M – Homotopy Functions and Their Equivalences in Algebraic Topology: A New Perspective

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Abstract:- This paper explores the complex realm of multivalued homotopy functions in algebraic topology. It covers four main areas: the foundational concepts of multivalued homotopy functions, the group operations within these functions, the action of the multivalued fundamental group on homotopy classes, and the criteria for the equivalence of multivalued homotopy functions. By examining these aspects, we reveal new algebraic structures, symmetries, and deeper connections between topological spaces and their algebraic invariants, providing a comprehensive framework for understanding and classifying multivalued homotopy functions.

Keywords*:* Homotopy, Homotopy Class, Path Lifting, Fundamental group.

1. Introduction

This paper delves into the intricate world of multivalued homotopy functions, exploring their properties and applications through four distinct sections. Each section addresses a critical aspect of multivalued homotopy, revealing the depth and breadth of this fascinating topic in algebraic topology.

The second section delves into the concept of multivalued homotopy functions, where each input can correspond to multiple output paths. This section establishes foundational definitions, explores unique properties, and provides illustrative examples to contrast multivalued mappings with traditional single-valued functions.

The third section extends the discussion to group operations in multivalued homotopy functions. By generalizing classical group operations to a multivalued context, we reveal new algebraic structures and symmetries. This section elucidates how these operations are defined and manipulated, highlighting their role in enhancing our understanding of multivalued homotopy.

The fourth section examines the multivalued fundamental group and its action on homotopy classes. We explore how the fundamental group, a pivotal concept in algebraic topology, is adapted to multivalued scenarios. The analysis focuses on the implications of multivalued operations for the structure of homotopy classes, uncovering intricate connections between topological spaces and their algebraic invariants.

Finally, we address the equivalence of multivalued homotopy functions. Establishing criteria for equivalence is crucial for determining when two multivalued homotopy functions are topologically indistinguishable. This section presents key theorems and proofs that characterize equivalence relations, providing a robust framework for comparing and classifying multivalued homotopy functions.

Through a comprehensive exploration of these four key areas, this paper aims to illuminate the complex relationships and algebraic structures underlying multivalued homotopy functions, contributing to a deeper understanding of their significance in algebraic topology.

2. $\mathcal{M}-$ Functions in Homotopy

In algebraic topology, multivalued functions and homotopy are fundamental concepts used to study the properties and structures of topological spaces. While standard functions map a single input to a single output, multivalued functions can map a single input to multiple outputs. This flexibility makes them useful in various scenarios, particularly in the context of covering spaces, homotopy, and path lifting. Here, we will explore how multivalued functions interact with homotopy in algebraic topology. When dealing with multivalued functions, the concept of homotopy can be extended to $\mathcal{M}-$ homotopy, which considers continuous deformations between multivalued functions. The classification of homotopy classes of multivalued maps can be more complex than single-valued maps. By studying the homotopy between multivalued functions, one can classify the different ways a space can be mapped to another, considering the multivalued nature of the maps. Multivalued functions expand the traditional notion of homotopy, allowing for mappings that can assign multiple outputs to a single input. This generalization is particularly useful in complex systems where single-valued functions fall short in capturing all the nuances of continuous deformations.

Definition 2.1. Consider two continuous functions $\mathcal{K}, \mathcal{L} : \mathcal{X} \rightarrow \mathcal{Y}$ between topological spaces $\mathcal{X}$ and $\mathcal{Y}$. A $\mathcal{M}-$ homotopy between $\mathcal{K}$ and $\mathcal{L}$ is a multivalued function $\mathcal{H} : \mathcal{X} \times [0,1] \rightarrow \mathcal{P}(\mathcal{Y})$. Where $\mathcal{P}(\mathcal{Y})$ denotes the power set of $\mathcal{Y}$ satisfying the following conditions, for each $x \in \mathcal{X}$ and $\mathrm{t} \in [0,1]$. $\mathcal{H}(x, \mathrm{t})$ is a non-empty subset of $\mathcal{Y}$ is a set containing all the intermediate states between $\mathcal{K}(\mathcal{X})$ and $\mathcal{L}(\mathcal{X})$ at time t.

1. **Initial and Final conditions:**
   - $\mathcal{H}(x, 0) = \mathcal{K}(\mathcal{X}) \ \forall \ x \in \mathcal{X}$.
   - $\mathcal{H}(x, 1) = \mathcal{L}(\mathcal{X}) \ \forall \ x \in \mathcal{X}$.
2. **Continuity:**
   For every $\mathrm{t} \in [0,1]$, the set-valued map $x \longmapsto \mathcal{H}(x, \mathrm{t})$ is continuous in the sense of multivalued functions. This means that for any $x \in \mathcal{X}$ and $\mathrm{t} \in [0,1]$, for any open set $\mathcal{V} \subseteq \mathcal{Y}$ such that $\mathcal{H}(x, \mathrm{t}) \subseteq \mathcal{V}$, there exists an open neighborhood $\mathcal{U} \subseteq \mathcal{X} \times [0,1])(x, \mathrm{t})$ such that for all $(x', \mathrm{t}') \in \mathcal{U}$, $\mathcal{H}(x', \mathrm{t}') \subseteq \mathcal{V}$.
3. **Path conditions:**
   For each fixed $x \in \mathcal{X}$ and for any $y \in \mathcal{H}(x, \mathrm{t})$ and $y' \in \mathcal{H}(x, \mathrm{t}')$ with $(\mathrm{t} \leq \mathrm{t}')$, there exists a continuous path in $\mathcal{Y}$ connecting y and y'.

