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A New Approach for Solving Initial Value Problems by Using Modified Energy Balance Method

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Abstract

This paper discusses and implements a newly an analytical approach based on Modified Energy Balance Method for solving several mathematical models inspired by applied mathematics that describe the motion of nonlinear oscillations.

After applying the method, we obtain a set of nonlinear algebraic equations, the solution of which requires an iterative method, for determine the initial solution.

In our work, we used one of the modifications of Homotopy Perturbation method, and through this modification we were able to determine the initial solution without using guess the initial solution.

The obtained results improve the exactness and the accuracy of the proposed combinations method is cabable to solve a large number of nonlinear initial value problems.

Key Words: Energy Blance Method, Hamilton, Modfid Energy Blance Method, nonlinear oscillator’s, Homotopy Perturbation method, nonlinear algebraic equations.

Introduction:

Nonlinear differential equations represent the modeling of many physical, mechanical, and engineering phenomena. Therefore, they have received great interest from researchers in order to find solutions of this problem using analytical or numerical methods.

The nonlinear oscillators have received a large share by researchers have proposed and developed several methods for solving nonlinear systems, such as the Homotopy Perturbation method [1,2,3], which has been widely used, the parameter-expanding method [4,5], the frequency-amplitude formulation [6],the Hamiltonian approach [7,8],the max-min approach [9], Energy Blance Method [10,11,12], by using first approximation was used in it, Harmonic Balance method [13,14,15] have been used first approximation.

Many researchers have been developed the Energy Blance Method by taking approximation of higher degree [16,17,18], and others have improved the compatibility Harmonic Balance method by making some modifications [19,20].

In our work, after applying the proposed method, we obtained a set of nonlinear algebraic equations, which are very difficult, especially in estimating the initial solution. We used Homotopy Perturbation techniques to overcome this problem, so that the initial solution is chosen from the set of nonlinear algebraic equations. To demonstrate the accuracy and effectiveness of the used method two nonlinear oscillators problems have been solved and the results were compared with other methods.

Energy Balance method Simple Description [10]:

Consider a second-order nonlinear differential equation as follows:

$$u'' + f(u) = 0 \quad (1)$$

Subject to initial conditions:

$$u(0) = A\,, u'(0) = 0 \quad (2)$$

Where: $f(u) = f(u, u', u'')$: is a nonlinear function, u and t represent dimensionless displacement and time respectively.

The Variational principle of Eq. (1) can be expressed as:

$$J(u) = \int_0^t (-\frac{1}{2}u'^2 + F(u))dt \quad (3)$$

Where $F(u) = \int f(u, u')du$

Its Hamiltonian can be written in the following form:

$$H: \frac{1}{2}u'^2 + F(u) = F(A) \quad (4)$$

Or

$$R(u) = \frac{1}{2}u'^2 + F(u) - F(A) = 0 \quad (5)$$

Considering the first approximate solution in the following form:

$$u = A\cos\omega_1 t \quad (6)$$

By substituting Eq. (6) into Eq. (5), gives the following residual equation:

$$\mathrm{R(t)} = \frac{A^2\omega^2}{2}(\sin\omega_1 t)^2 + F(A\cos\omega_1 t) - F(A) = 0 \quad (7)$$

Generally, the selected function (6) is just an approximation to the exact solution of problem (1). It follows that $R(t)$ is different from zero for the vast majority of $R(t)$ values. The frequency ω_1 is determined

by using collocation at $\omega_1 t = \frac{\pi}{4}$, that means by imposing the condition.$R\left(\omega_1 t = \frac{\pi}{4}\right)$, This yields

$$\omega_1(A) = \frac{2}{A}\sqrt{F(A) - F(\frac{A}{\sqrt{2}})} \quad (8)$$

and, as a consequence:

$$u(t) = A\cos(\frac{2}{A}\sqrt{F(A) - F(\frac{A}{\sqrt{2}})}\,t) \quad (9)$$

1- Improved energy balance method:

Let's consider the second approximate solution of Eq. (1) as the following form:

$$u(t) = A((1-v)\cos\omega_2 t + v\cos 3\omega_2 t) \quad (10)$$

Where A, v and ω_2 are constants, then the solution equation (10) readily satisfies the initial conditions given in Eq. (1).

