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Covariance and One-Step Error Prediction of Difference Equations Through Convolution and Deconvolution

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Abstract: This paper determines a cohesive framework for addressing the algorithmic filtering drawback faced in filtering the noise in the digital filter. Digital filters for nonlinear time-varying systems victimizing the noise reduction and this can be established through difference equations. The new generalized projected filter is adopted into the algorithmic filter structure administrated by identifying two different embedded systems of stochastic type and Riccati type nonlinear random equations. The gain matrix of the proposed filter is calculated by minimizing the covariance trace of the filtering error $P_{k+1|k+1}$, k^{th} element. Simultaneously the gain of the filter is identified and minimizing the filtering error $P_{k+1|k+1}$. Taking this partial differential coefficient and its first derivative of $tr\,\{P_{k+1|k+1}\}$ with relevancy K_{k+1} and set the derivative to zero, at some stage the higher order coefficient leads to the reduction of the noise in the digital filter. Further the, solutions to the non homogeneous difference equation problem using convolution and deconvolution methods are analyzed along with the numerical examples.

Key Words**:** Convolution, deconvolution, difference equation, Digital filters, nonlinear time varying systems, digital signal processing.

I Introduction

A generalized system and mathematical model has been analyzed by various researchers subject to random delay which reflects deterministic and nondeterministic nonlinearities [3]. Few researchers are developed and extend these theories in deception attacks [1]. By utilizing various filters along with the Riccati like equation techniques for filtering the errors and fixing the boundary of errors are developed in the recent days [5]. The corresponding covariance matrices values minimizing the upper bound and arrives optimal filter gain in a recursive manner [6, 10]. The error pattern analyses were focused along with random access protocol and independent and identically distributed deterministic and non deterministic variables [7, 16]. These are validated the upper bound by analyzing the state estimates and error covariance. Convolution is a fundamental mathematical operation which takes two functions and produces a third function that represents the amount of overlap between one of the functions and a reversed and translated version of the other function [9, 14]. Similarly, the deconvolution techniques have a great importance in solving different kinds of equations [2, 13]. Deconvolution is the inversion of convolution equation. This method can be implemented in many applications [11,18]. Here we used discrete convolution and deconvolution to obtain solution to the difference equations. Solving the difference equation using these concepts makes great interest. In section II we discussed on the notation of discrete convolution and deconvolution of finite and infinite sequences. In section III we presented our main concepts which are the unique solution to the non homogeneous linear difference equation. Further we used these concepts to solve the numerical solution of initial value problem and boundary value problem with suitable examples. Section IV focuses the covariance of the one-step prediction error and error calculations. Finally section V concludes the paper.

II Preliminaries

Fourier transform of periodic signal having Fourier series coefficient is a chain of impulses, occurring at multiplies of fundamental frequency, which is the strength of the impulse [4]. A system is an abstract module that accepts input signals and produces output signals in response [15]. A function $f(t)$ be the continuous-time signal, where $t \in R$. A function $f[n]$ be the discrete-time signal, where $n \in Z$. A continuous-time signal $f(t)$ is periodic with period T if, $f(t) = f(t+T)$ for all $t \in R$. A discrete-time signal $f[n]$ is periodic with period N if $f[n] = f[n+N]$ for all $n \in Z$.

A signal with the smallest period for which the signal is periodic is called fundamental period. Consider the signal of the form $f(t) = Ce^{at}$, where C is the complex number if both C and a are real, then there are three possible cases [8, 17]:

Case (i): If a < 0, then as t →∞, the signal tends to zero

Case (ii): If a > 0, then as t →∞, the signal tends to ∞

Case (iii): and for a = 0, the signal becomes constant.

Suppose $f(t+T) = e^{j(w_0 t+\varphi)}$ for some positive real w_0 and φ, where $C = e^{j\varphi}$ and $a = jw_0$. Note that $f(t+T) = e^{j(w_0(t+T)+\varphi)} = e^{j(w_0 t+\varphi)}e^{jw_0 T}$. If w_0T is multiple of 2π, then $e^{jw_0 T} = 1$ and the signal is of periodicity T. Thus, the fundamental period of this signal is $T = \frac{2\pi}{w_0}$. Also if $w_0 = 0$, then $f(t) = 1$ and this is periodic with any period and we see that $f(t) = e^{-jw_0 T}$ with period T_0. Then w_0 is known as the fundamental frequency of the input signal.