Remark 2.2. This definition extends the classical idea of a homotopy by allowing the intermediate "points" between $\mathcal{K}(\mathcal{X})$ and $\mathcal{L}(\mathcal{X})$ to be sets of points rather than single points. The continuity condition for multivalued functions ensures that these sets change in a controlled manner, maintaining a form of continuous deformation. The path connection condition guarantees that, even though we're dealing with sets, there is a coherent way to "travel" through these sets continuously.

Example 2.3. we'll identify a scenario where the conditions for a $\mathcal{M}-$ homotopy are not satisfied. Specifically, we can consider a case where the continuity condition or the path connection condition fails.

Let $\mathcal{X} = \{0,1\}$ a simple two-point space and let $\mathcal{Y} = \mathbb{R}$ the real line. Now define $\mathcal{K}, \mathcal{L} : \mathcal{X} \rightarrow \mathcal{Y}$ as:

- $\mathcal{K}(0) = 0, \ \mathcal{K}(1) = 2$
- $\mathcal{L}(0) = 1, \ \mathcal{L}(0) = 3$

The $\mathcal{M}-$ Homotopy $\mathcal{H}$, Suppose we define $\mathcal{H}:\mathcal{X}\times[0,1]\to\mathcal{P}(\mathcal{Y})$, Where $\mathcal{P}(\mathcal{Y})$ denotes the power set of $\mathcal{Y}$ satisfying the follows:

- $\mathcal{H}(0,\mathrm{t})=[0,1]\ \forall\ \mathrm{t}\in[0,1]$
- $\mathcal{H}(1,\mathrm{t})=\{2+\mathrm{t}\}\ \forall\ \mathrm{t}\in[0,1]$

1. **Initial Conditions** For $(\mathrm{t}=0)$:
   - $\mathcal{H}(0,0)=[0,1]$ which does not equal $\{\mathcal{K}(0)=0\}$. . This violates the condition $\mathcal{H}(x,0)=\{\mathcal{K}(\mathcal{X})\}$. And $\mathcal{H}(1,0)=\{2\}$, which equals $\{\mathcal{K}(1)=2\}$.
2. **Final Conditions** For $(\mathrm{t}=1)$:
   - $\mathcal{H}(0,1)=[0,1]$ which does not equal $\{\mathcal{L}(0)=1\}$. This violates the condition $\mathcal{H}(x,1)=\{\mathcal{L}(\mathcal{X})\}$. And $\mathcal{H}(1,1)=\{3\}$, which equals $\{\mathcal{L}(1)=3\}$.
3. **Continuity:** Continuity of the multivalued function $\mathcal{H}$ is not continuous in the usual sense of multivalued functions because the set-valued map $\mathcal{H}(0,\mathrm{t})=[0,1]$ does not continuously deform to a single value. For a continuous multivalued function, the values should smoothly transition from $\mathcal{K}(\mathcal{X})$ to $\mathcal{L}(\mathcal{X})$, which is not the case here since $\mathcal{H}(0,\mathrm{t})$ remains a fixed interval $[0,1]$.
4. **Path Connection:** For $(x=0)$, $\mathcal{H}(0,\mathrm{t})=[0,1]$ does not specify a clear path between any specific points in $[0,1]$ and $\mathcal{L}(0)=1$. For $(x=1)$, $\mathcal{H}(1,\mathrm{t})=\{2+\mathrm{t}\}$ does form a valid continuous path.

This example problem demonstrates that $\mathcal{H}$ fails to satisfy the $\mathcal{M}-$ homotopy conditions because: It does not meet the initial and final conditions for $(x=0)$. It fails the continuity requirement for $(x=0)$. The path connection condition is not clearly defined for $(x=0)$. This serves as a counterexample showing that not every multivalued function $\mathcal{H}$ between $\mathscr{f}$ and $\mathscr{g}$ qualifies as a valid $\mathcal{M}-$ homotopy.

Example 2.4. Consider multivalued functions $\mathcal{K},\ \mathcal{L}:S^1\to\mathcal{P}(\mathbb{C})$ where $\mathcal{K}(\theta)=\{\mathrm{e}^{\mathrm{i}\theta},\ \mathrm{e}^{-\mathrm{i}\theta}\}$ and $\mathcal{L}(\theta)=\{\mathrm{e}^{\mathrm{i}(\theta+\pi)},\ \mathrm{e}^{-\mathrm{i}(\theta+\pi)}\}$. A $\mathcal{M}-$ homotopy $\mathcal{H}_{\mathrm{t}}$ could continuously transform $\mathcal{K}$ into $\mathcal{L}$.

Example 2.5. Suppose $\mathcal{X}$ is the unit interval $[0,1]$ and $\mathcal{Y}$ is the Euclidean plane $\mathbb{R}^2$. Let $\mathcal{K}$ and $\mathcal{L}$ be two multivalued functions defined by,

- $\mathcal{K}(\mathcal{X})=\{(\mathcal{X},0)\ for\ (\mathcal{X}\ [0,1])\}$
- $\mathcal{L}(\mathcal{X})=\{(\mathcal{X},1)\ for\ (\mathcal{X}\ [0,1])\}$

A homotopy $\mathcal{H}:[0,1]\times[0,1]\to\mathbb{R}^2$ between $\mathcal{K}$ and $\mathcal{L}$ can be defined by, $\mathcal{H}(x,\mathrm{t})=\{x,\mathrm{t}\}$ for each $x\in[0,1]$ and $\in[0,1]$, $\mathcal{H}(x,\mathrm{t})$ provides a continuous deformation from $(x,0)$ to $(x,1)$.

Problem 2.6. Given a covering space $\phi:A\to B$ and a loop $\gamma:[0,1]\to B$ based at $\beta\in B$, find a multivalued homotopy lifting γ to a loop in A.