Substituting Eq. (10) into Eq. (5) can be obtained:

$$R(t) = \frac{A^2{\omega_2}^2}{2}\left((1-v)sin\omega_2 t + 3v\sin 3\omega_2 t\right)^2 + F\left((A(1-v)\cos\omega_2 t + v\cos 3\omega_2 t)\right) - F(A) = 0 \quad (11)$$

Now, taking collocation at $,\frac{\pi}{8},\frac{\pi}{4}$. the following nonlinear algebraic equations can be obtained

$$\frac{A^2{\omega_2}^2}{2}\left((1-v)sin\frac{\pi}{4}t + 3v\sin 3\frac{\pi}{4}t\right)^2 + F\left(\left(A(1-v)\cos\frac{\pi}{4}t + v\cos 3\frac{\pi}{4}t\right)\right) - F(A) = 0 \quad (12)$$

$$\frac{A^2{\omega_2}^2}{2}\left((1-v)sin\frac{\pi}{8}t+3v\sin 3\frac{\pi}{8}t\right)^2+F\left(\left(A(1-v)\cos\frac{\pi}{8}t+v\cos 3\frac{\pi}{8}t\right)\right)-F(A)=0 \quad (13)$$

Eliminating ω^2 from Eq. (12) and substituting into Eq. (13) and then Eq. (13) take the following form

$$f(A,u,u^2,u^3,u^4,\ldots)=0 \qquad (14)$$

Where Eq. (14) represent nonlinear algebraic equation, Now applying the iterative procedure based on the Homotopy perturbation method [21], one can obtain the values of v from Eq. (14) which are:

$$v=\lim_{p\to 1} v_0+pv_1+p^2v_2+\cdots (15)$$

Where v_0 is the initial approximation and the unknowns $v_1, v_2, v_3, ..$ determine from the homotopy perturbation method

Finally, substituting the value of v from Eq. (15) into Eq. (12), the angular frequency ω_2 is determined.

4-Test Examples:

The following section presents a descriptive example of the proposed method.

4-1: Consider nonlinear oscillator equation [22,23,24]:

$$u''+u+au^2u''+auu'^2+bu^3=0 \quad (16)$$

With initial condition:

$$u(0)=A\,,u'(0)=0 \quad (17)$$

This system describes the uni-modal large-amplitude free vibrations of a slender inextensible cantilever beam carrying an intermediate lumped mass with a rotary inertia. The third and fourth terms in Eq. (16) represent inertia-type cubic non-linearity arising from the inextensibility assumption. The last term is a static-type cubic non-linearity associated with the potential energy stored in bending. The modal constants a and b result from the discretization procedure and they have specific values for each mode as described in [22].

By integrating Eq. (16), we can readily obtain the Hamiltonian formulation as following:

$$H{:}\frac{1}{2}u'^2+\frac{1}{2}u^2+\frac{a}{2}u^2u'^2+\frac{b}{4}u^4=\frac{A^2}{2}+\frac{bA^4}{4} \quad (18)$$

Or

$$R(u){:}\,\frac{1}{2}u'^2+\frac{1}{2}u^2+\frac{a}{2}u^2u'^2+\frac{b}{4}u^4-\frac{A^2}{2}-\frac{bA^4}{4}=0 \quad (19)$$

