L V)\xi, \xi) = 0 \tag{6.12}$$

Again, As we know that

$$g(\breve{R}(e_i, V)\xi, \xi) + g(\breve{R}(\xi, \xi)V, e_i) = 0 \tag{6.13}$$

Therefore, we have

$$g(\nabla_L \breve{R}(e_i, V)\xi, \xi) + g(\breve{R}(e_i, V)\xi, \nabla_L \xi) = 0 \tag{6.14}$$

Making use of above equation (6.14) in the equation (6.11), we obtain

$$g((\nabla_L \breve{R})(e_i, V)\xi, \xi) = 0 \tag{6.15}$$

Also, we know that,

$$\eta((\nabla_L \breve{Q})e_i)\eta(e_i) = g(((\nabla_L \breve{Q})e_i), \xi)g(e_i, \xi) \tag{6.16}$$

Using the equations (2.8) and (2.14) in above equation, we get

$$\eta((\nabla_L \breve{Q})e_i)\eta(e_i) = 0 \tag{6.17}$$

Considering equations (6.12) and (6.13), equation (6.10) results in

$$(\nabla_L \breve{S})(V,\xi) = -\frac{1}{2(n-2)}dr\breve{L}\eta(V) - \frac{(n-1)}{(n-2)}dr\omega \tag{6.18}$$

We now substitute V with ξ in the above expression and apply equations (2.9) and (2.15), resulting in

$$dr(\omega) = -\frac{dr\breve{L}}{2(n-1)} \tag{6.19}$$

which shows that r is constant.

Since,

$$(\nabla_L \breve{S})(V,\xi) = \nabla_L \breve{S}(V,\xi) - \breve{S}(\nabla_L V,\xi) - S(V,\nabla_L \xi) \tag{6.20}$$

Applying equations (2.8), (2.9), and (2.14) to the expression above, we deduce that

$$(\nabla_L \breve{S})(V,\xi) = \beta \breve{S}(V,L) \tag{6.21}$$

therefore, from equations (6.14), (6.15) and (6.16), we obtain

$$S(V,L) = -\beta^2(n-1)g(V,L). \tag{6.22}$$

This proves the theorem.

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