Solution:

1. Identify the set of points $\tilde{\beta}\in\phi^{-1}(\beta)$ as the possible starting points for the lifted loop.
2. Define a multivalued function $\mathcal{K}$ that maps each $\mathrm{t}\in[0,1]$ to the set of points in $\phi^{-1}(\gamma(\mathrm{t}))$.
3. Construct a $\mathcal{M}-$ homotopy $\mathcal{H}_{\mathrm{t}}$ that continuously deforms the initial point $\tilde{\beta}$ to the endpoint, ensuring $\mathcal{H}_0=\tilde{\beta}$ and $\mathcal{H}_1=\tilde{\beta}$.

Problem 2.7. Let $\mathcal{X}=S^1$ unit circle in $\mathbb{R}^2$ and $\mathcal{Y}=\mathbb{R}^2$. Define tow multivalued function $\mathcal{K},\ \mathcal{L}:S^1\to\mathcal{Y}$ as follows;

- $\mathcal{K}(\theta)=\{cos\theta,\ sin\theta\}$ Representing the unit circles.

- $\mathcal{L}(\theta) = \{2cos\theta,\ 2sin\theta\}$ Representing a circle of radius 2.

A homotopy $\mathcal{H} : S^1 \times [0,1] \to \mathbb{R}^2$ between $\mathcal{K}$ and $\mathcal{L}$ can be defined by, $\mathcal{H}(\theta, \mathrm{t}) = \{(1+\mathrm{t})cos\theta,\ (1+\mathrm{t})sin\theta\}$. Here, $\mathcal{H}(\theta, 0) = \mathcal{K}(\theta)$ and $\mathcal{H}(\theta, 1) = \mathcal{L}(\theta)$, for each $\mathrm{t} \in [0,1]$. $\mathcal{H}(\theta,\ \mathrm{t})$ describes a circle with radius $(1+\mathrm{t})$, continuously deforming the unit circle into the circle of radius 2.

Problem 2.8. Let $\mathcal{X} = [0,1]$, and $\mathcal{Y} = \mathbb{R}^2$. Define tow multivalued function $\mathcal{K}, \mathcal{L} : [0,1] \to \mathcal{Y}$ as follows;

- $\mathcal{K}(\mathcal{X}) = \{(\mathcal{X}, 0),\ (\mathcal{X}, 1)\}$ Representing a pair of interval line segments at each $x \in [0,1]$.
- $\mathcal{L}(\mathcal{X}) = \{(x, 0)\}$ Representing a single horizontal line segment from $(0,0)$ to $(1,0)$.

A homotopy $\mathcal{H} : [0,\ 1] \times [0,1] \to \mathbb{R}^2$ between $\mathcal{K}$ and $\mathcal{L}$ can be defined by, $\mathcal{H}(x, \mathrm{t}) = \{(x, 0),\ (x,\ (1-\mathrm{t}))\}$. Here, $\mathcal{H}(\mathcal{X}, 0) = \mathcal{K}(\mathcal{X})$ and $\mathcal{H}(\mathcal{X}, 1) = \mathcal{L}(\mathcal{X})$, for each $\mathrm{t} \in [0,1]$, $\mathcal{H}(x,\ \mathrm{t})$ provides a continuous deformation where the point $(x, 1)$ moves linearly downward to $(x, 1)$, effectively collapsing the vertical segments into the horizontal line segment.

Theorem 2.9. If $\mathcal{X}$ and $\mathcal{Y}$ are path-connected spaces, then any two multivalued functions $\mathcal{K}, \mathcal{L} \to \mathcal{P}(\mathcal{Y})$ that map each point in $\mathcal{X}$ to a single set in $\mathcal{Y}$ can be connected by a $\mathcal{M} -$ homotopy.

Proof: Since $\mathcal{X}$ and $\mathcal{Y}$ are path-connected, there exists a path between any two points in $\mathcal{Y}$. Construct a continuous family of multivalued functions $\mathcal{H}_{\mathrm{t}}$ such that $\mathcal{H}_0 = \mathcal{K}$ and $\mathcal{H}_1 = \mathcal{L}$, where $\mathcal{H}_{\mathrm{t}}$ is defined by a continuous deformation between the sets $\mathcal{K}(\mathcal{X})$ and $\mathcal{L}(\mathcal{X})$ for each $x \in \mathcal{X}$. The continuity of $\mathcal{H}_{\mathrm{t}}$ follows from the path-connectedness of $\mathcal{Y}$.

Theorem 2.10. The set of homotopy classes of multivalued loops at a base point $\beta \in B$ forms a group under the operation of concatenation.

Proof: Let $[\mathcal{K}]$ and $[\mathcal{L}]$ be homotopy classes of multivalued loops based at β. Define the concatenation $([\mathcal{K}] * [\mathcal{L}])$ by concatenating representatives of these classes. This operation is associative, has an identity element (the constant multivalued function at β, and each element has an inverse (the multivalued function retracing the loop in reverse). Therefore, the set of homotopy classes under concatenation forms a group.

Example 2.11. Consider $\mathcal{X} = [0,1]$ and $\mathcal{Y} = \mathbb{R}$. Let $\mathcal{K}$ be a multivalued function such that $\mathcal{K}(\mathrm{t}) = \{\mathrm{t}, 2\mathrm{t}\}$ and $\mathcal{L}(\mathrm{t}) = \{\mathrm{t},\ \mathrm{t}+1\}$. A $\mathcal{M} -$ homotopy $\mathcal{H}_{\mathrm{t}}$ connecting $\mathcal{K}$ and $\mathcal{L}$ can be defined as $\mathcal{H}_{\mathrm{t}(\mathrm{t})} = \{(1-\mathrm{t})\mathrm{t} + \mathrm{t}(\mathrm{t}+1),$ $(1-\mathrm{t})2\mathrm{t} + \mathrm{t}(\mathrm{t}+1)\}$.