Assume the initial approximate guess can be expressed as:

$$u=A((1-v)\cos\omega_2 t+v\ \cos 3\omega_2 t \qquad (20)$$

Substituting Eq. (20) into Eq. (24), yields:

$$\begin{aligned}R(t)=\frac{A^2\omega_2}{2}&\left((1-v)\sin(\omega_2 t)+3v\sin(3\omega_2 t)\right)^2+\frac{A^2}{2}\left((1-v)\cos(\omega_2 t)+v\cos(3\omega_2 t)\right)^2\\&+\frac{aA^4\omega_2^2}{2}\left((1-v)\sin(\omega_2 t)+3v\sin(3\omega_2 t)\right)^2\left((1-v)\cos(\omega_2 t)+v\cos(3\omega_2 t)\right)^2\\&+\frac{bA^4}{4}\left((1-v)\cos(\omega_2 t)+v\cos(3\omega_2 t)\right)^4-\frac{A^2}{2}-\frac{bA^4}{4}=0 \qquad (21)\end{aligned}$$

Now, taking collocation at $\frac{\pi}{4},\frac{\pi}{8}$, the following nonlinear algebraic equations can be obtained:

$$\begin{aligned}(32aA^2v^4-16A^2av^2+2A^2a+16v^2+16v+4)\omega_2^2+16bA^2v^4-32A^2bv^3+24bA^2v^2-8bA^2v-3A^2b\\+16v^2-16v-4=0 \quad (22)\end{aligned}$$

$$(1-32aA^2v^4+32aA^2v^3-16aA^2v^2-16aA^2v)\sqrt{2}+72aA^2v^4-128aA^2v^3+72aA^2v^2+2aA^2 + (8v^2+32v-4)\sqrt{2}+80v^2-16v+8)\,{\omega_2}^2+(-8bA^2v^4+8bA^2v^3-4bA^2v+2bA^2 - 8v^2+4)\sqrt{2}+12bA^2\,v^4-16bA^2\,v^3+12bA^2v^2-8bA^2v-5bA^2+16v^2-16v-8 = 0 \quad (23)$$

We may ignore greater than second-order terms of v:

$$(-16aA^2v^2+2aA^2+16v^2+16v+4)\omega_2^2+24bA^2v^2-8bA^2v-3bA^2+16v^2-16v-4=0 \quad (24)$$

$$\left[(-16aA^2v^2+16aA^2v+8v^2+32v-4)\sqrt{2}+72aA^2v^2+2aA^2+80v^2-16v+8\right]\omega_2^2 + (-4bA^2v+2bA^2-8v^2+4)\sqrt{2}+12bA^2v^2-8bA^2v-5bA^2+16v^2-16v-8 = 0 \quad (25)$$

Eliminating $\omega_2^2$from Eq. (29), we obtain the following equation as

$$\omega_2^2=\frac{4+16v-16v^2+3bA^2+8bA^2v-24bA^2v^2}{4+16v+16v^2+2aA^2-16aA^2v^2} \qquad (26)$$

substituting Eq. (26) into Eq (25) and after simplifying we get nonlinear algebraic equation of v:

$$f(v):\left(384abA^4\sqrt{2}-1920abA^4+384aA^2\sqrt{2}-192bA^2\sqrt{2}-1408aA^2-1728bA^2-256\sqrt{2}-1024\right)v^4 + \left(-448abA^4\sqrt{2}+704abA^4-512aA^2\sqrt{2}-768bA^2\sqrt{2}+1408aA^2+1088bA^2-512\sqrt{2} + 1536\right)v^3 + \left(48abA^4\sqrt{2}+272abA^4+112aA^2\sqrt{2}+344bA^2\sqrt{2}+416aA^2-240bA^2+640\sqrt{2} - 384\right)v^2+\left(40abA^4\sqrt{2}+64aA^2\sqrt{2}+80bA^2\sqrt{2}-96bA^2+128\sqrt{2}-128\right)v+4A^4ab\sqrt{2} - 4abA^4+8\sqrt{2}aA^2-4A^2b\sqrt{2}-8aA^2+4aA^2=0 \qquad (27)$$