III Linear Discrete Convolution and Deconvolution

Let $a = (a_0, a_1, a_2, \ldots\ldots a_n)$ and $b = (b_0, b_1, b_2, \ldots\ldots b_n)$ are the finite sequence with same number of elements. The discrete convolution of these sequences is given by

$$c = a * b = (c_0, c_1, c_2, \ldots\ldots c_n) \qquad (1)$$

where c is a finite sequence and defined as below

$$c_0 = a_0 b_0$$

$$c_1 = a_1 b_0 + a_0 b_1$$

……..

$$c_k = \sum_{i=o}^{n} a_{n-i} b_i \qquad (2)$$

Here a and c are known finite sequences with $a_0 \neq 0$, we can determine the finite sequence b provided the condition (i) to be satisfied. This is known as deconvolution of the sequence c by the sequence a.This was denoted by $b = c/a$ (3)

The relations are defined as follows

$$b_0 = c_0/a_0$$

$$b_1 = \frac{1}{a_0}(c_1 - a_1 b_0)$$

…………

$$b_k = \frac{1}{a_0}(c_k - \sum_{i=0}^{n-1} a_{n-i}\, b_i) \qquad (4)$$

Inverse of the finite sequence of a is given by $a^{-I} = \delta/a$, such that

$$\frac{c}{a} = c * a^{-I} \qquad (5)$$

We consider the infinite case with same definition, with arbitrary n. Therefore

$$(a_0, a_1, a_2, \dots\dots a_n \dots.) * (b_0, b_1, b_2, \dots\dots b_n \dots.) = (\big((a_0, a_1, a_2, \dots\dots a_n \dots.) * (b_0, b_1, b_2, \dots\dots b_n \dots.)\big) : n = 0, 1, 2 \dots.$$

3.1 Linear Differnce Equation With Constant Coefficient

Consider the following non homogeneous linear difference equation

$$\sum_{i=0}^{k} a_{n-i} u_{i+n} = b_n, \quad n = 1,2, \dots. \qquad (6)$$

with the coefficients $a_0 \neq 0$ and we denote

$$a = (a_0, a_1, a_2, \dots\dots a_n, \dots.) \qquad (7)$$

such that $k = 0$ if $n > k$ and

$$b = (b_0, b_1, b_2, \dots\dots b_n, \dots..) \qquad (8)$$

Theorem 3.1.1 : The unique solution $u = (u_0, u_1, u_2, \dots\dots u_n, \dots)$ of the equation (6) with the initial values $u_0, u_1, \dots\dots u_{k-1}$ is given by

$$u = ((a_0, a_1, a_2, \dots\dots a_{k-1}) * ((u_0, u_1, u_2, \dots\dots u_{k-1}), b) * a^{-1} \qquad (9)$$

Proof : Let us denote $c = (c_0, c_1, c_2, \dots\dots c_n, \dots.) = a * u$ (10)

Using convolution product, we get

$$c_0 = a_0 u_0$$

$$c_1 = a_1 u_0 + a_0 u_1$$

……..

$$c_{k-1} = \sum_{i=o}^{k-1} a_{k-1-i} u_i \qquad (11)$$

Therefore

$$c = (c_0, c_1, c_2, \dots\dots c_{k-1}) = (a_0, a_1, a_2, \dots\dots a_{k-1}) * (u_0, u_1, u_2, \dots\dots u_{k-1}) \qquad (12)$$

Changing the index $i = j + k$, and take $a_{k+n} = 0, n = 1,2, \dots.$ which implies

$$c_{k+n} = \sum_{i=o}^{k+n} a_{k+n-i} u_i$$

$$= \sum_{j=-k}^{n} a_{n-j} u_{j+k}$$

$$= \sum_{j=0}^{n} a_{n-j} u_{j+k}$$

$$= b_k \qquad n = 1,2, \dots. \qquad (13)$$

Hence, the equation (12) and (13) gives

$$u = ((a_0, a_1, a_2, \dots\dots a_{k-1}) * ((u_0, u_1, u_2, \dots\dots u_{k-1}), b)$$

$$= (a_0 u_0, a_1 u_0 + a_0 u_1, \dots\dots,, b_0, b_1, b_2, \dots\dots b_n, \dots.$$

$$= b * \sum_{i=o}^{k-1} a_{k-1-i} u_i \qquad (14)$$

where (11) and (14) gives the solution to the non homogeneous linear difference equation (6) and the converse is also true.

EXAMPLE 3.1.2 : The linear difference equation given by $u_{n+2} - 2u_{n-1} - 3u_n = b_n$, where $n = 0,1,2, ...$ has the initial conditions $u_0 = u_1 = 1$ with $a = (1, -2, -3,0,0, ...)$ and $b = (0,1,2, ...)$.

Solution: We know that

$$u = ((a_0, a_1) * (u_0, u_1), b)$$

$$= ((1, -2) * (1,1), b)$$

$$= (1, -1,0,1,2,3, ...)$$

This results the sequence

$$u = \frac{c}{a} = (1,15,14,45, ...)$$

Hence the solution.