Example 2.12. In the covering space $\phi : \mathbb{R} \to S^1$, where $\phi(\mathrm{t}) = \mathrm{e}^{2\,\mathrm{it}}$, consider the multivalued function $\mathcal{K}$ that assigns to each point in S^1 the set of all its preimages in $\mathbb{R}$. This function is homotopic to itself under any continuous deformation that lifts to $\mathbb{R}$.

Remark 2.13. The homotopy classes of multivalued functions can reveal more intricate topological properties of spaces than single-valued functions. They provide a richer structure for studying spaces with branching behaviors or complex coverings.

5. Group Operations in $\mathcal{M} -$ Homotopic Functions

In algebraic topology, group operations play a fundamental role, particularly in the context of fundamental groups and homotopy classes. When considering $\mathcal{M} -$ homotopic functions, we can explore how group operations might be applied or extended to these more complex mappings. Here's a detailed examination of how group operations can be used with $\mathcal{M} -$ homotopic functions. We often use group operations to understand the structure and relationships between homotopy classes. This concept can be extended to $\mathcal{M}-$ homotopic functions. Below are the definitions and explanations on how group operations can be used with multivalued homotopic functions. Group operations can indeed be applied to $\mathcal{M}-$ homotopic functions, primarily through the concatenation of paths and loops. This extends the classical notion of the fundamental group to settings where multivalued functions are considered. The resulting structures provide a richer framework for exploring the topological properties of spaces and their coverings, enhancing our understanding of $\mathcal{M}-$ homotopy and its applications in algebraic topology. The group operations within the context of $\mathcal{M}-$ homotopic functions necessitate a reevaluation of algebraic structures. These operations must be defined carefully to ensure that the set of $\mathcal{M}-$ homotopy classes retains the necessary algebraic properties, such as closure, associativity, and the existence of an identity element.

Concatenation of Multivalued Paths:

- Consider two multivalued paths $\mathcal{K}$ and $\mathcal{L}$ in $\mathcal{X}$ from a base point β. If $\mathcal{K}$ maps each point t in $[0,1]$ to a set of points $\mathcal{K}(\mathrm{t}) \subset \mathcal{Y}$, and $\mathcal{L}$ similarly maps $[0,1]$ to sets in $\mathcal{Y}$, we can define a concatenated path $(\mathcal{K} * \mathcal{L})$ by:

$$(\mathcal{K} * \mathcal{L})(\mathrm{t}) = \begin{cases} \mathcal{K}(2\mathrm{t}) & if \quad 0 \leq \mathrm{t} \leq 0.5 \\ \mathcal{L}(2\mathrm{t}-1) & if \quad 0.5 < \mathrm{t} \leq 1 \end{cases}$$

- This concatenation operation combines the multivalued paths, effectively defining a new multivalued path that traverses $\mathcal{K}$ first and then $\mathcal{L}$.

1. **Homotopy Classes and Group Structure:**
   - Just as with single-valued functions, we can consider homotopy classes of multivalued functions. Two multivalued functions $\mathcal{K}$ and $\mathcal{L}$ are homotopic if there exists a continuous family of multivalued functions $\mathcal{H}_\mathrm{t}$ connecting them.
   - The set of homotopy classes of multivalued loops at a point β can be endowed with a group structure by using the concatenation of loops as the group operation.
2. **Group operation on homotopy class:**
   - The homotopy classes of these multivalued loops can be combined using concatenation, forming a group structure analogous to the fundamental group. The identity element is the class of the constant loop, and the inverse of a loop is its traversal in the opposite direction.

Definition 3.1. Group Operations on $\mathcal{M}-$ Homotopic Functions is a set of multivalued functions from $\mathcal{X} \to \mathcal{P}(\mathcal{Y})$ (denoted as $\mathcal{F}$), we can define group operations as follows, assuming that $\mathcal{Y}$ has a group structure, If $(\mathcal{Y},+)$ is an abelian group, the addition of two $\mathcal{M}-$ homotopic functions $(\mathcal{K},\ \mathcal{L} \in \mathcal{F})$ is defined by: $[(\mathcal{K}+\mathcal{L})(x) = \{\mathscr{f} + \mathscr{g} \mid \mathscr{f} \in\ \mathcal{G}(x)\}]$ for all $(x \in \mathcal{X})$. The resulting function $(\mathcal{K}+\mathcal{L})$ is also a multivalued function. If $(\mathcal{Y},\ *)$ is a group, the multiplication of two $\mathcal{M}-$ homotopic functions $(\mathcal{K},\ \mathcal{L} \in \mathcal{F})$ is defined by: $[(\mathcal{K} * \mathcal{L})(x) = \{\mathscr{f} * \mathscr{g} \mid \mathscr{f} \in\ \mathcal{G}(x)\}]$ for all $(x \in \mathcal{X})$. The resulting function $(\mathcal{K} * \mathcal{L})$ is also a multivalued function. If $(\mathcal{Y},\ *)$ is a group, the inverse of a $\mathcal{M}-$ homotopic $(\mathcal{K} \in \mathcal{F})$ function is defined by, $\mathcal{K}^{-1}(x) = \{\mathscr{f}^{-1} \mid \mathscr{f} \in \mathcal{K}(x)\}$ for all $(x \in \mathcal{X})$. The resulting function $\mathcal{K}^{-1}$ is also a multivalued function.

Definition 3.2. A $\mathcal{M}-$ homotopy between two multivalued functions $\mathcal{K}, \mathcal{L} : \mathcal{X} \to \mathcal{P}(\mathcal{Y})$ is a continuous family of multivalued functions $\mathcal{H}_\mathrm{t} : \mathcal{X} \to \mathcal{P}(\mathcal{Y})$ for $\mathrm{t} \in [0,1]$, such that: $\mathcal{H}_0 = \mathcal{K}$ and $\mathcal{H}_1 = \mathcal{L}$.