For solving nonlinear algebraic equation, we will be applying the modified Homotopy perturbation method [21], wherefore Eq. (27) can be written into another form as

$$f(v):v+\frac{1}{K}\Big(4A^4ab\sqrt{2}-4abA^4+8\sqrt{2}aA^2-4A^2b\sqrt{2}-8aA^2+4aA^2 + p\Big(\big(384abA^4\sqrt{2}-1920abA^4+384aA^2\sqrt{2}-192bA^2\sqrt{2}-1408aA^2-1728bA^2 - 256\sqrt{2}-1024\big)v^4 + \big(-448abA^4\sqrt{2}+704abA^4-512aA^2\sqrt{2}-768bA^2\sqrt{2}+1408aA^2+1088bA^2-512\sqrt{2} + 1536\big)v^3 + \big(48abA^4\sqrt{2}+272abA^4+112aA^2\sqrt{2}+344bA^2\sqrt{2}+416aA^2-240bA^2+640\sqrt{2} - 384\big)v^2\Big)\Big)=0 \quad (28)$$

Where:

$$K=40abA^4\sqrt{2}+64aA^2\sqrt{2}+80bA^2\sqrt{2}-96bA^2+128\sqrt{2}-128 \quad (29)$$

Substituting Eq. (15) into Eq. (28), and equating highest power of p will result in:

$$p^0: v_0+\tfrac{1}{K}\left(-8aA^2+4bA^2+8\sqrt{2}aA^2-4\sqrt{2}bA^2-4abA^4+4\sqrt{2}bA^4a\right)=0$$

$$p^1: v_1+\tfrac{1}{K}(-1024v_0^4+1536v_0^3-384v_0^2-1408av_0^4-1728bA^2v_0^2+\cdots)=0$$

(30)

$$p^2: v_2+\frac{1}{K}(3264bA^2v_0^2v_1+832aA^2v_0v_1+\cdots)=0$$

From (30) we find v_0, v_1, v_2, ...,thus

$$v=\ v_0+v_1+v_2+\cdots \qquad (31)$$

Now substituting the value of v from Eq. (31) into Eq. (26), can be obtain ω_2 and substituting the values of v and ω_2 into Eq. (20) we obtained the second-order approximate solution of Eq, (16).

Table (1): comparison of the approximate frequency obtained in this paper with exact and other existing frequencies for $a = 1, b = 2$.

A	Exact [24] ω_{ex}	[23] ω_2 (% error)	Present study ω_2 (% error)
0.5	1.10608	1.10554 0.049	1.10562 0.042
1	1.30158	1.29099 0.814	1.29455 0.54
5	1.87549	1.68874 9.957	1.75634 6.353
50	1.99748	1.73159 13.311	1.81386 9.193
100	1.99928	1.73194 13.372	1.81433 9.251
500	1.99996	1.73205 13.396	1.81448 9.274
1000	1.99999	1.73205 13.396	1.81448 9.275

Where (% error) denotes the absolute percentage error.

Table (2) comparison of the approximate frequency obtained in this paper with exact and other existing frequencies for $a = b = 2$.

A	Exact[24] ω_{ex}	ω_2 [23] (% error)	present study (% error) ω_2
0.5	1.05224	1.05165 0.056	1.05087 0.130
0.1	1.14348	1.13291 0.924	1.13183 1.019
5	1.37132	1.2579 8.274	1.27057 7.347
50	1.41335	1.26791	1.28291

		10.290	9.242
100	1.41397	1.26799 10.324	1.28300 9.263
500	1.4142	1.26802 10.337	1.28304 9.274
1000	1.41421	1.26802 10.337	1.28304 9.274

Where (% error) denotes the absolute percentage error.