EXAMPLE 3.1.3 : The boundary value problem formed by the difference equation

$u_{n+2} + u_n = b_n, n = 0,1,2, ...$ with $b = (2,0, -2,0,2,0, -2,0, ...)$ has $a = (1,0,1,0,1,0,1,0 ...)$ and $a^{-1} = (1,0, -1,0,1,0, -1,0 ...)$

Solution : Consider

$$u = (0,0,1 * a^{-1}) + u_0 . (1,0,0 ...) * a^{-1} + u_1 . (0,1,0,0 ...) * a^{-1}$$

$$= (0,0,2,0 - 4,0,6,0, ...) + u_0 . (1,0, -1,0,1,0, -1,0, ...) + u_1 . (0,1,0 - 1,0,1,0, -1 ...)$$

Case 1: If $u_3 = 0, u_4 = -3$, which gives $u_1 = -u_3 = 0$ and $u_0 = 1$

Hence the boundary value problem has the unique solution

$$u = (1,0,1,0, -3,0,5,0, ...)$$

Case 2: If $u_2 = 1, u_4 = -3$, $u_2 = 2 - u_0 = 1$ and $u_4 = -4 + u_3 = -3$. Then $u_0 = 1$. Hence the boundary value problem has infinite solutions by the following relation

$$u = (1,0,1,0, -3,0,5,0, ...) + u_1 . (0,1,0 - 1,0,1,0, -1 ...)$$

where u_1 is arbitrary.

Case 3: If $u_2 \neq 1, u_4 = -3$, which gives both $u_0 = 1$ and $u_0 \neq 1$. This leads to the contradiction. Therefore it has no solution.

IV One step error prediction and covariance

The gain matrix of the filter K_{k+1} is calculated by minimizing the covariance trace of the filtering error $P_{k+1|k+1}$. The gain matrix and filter's mechanism works recursively within the theoretical and reasonable significance and to attenuate $tr \{P_{k+1|k+1}\}$. Algorithmic filters are used to attenuate the conditional covariance matrix $\mathbb{E}\{e_{k+1|k+1} e^T_{k+1|k+1} | \xi(k+1)\}$, wherever the gain of the filter K_{k+1} is calculated piece wise [12] and the same can be extended in the bio signal processing.

We denote the prediction error in one step as $e_{k+1|k} = x_{k+1} - \hat{x}_{k+1|k}$ and the filter error is $e_{k+1|k+1} = x_{k+1} - x_{k+1|k+1}$. Similarly, the filter error is often written as:

$$e_{k+1|k} = A_k e_{k|k} + B_k \omega_k \qquad (15)$$

$$e_{k+1|k+1} = (I - K_{k+1} \Phi_{\xi(k+1)} C_{k+1}) e_{k+1|k} - K_{k+1} \Phi_{\xi(k+1)} v_{k+1} \qquad (16)$$

Where A , B and C are matrices and $\Phi_{\xi(k+1)}$ denoted as difference between the covariance of the matrices A and B.

The covariance and one-step error prediction is calculated from the following theorems.

Theorem 4.1: Consider the filtering system's (16) error dynamics. The covariance of the predicted error in one step $P_{k+1|k} \triangleq E\{e_{k+1|k}e_{k+1|k}^T\}$ and the covariance of the filtering error $P_{k+1|k+1} \triangleq E\{e_{k+1|k+1}e_{k+1|k+1}^T\}$(with the initial condition $P_{0|0}$) and is given by the subsequent equations

$$P_{k+1|k} = A_k P_{k|k} A_k^T + B_k Q_k B_k^T \tag{17}$$

$$P_{k+1|k+1} = \sum_{i=1}^N p_i\, \mu_{i,k+1} P_{k+1|k} \mu_{i,k+1}^T + \sum_{i=1}^N p_i\, K_{k+1}\Phi_i R_{k+1}\Phi_i K_{k+1}^T \tag{18}$$

here $\mu_{i,k+1}^T = I - K_{k+1}\Phi_i C_{k+1}$ and $P_{k|k}$, R_{k+1} and Q_k are Riccati type matrices with positive coefficients. In addition, $P_{k+1|k+1}$ is decreased by the subsequent filter gain:

$$K_{k+1} = P_{k+1|k} C_{k+1}^T \overline{\Phi} (\textstyle\sum_{i=1}^N p_i\, \Phi_i \mathcal{R}_{k+1} \Phi_i)^{-1} \tag{19}$$

where $\mathcal{R}_{k+1} = C_{k+1} P_{k+1|k} C_{k+1}^T + R_{k+1}$

Proof: Initially, consider the error covariance of the one - step error prediction $P_{k+1|k}$ and this satisfies

$$P_{k+1|k} = A_k P_{k|k} A_k^T + B_k Q_k B_k^T \tag{20}$$

Then consider the covariance of the filtering error $P_{k+1|k+1}$, and considering (17), we have

$$P_{k+1|k+1} = \mathbb{E}\{e_{k+1|k+1}e_{k+1|k+1}^T\}$$

$$= \{\mu_{\xi(k+1),k+1} e_{k+1|k} e_{k+1|k}^T \mu_{\xi(k+1),k+1}^T + K_{k+1}\Phi_{\xi(k+1)} v_{k+1} v_{k+1}^T \Phi_{\xi(k+1)}^T K_{k+1}^T\} \tag{21}$$