Remark 3.3. A $\mathcal{M}-$ homotopies generalize the concept of single-valued homotopies by allowing the image of a point to be a set. This is particularly useful in contexts where functions may naturally have multiple values, such as in covering spaces or when considering multi-valued solutions to equations.

Corollary 3.4. The set of homotopy classes of multivalued loops at a base point β forms a group under concatenation, with the identity element being the class of the constant loop and the inverse of a class being the class of the reverse loop.

Theorem 3.5. The set of homotopy classes of maps from a pointed topological space $(\mathcal{X},\ x_0)$ to a pointed topological space $(\mathcal{Y},\ y_0)$, denoted by $[\mathcal{X},\mathcal{Y}]$, forms a group under the operation of concatenation of loops (when $\mathcal{Y}$ is a loop space).

Proof: Closure, If $[f]$, $[g] \in [\mathcal{X},\mathcal{Y}]$, their concatenation $[f \cdot g]$ is also in $[\mathcal{X},\mathcal{Y}]$. Here, $[f \cdot g]$ represents the map obtained by first applying f and g. Since f and g are continuous, their concatenation $(f \cdot g)$ is also continuous. Associativity: For $[f],[g],[k] \in [\mathcal{X},\mathcal{Y}]$, the concatenation operation is associative, $((f \cdot g) \cdot k) = (f \cdot (g \cdot k))$. This follows directly from the associativity of function composition. Identity Element: The constant map $e:\ \mathcal{X} \to \mathcal{Y}$ defined by $e(x) = y_0$ for all $x \in \mathcal{X}$ serves as the identity element in $[\mathcal{X},\mathcal{Y}]$. For any $[f] \in [\mathcal{X},\mathcal{Y}]$, $(f \cdot e) = f$ and $(e \cdot f) = f)$. Inverse Element, For each $[f] \in [\mathcal{X},\mathcal{Y}]$, there exists an inverse element $[f^{-1}]$ such that $(f \cdot f^{-1} \sim e)$ and $(f^{-1} \cdot f \sim e)$. Here, (f^{-1}) represents the map that "reverses" the path defined by f. Hence, $[\mathcal{X},\mathcal{Y}]$ forms a group under concatenation.

Theorem 3.6. Homotopy Groups are Abelian for $(n \geq 2)$. For $(n \geq 2)$, the n^{th} homotopy group $\pi_n(\mathcal{Y},y_0)$ is abelian.

Proof: Consider two maps $f,g : S^n \to \mathcal{Y}$ based at y_0. To show commutativity, we need to demonstrate that the homotopy classes of their concatenations are equal: $[f \cdot g] = [g \cdot f]$. Cube Construction, Construct a map $\mathcal{H} : I^n \to \mathcal{Y}$ such that $\mathcal{H}|\{\partial I^n\} = (f \cdot g)$ and $\mathcal{H}|\{\partial I^n\} = (g \cdot f)$. Here, I^n is the $n-$dimensional cube, and ∂I^n represent different ways of gluing the boundary. Homotopy, The map $\mathcal{H}$ provides a homotopy between $(f \cdot g)$ and $(g \cdot f)$, demonstrating that $[f \cdot g] = [g \cdot f]$ in $\pi_n(\mathcal{Y},y_0)$. Since concatenation is commutative for $(n \geq 2)$, $\pi_n(\mathcal{Y},y_0)$ is abelian.

Theorem 3. 7. (Hurewicz Theorem). For a simply connected space $\mathcal{X}$, the first nontrivial homotopy group $\pi_n(\mathcal{X})$ is isomorphic to the first nontrivial homology group $\mathcal{H}_n(\mathcal{X})$ for $(n \geq 2)$.

Proof: Define the Hurewicz homomorphism $h : \pi_n(\mathcal{X}) \to \mathcal{H}_n(\mathcal{X})$ that maps a homotopy class of maps $[f]$ to the corresponding homology class. Isomorphism, For $(n \geq 2)$, h is an isomorphism. This is proven using the exact sequences of homotopy and homology groups, and by showing that h is both injective and surjective. Injectivity, If $h[f] = 0$, then f is homotopic to a constant map, implying $[f] = 0$ in $\pi_n(\mathcal{X})$. Surjectivity, For any homology class $c \in \mathcal{H}_n(\mathcal{X})$, there exists a map $f : S^n \to \mathcal{X}$ such that $h([f]) = [c]$. Hence, $\pi_n(\mathcal{X})$ for $(n \geq 2)$.

6. $\mathcal{M}-$ functions in Fundamental Group

Extending the concept of the fundamental group to include multivalued functions enriches our understanding of topological spaces. This approach allows for a more flexible and comprehensive analysis of loops and paths, accommodating scenarios where traditional single-valued paths are insufficient. Establishing equivalence among $\mathcal{M}-$ homotopy functions involves defining suitable equivalence relations that respect the multivalued nature of the functions. This process ensures that the generalized homotopy classes form a coherent and meaningful extension of classical homotopy classes. In the context of single-valued functions, homotopy classes of loops at a base point $(\beta \in B)$ form a group known as the fundamental group $\pi_1(B,\beta)$. The group operation is the concatenation of loops. This concept can be extended to $\mathcal{M}-$ homotopy classes, though with some nuances. Consider the universal covering space $\phi : A \to B$ and let $(\beta \in B)$ with $\alpha \in \phi^{-1}(\beta)$.