4-2: Antisymmetric quadratic nonlinear oscillator [25,26]:

Let us consider an antisymmetric quadratic nonlinear oscillator as in the following from

$$u'' + sign(u)\, u^2 = 0 \quad (32)$$

With initial condition:

$$u(0) = A\,, u'(0) = 0 \quad (33)$$

According to the variational principle Eq. (32) can be written as:

$$J(u) = \int_0^t \left(-\frac{1}{2}u'^2 + \frac{1}{3}u^3\right) dt \quad (34)$$

The Hamiltonian of Eq. (31) becomes

$$H: \frac{1}{2}u'^2 + \frac{1}{3}u^3 = \frac{A^3}{3} \quad (35)$$

Residual of the Eq. (35) is

$$R(t): \frac{1}{2}u'^2 + \frac{1}{3}u^3 - \frac{A^3}{3} = 0 \quad (36)$$

Substituting Eq. (10) into Eq. (36) residua becomes

$$R(t) = \frac{A^2\omega_2^2}{2}\left((1-v)sin\,\omega_2 t + 3v\sin 3\omega_2 t\right)^2 + \frac{A^3}{3}\left((1-v)cos\,\omega_2 t + v\cos 3\omega_2 t\,\right)^3 - \frac{A^3}{3} = 0 \quad (37)$$

Now, taking collocation at $\frac{\pi}{4}, \frac{\pi}{8}$,the following nonlinear algebraic equations can be obtained:

$$(12v^2 + 12v + 3)\omega_2^2 - 11.314Av^3 + 16.970Av^2 - 8.4852Av - 2.5858A = 0 \quad (38)$$

$$(0.8787 + 10.9724v + 34.241v^2)\omega_2^2 + 3.2475Av^2 - 0.6341Av^3 - 0.8456A - 5.5435Av = 0 \quad (39)$$

Eliminating ω_2^2 between Eq. (38) and Eq. (39) and after simplification we get nonlinear algebraic equation of v as

$$f(v): 397.793474v^5 - 425.571762v^4 + 84.8342050v^3 + 99.8017532v^2 + 9.04964284v - 0.26465754 = 0 \quad (40)$$

For solving nonlinear algebraic equation (40), we will be applying the modified Homotopy perturbation method [21], by rewrite above equation in the form:

$$f(v): v - 0.029245 + 41.968v^5 - 47.026v^4 + 9.3743v^3 + 11.028u^2 = 0 \quad (43)$$

We can assume that the solution of Eq. (43) can be express as a series in p

$v = p^0 v_0 + p v_1 + p^2 v_2 + \cdots \qquad (44)$

Substituting Eq. (44) into Eq. (46), and equating highest power of p will result in:

$$\begin{cases} p^0: v_0 - 0.029245 = 0 \\ p^1: v_1 + 11.028 v_0^2 + 9.3743 v_0^3 = 0 \\ p^2: v_2 + 22.056 v_0 v_1 + 28.1229 v_0^2 v_1 = 0 \\ \quad \vdots \end{cases} \qquad (45)$$

From (45) we find $v_0, v_1, v_2, \ldots$,thus the value of v is in the form

$$v = v_0 + v_1 + v_2 + \cdots \qquad (46)$$

Now substituting the value of v from Eq. (46) into Eq. (38), can be obtain ω_2 and substituting the values of v and ω_2 into Eq. (10) we obtained the second-order approximate solution of Eq, (32).

Table 3: comparison of the present solution with exact and Belendez et al [25] which A=1.

t	Exact x_{ex}	Belendez et at.[25] x_2 (% error)	Present study x_2 (% error)
1.25	0.37892	0.37302 1.557	0.37572 0.844
1.75	-0.02694	-0.03736 38.678	-0.031174 15.716
2.25	-0.43074	-0.44435 3.159	-0.43552 1.110
2.75	-0.78279	-0.79403 1.435	-0.78622 0.438
3.25	-0.98311	-0.98704 0.399	-0.98466 0.158
3.75	-0.9509	-0.94227 0.917	-0.94698 0.412
4.25	-0.70048	-0.68017 2.899	-0.69061 1.398

Where (% error) denotes the absolute percentage error.

5- conclusion:

For most of the applications which have been studied in literature, the present study has provided more precise solutions with fewer iteration, compared to other methods.

For future reseach work, it is recommended to solve another initial value problems by Modified energy balance method.

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