Further (21) can be extended according to $\Phi_{\xi(k+1)}$ as follows:

$$\Phi_{\xi(k+1)} = \textstyle\sum_{i=1}^N \delta(\xi(k+1) - i)\, \Phi_i \tag{22}$$

The corresponding error prediction can be written like

$$\mathbb{E}\{\mathcal{K}_{\xi(k+1),k+1} e_{k+1|k} e_{k+1|k}^T \mu_{\xi(k+1),k+1}^T\} = \mathbb{E}\left\{\begin{matrix} \left(I - K_{k+1}\sum_{i=1}^N \delta(\xi(k+1) - i)\, \Phi_i C_{k+1}\right) e_{k+1|k} e_{k+1|k}^T + \\ (I - K_{k+1}\sum_{i=1}^N \delta(\xi(k+1) - i)\Phi_i C_{k+1})^T \end{matrix}\right\}$$

$$= P_{k+1|k} - \mathbb{E}\left\{\sum_{i=1}^N \delta(\xi(k+1) - i)\, K_{k+1}\Phi_i C_{k+1} e_{k+1|k} \times e_{k+1|k}^T\right\}$$

$$- \mathbb{E}\left\{\sum_{i=1}^N \delta(\xi(k+1) - i)\, e_{k+1|k} e_{k+1|k}^T C_{k+1}^T \Phi_i K_{k+1}^T\right\}$$

$$+ \mathbb{E}\left\{\sum_{i=1}^N \delta(\xi(k+1) - i)\, K_{k+1}\Phi_i C_{k+1} \times e_{k+1|k} e_{k+1|k}^T C_{k+1}^T \Phi_i K_{k+1}^T\right\}$$

$$= \textstyle\sum_{i=1}^N p_i (I - K_{k+1}\Phi_i C_{k+1})\, P_{k+1|k} (I - K_{k+1}\Phi_i C_{k+1})^T$$

and

$$\mathbb{E}\{K_{k+1}\Phi_{\xi(k+1)} v_{k+1} v_{k+1}^T \Phi_{\xi(k+1)}^T K_{k+1}^T\} = \mathbb{E}\{\textstyle\sum_{i=1}^N p_i K_{k+1}\Phi_i R_{k+1}\Phi_i K_{k+1}^T\}$$

$$P_{k+1|k+1} = \sum_{i=1}^N p_i\, \mathcal{K}_{i,k+1} P_{k+1|k} \mathcal{K}_{i,k+1}^T + \sum_{i=1}^N p_i\, K_{k+1}\Phi_i R_{k+1}\Phi_i K_{k+1}^T$$

The corresponding traces are

$$\frac{\partial tr\{P_{k+1|k+1}\}}{\partial K_{k+1}} = -2\textstyle\sum_{i=1}^N p_i (\mathcal{K}_{i,k+1} P_{k+1|k} C_{i,k+1}^T \Phi_i) + 2K_{k+1}\sum_{i=1}^N p_i (\Phi_i R_{k+1} \Phi_i) = 0$$

Based on the higher order equations, the optimum filter gain K_{k+1} is determined and predicted as

$$K_{k+1} = P_{k+1|k} C_{k+1}^{T} \overline{\Phi} (\sum_{i=1}^{N} p_i \Phi_i \mathcal{R}_{k+1} \Phi_i)^{-1}$$

This completes the proof.

V Conclusion

The deterministic and non deterministic cohesive framework for identifying the filtering problems and its drawback against minimizing the random noise was analyzed. Further, nonlinear time-varying systems victimizing the noise reduction in the digital signal through Riccati type difference equations. Generalized projected filter is adopted into the algorithmic filter structure administrated by identifying two different embedded systems of stochastic type and Riccati type nonlinear random equations. Simultaneously the gain matrix of the proposed filter is calculated by minimizing the covariance trace of the filtering error $P_{k+1|k+1}$, k^{th} element. Further we applied the convolution and deconvolution concepts along with the initial conditions in the difference equations. We also arrived that how the discrete convolution and deconvolution, be used to compute the numerical values of linear non-homogeneous difference and with constant coefficients, through some solved examples.

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