1. **Multivalued loop lifting: A loop $\gamma : [0,1] \to B$** based at β can be lifted to a multivalued loop in A. Each point in the fiber $\phi^{-1}(\beta)$ can be connected by paths corresponding to elements of the fundamental group $\pi_1(B,\beta)$.
2. **Concatenation of Multivalued loops:** If γ_1 and γ_2 are two loops in B based at β their concatenation $(\gamma_1 * \gamma_2)$ can be lifted to multivalued loops in A resulting in a $\mathcal{M} -$ function representing the combined path.
3. **Homotopy classes of multivalued loops:** The homotopy classes of multivalued loops at a base point $(\beta \in B)$ form a group under the operation of concatenation. Specifically:
   - Identity element: The constant multivalued function at β serves as the identity element.
   - Inverse: The inverse of a multivalued path $\mathcal{K}$ is a path that retraces $\mathcal{K}$ in the opposite direction.

Definition 4.1. A multivalued path in a topological space $\mathcal{X}$ from a point α_0 to a point α_1 is a continuous multivalued function $\Gamma : [0,1] \to \mathcal{X}$ such that $\Gamma(0) = \alpha_0$ and . $\Gamma(1) = \alpha_1$.

Example 4.2. Consider the space $\mathbb{R}^2$ and points $(0,0)$ and $(1,1)$. A multivalued path from $(0,0)$ to $(1,1)$. could be a function Γ that, at each time $t \in [0,1]$, includes all points on the line segment connecting $(0,0)$ and $(1,1)$. Thus, $\Gamma(t) = \{(1-t)(0,0) + t(1,1)\}$.

Example 4.3. In a space $\mathcal{X}$, consider two loops α and β based at x_0. Let be a multivalued homotopy between α and β such that $t \in [0,1]$, for each $\mathcal{H}_{(t)}$ represents a continuous deformation of α into β via multivalued intermediate loops.

Theorem 4.4. (Lifting of $\mathcal{M} -$ Homotopies) Given a covering space $\phi : A \to B$ and a $\mathcal{M} -$ homotopy $\mathcal{H}_t : \mathcal{X} \to \mathcal{P}(\mathcal{Y})$, there exists a lifted $\mathcal{M} -$ homotopy $\widetilde{\mathcal{H}}_t : \widetilde{\mathcal{X}} \to \mathcal{P}(\widetilde{\mathcal{Y}})$ such that $(\phi \circ \widetilde{\mathcal{H}}_t) = \mathcal{H}_t$.

Proof: Since ϕ is a covering map, each point in $\mathcal{X}$ has a neighborhood λ such that $\phi^{-1}(\lambda)$ is a disjoint union of open sets in $\widetilde{\mathcal{X}}$. The continuity of $\mathcal{H}_t$ and the lifting properties of ϕ ensure that $\widetilde{\mathcal{H}}_t$ can be defined continuously on $\widetilde{\mathcal{X}}$ in a manner that respects the multivalued nature of the functions.

Problem 4.5. Given two $\mathcal{M} -$ functions $\mathcal{K}, \mathcal{L} : \mathcal{X} \to \mathcal{P}(\mathcal{Y})$, determine if there exists a $\mathcal{M} -$ homotopy $\mathcal{H}_t : \mathcal{X} \to \mathcal{P}(\mathcal{Y})$ connecting $\mathcal{K}$ and $\mathcal{L}$.

Solution: To determine if there exists a $\mathcal{M} -$ homotopy connecting $\mathcal{K}$ and $\mathcal{L}$ we need to check if there exists a continuous function $\mathcal{H}_t : \mathcal{X} \times [0,1] \to \mathcal{P}(\mathcal{Y})$ such that;

1. $\mathcal{H}_0(x) = \mathcal{K}(x) \ \forall \ x \in \mathcal{X}$
2. $\mathcal{H}_1(x) = \mathcal{L}(x) \ \forall \ x \in \mathcal{X}$
3. $\mathcal{H}_t$ is continuous for all $t \in [0,1]$

A multivalued homotopy exists if and only if these conditions are satisfied.

Problem 4.6. Consider the covering space $\phi : A \to B$. Describe the set of $\mathcal{M} -$ functions that represent lifts of a given path in B. Determine the homotopy classes of these multivalued functions.

Solution: For the covering space $\phi : A \to B$ the set of $\mathcal{M} -$ functions representing lifts of a given path in B is described by the fibers of the covering map ϕ. The homotopy classes of these multivalued functions depend on the homotopy classes of the paths in B and the properties of the covering map ϕ. To describe the set of multivalued functions representing lifts of a given path in B we need to consider the Preimage of the path under the covering map ϕ. The homotopy classes of these multivalued functions correspond to the homotopy classes of the paths in $\mathcal{X}$ and are determined by the properties of the covering map ϕ.

Theorem 4.7. Let $\mathcal{X}$ be a topological space and $x_0 \in \mathcal{X}$. The set of multivalued loops based at x_0, under the operation of concatenation followed by $\mathcal{M}-$ homotopy equivalence, forms a group, denoted $\{\pi^{-1}(\mathcal{M})\}(\mathcal{X}, x_0)$

Proof: Closure: Let α and β be two elements in $\{\pi^{-1}(\mathcal{M})\}(\mathcal{X}, x_0)$, where α and β are multivalued loops based at x_0. The concatenation $(\alpha * \beta)$ is also a multivalued loop based at x_0. Hence, the set of multivalued loops is closed under concatenation.

Associativity: For multivalued loops α, β and γ based at x_0, the concatenation operation is associative, i.e., $(\alpha * \beta) * \gamma \sim \alpha * (\beta \sim \gamma)$. This follows from the properties of path concatenation in classical homotopy theory and holds for multivalued functions as well since $\mathcal{M}-$ homotopies can interpolate between any two such concatenations.

Identity Element: The constant multivalued loop $e_{x_0}(\mathrm{t}) = x_0$ for all $\mathrm{t} \in [0,1]$ acts as the identity element. For any multivalued loop α, the concatenations $(e_{x_0} * \alpha)$ and $(\alpha * e_{x_0})$ are homotopic to α.

Inverse Element: For a multivalued loop α, define the inverse loop α^{-1} by $(\alpha^{-1}(\mathrm{t})) = \alpha(1-\mathrm{t})$. Then, $(\alpha * \alpha^{-1} \sim e_{x_0})$ and $(\alpha^{-1} * \alpha \sim e_{x_0})$. Hence, $\{\pi^{-1}(\mathcal{M})\}(\mathcal{X}, x_0)$ satisfies the group axioms.

Theorem 4.8. Two topological spaces $\mathcal{X}$ and $\mathcal{Y}$ are $\mathcal{M}-$ homotopy equivalent if there exist multivalued functions $f: \mathcal{X} \to \mathcal{Y}$ and $g: \mathcal{Y} \to \mathcal{X}$ such that $(f \circ g)$ and are multivalued homotopic to the identity maps on $\mathcal{X}$ and $\mathcal{Y}$, respectively.

Proof: Existence of Multivalued Functions, Suppose there exist multivalued functions $f: \mathcal{X} \to \mathcal{Y}$ and $g: \mathcal{Y} \to \mathcal{X}$. $\mathcal{M}$ –Homotopy to Identity, Assume $(g \circ f \sim id_{\mathcal{X}})$ and $(f \circ g \sim id_{\mathcal{Y}})$, where $\sim$ denotes $\mathcal{M}-$ homotopy equivalence. This means there exist $\mathcal{M}-$ homotopies $\mathcal{H}: \mathcal{X} \times [0,1] \to \mathcal{X}$ and $\mathcal{H}: \mathcal{Y} \times [0,1] \to \mathcal{Y}$ such that, $\mathcal{H}(x, 0) = (g \circ f)(x)$ and $\mathcal{H}(x, 1) = (x)$ for all $x \in \mathcal{X}$. $\mathcal{K}(y, 0) = (g \circ f)(y)$ and $\mathcal{K}(y, 1) = (y)$ for all $y \in \mathcal{Y}$ Homotopy Inverses, By definition of $\mathcal{M}-$ homotopies, $\mathcal{H}$ and $\mathcal{K}$ provide continuous deformations from $(g \circ f) \to id_{\mathcal{X}}$ and from $(f \circ g) \to id_{\mathcal{Y}}$. Therefore, $\mathcal{X}$ and $\mathcal{Y}$ are $\mathcal{M}-$ homotopy equivalent, as the existence of these $\mathcal{M}-$ homotopies ensures the spaces can be continuously deformed into each other through multivalued functions.

Corollary 4.9. If $\mathcal{X}$ and $\mathcal{Y}$ are path-connected, then any two multivalued functions from $\mathcal{X}$ to $\mathcal{Y}$ can be connected by a $\mathcal{M}-$ homotopy.

5. Equivalence class of $\mathcal{M}-$ Homotopy Functions

In algebraic topology, homotopy theory investigates the properties of spaces that are invariant under continuous deformations. Traditional homotopy functions consider single-valued maps between spaces, but in more complex scenarios, multivalued functions are essential. These functions, which map each point in one space to a set of points in another, enable a richer framework for studying topological properties. Equivalence of $\mathcal{M}-$ homotopy functions aims to classify multivalued functions based on their homotopy classes. Two multivalued functions $\mathcal{K}$ and $\mathcal{L}$ are considered equivalent if there exists a $\mathcal{M}-$ homotopy $\mathcal{H}_{\mathrm{t}}$ that continuously transforms $\mathcal{K}$ into $\mathcal{L}$. Group operations can be defined on the set of equivalence classes of $\mathcal{M}-$ homotopy functions, analogous to the operations on the fundamental group. This includes the concatenation of multivalued paths and the consideration of homotopy classes of loops, which form a group under the operation of concatenation. Equivalence of $\mathcal{M}-$ homotopy functions offers a powerful framework for studying topological spaces. By extending classical homotopy concepts to multivalued settings, mathematicians can explore more intricate properties of spaces, uncovering new relationships and invariants that are not apparent through single-valued functions alone. This approach is particularly useful in the study of complex spaces, such as those encountered in covering space theory and other advanced topics in algebraic topology.

Importance of Equivalence:

- Topological Invariants: Understanding the equivalence of $\mathcal{M}-$ homotopy functions helps in identifying topological invariants that remain unchanged under continuous deformations, providing deeper insights into the fundamental structure of spaces.
- Classification: Classifying spaces using $\mathcal{M}-$ homotopy equivalence extends the traditional methods, allowing for a more nuanced understanding of spaces that may exhibit complex or branching behaviors.

Definition 5.1. A multivalued function $\mathcal{K}$ from a topological space $\mathcal{X}$ to a topological space $\mathcal{Y}$ is a relation $\mathcal{K} \subseteq \mathcal{X} \times \mathcal{Y}$ such that each $x \in \mathcal{X}$ is associated with a subset $\mathcal{K}(x) \subseteq \mathcal{Y}$. Two continuous functions $f, g : \mathcal{X} \to \mathcal{Y}$ are homotopy equivalent if there exists a continuous function $\mathcal{H} : \mathcal{X} \times [0,1] \to \mathcal{Y}$ such that,

- $\mathcal{H}(x, 0) = f(x)\ \forall x \in \mathcal{X}$
- $\mathcal{H}(x, 1) = g(x)\ \forall x \in \mathcal{X}$

Definition 5.2. A homotopy equivalence class functions $[f]$ between two topological spaces $\mathcal{X}$ and $\mathcal{Y}$ can be defined a function $f : \mathcal{X} \to \mathcal{Y}$ consist of all functions $g : \mathcal{X} \to \mathcal{Y}$ that are homotopy equivalent to $\mathcal{K}$. In terms of $\mathcal{M}-$ homotopy functions we can represent $[f]$ as, $\mathcal{K} : \mathcal{X} \to \mathcal{P}(\mathcal{Y})$ defined by, $\mathcal{F}(x) = \{y \in \mathcal{Y} \mid \exists\ g \in [f], g(x) = y\}$.

Remark 5.3. $\mathcal{F}$ maps each point $x \in \mathcal{X}$ to the set of points in $\mathcal{Y}$ that can be reached by applying any function g that is homotopy equivalent to f. Thus $\mathcal{F}$ encapsulates all the possible outcomes in $\mathcal{Y}$ for each equivalence class of f.

Problem 5.4. Maps between the circle S^1 and itself.

Solution: Consider the circle S^1. Let define $f : S^1 \to S^1$ as the identity map, where $f(\mathcal{X}) = x\ \forall x \in S^1$ and let define $g : S^1 \to S^1$ as a rotation by a fixed angle θ. So $g(\mathcal{X}) = \mathbb{R}\theta(x)$, where $\mathbb{R}\theta$ represents the rotation of $\mathcal{X}$ by the angle θ. These two functions f and g are homotopic if θ is a multiple of 2π, meaning g is essentially the identity map (a full rotation brings every point back to itself). The homotopy $\mathcal{H} : S^1 \times [0,1] \to S^1$ can be defined by $\mathcal{H}(x, \mathrm{t}) = \mathbb{R} - \{\mathrm{t}\theta(x)\}$, which continuously rotates each point x from 0 to θ as t goes from 0 to 1. The homotopy equivalence class $[f]$ of the identity map f induces all maps g that can be written as rotation by integer multiples of 2π.

Problem 5.5. Maps between $\mathbb{R}^n$ and a point.

Solution: Consider the Euclidean space $\mathbb{R}^n$ and a single point space $\{\star\}$. Define $f : \mathbb{R}^n \to \{\star\}$ as the constant map sending every point in $\mathbb{R}^n$ to the point $\{\star\}$ and any other constant map $g : \mathbb{R}^n \to \{\star\}$ also sends every point in $\mathbb{R}^n$ to the point $\{\star\}$. Both f and g are clearly homotopic because the only map possible from $\mathbb{R}^n$ to a single point is the constant map. A homotopy $\mathcal{H}: \mathbb{R}^n \times [0,1] \to \{\star\}$ can be trivially defined as $\mathcal{H}(x, \mathrm{t}) = \{\star\} \forall x \in \mathbb{R}^n$ and $\mathrm{t} \in [0,1]$. The homotopy equivalence class $[f]$ of the constant map f includes only the constant maps, as these are the only possible maps between $\mathbb{R}^n$ and $\{\star\}$.

Problem 5.6. Maps from the unit interval $[0,1]$ to itself.

Solution: Consider the unit interval $[0,1]$. Define the function $f : [0,1] \to [0,1]$ by $f(x) = x$ is the identity map. and define the map $g : [0,1] \to [0,1]$ by $g(x) = x^2$. The functions f and g are homotopic . A homotopy $\mathcal{H} : [0,1] \times [0,1] \to [0,1]$ can be given by $\mathcal{H}(x,t) = (1-t)x + tx^2$, which continuously deforms $f(x) = x$ to $g(x) = x^2$. As t goes from 0 to 1. The homotopy equivalence class $[f]$ of the identity map f includes all functions g that can be continuously deformed to f.

Example 5.7. Consider the universal covering space $\boldsymbol{\phi} : \boldsymbol{A} \to \boldsymbol{B}$. If $\boldsymbol{\gamma} : [\mathbf{0},\mathbf{1}] \to \boldsymbol{B}$ is a loop based at $\boldsymbol{\beta}$, the set of all possible lifts of $\boldsymbol{\gamma}$ to $\boldsymbol{A}$ represents a multivalued function. The equivalence of these $\boldsymbol{\mathcal{M}}$ −functions under homotopy provides information about the structure of $\boldsymbol{B}$ and the behavior of its covering spaces.

Problem 5.8. For a multivalued function $\mathcal{K} : \mathcal{X} \to \mathcal{P}(\mathcal{Y})$, find the homotopy class $[\mathcal{K}]$ and describe its properties. Determine the conditions under which two multivalued functions $\mathcal{K}$ and $\mathcal{L}$ are in the same homotopy class.

Solution: To find the homotopy class $[\mathcal{K}]$ of a multivalued function $\mathcal{K} : \mathcal{X} \to \mathcal{P}(\mathcal{Y})$, we consider all continuous deformations (homotopies) of $\mathcal{K}$ within the space of multivalued functions. Two multivalued functions $\mathcal{K}$ and $\mathcal{L}$ are in the same homotopy class if there exists a continuous function $\mathcal{H}_t$ connecting them, i.e., $\mathcal{K}$ and $\mathcal{L}$ are homotopic. Properties of the homotopy class $[\mathcal{K}]$ may include its fundamental group, homology groups, and other algebraic invariants associated with the space $\mathcal{X}$ and $\mathcal{Y}$. Two multivalued functions $\mathcal{K}$ and $\mathcal{L}$ are in the same homotopy class if and only if they are homotopic.

Conclusion:

The $\mathcal{M}-$ homotopy functions represent a significant and promising development in algebraic topology. By broadening the scope of homotopy theory, we can tackle more complex topological problems and gain deeper insights into the fundamental nature of topological spaces. Future research in this area is likely to reveal further applications and theoretical advancements, solidifying the importance of multivalued approaches in the broader context of mathematical topology.

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