Tuijin Jishu/Journal of Propulsion Technology
ISSN: 1001-4055
Vol. 45 No. 3 (2024)

A Study on W8 - Curvature Tensor in LP - Kenmotsu Manifolds

Rajesh Chaudhary[1] and P. N. Singh[2]

Research scholar[1] and Professor[2]

Abstract: The objective of the present paper is to study the curvature properties of LP -Kenmotsu manifolds satisfying the conditions W_8 - flatness, $\varphi - W_8$- semisymmetric, $W_8 \cdot Q$=0 and found some interesting results.

Mathematics Subject Classification 2010 : 53C15, 53C25, 53D10.

Keyword and Phrases: Almost contact manifolds, Kenmotsu manifolds, LP - Kenmotsu manifolds, φ- symmetric, φ-semisymmetric, W_8-curvature tensor, Einstein manifold, η-Einstein manifold.

1 Introduction

The concept of Lorentzian paracontact, specifically Lorentzian para-Sasakian (LP-Sasakian) manifolds, was first presented by K. Matsumoto [7] in 1989. Subsequently, numerous geometers, including Matsumoto and Mihai [8], Mihai and Rosca [6], Mihai, Shaikh and De [5], Venkatesha, Pradeep Kumar, and Bagewadi [15], Venkatesha, and Bagewadi [16, 17], and obtained several outcomes from these manifolds. F. O¨zen Zengin studied the nature of LP - Sasakian manifolds admitting the $M-$ projective curvature tensor and examined whether this manifold satisfies the condition $W(X,Y) \cdot R = 0$. Moreover, he proved that in the $M-$ projectively flat LP - Sasakian manifolds, the conditions $R(X,Y) \cdot R = 0$ and $R(X,Y) \cdot S = 0$ are satisfied and then he introduced the concept of $M-$ projectively flat space-time. A class of virtually paracontact metric manifolds, called para-Kenmotsu (abbreviated P-Kenmotsu) and special para-Kenmotsu (abbreviated SP-Kenmotsu) manifolds, was developed by Sinha and Sai Prasad [2] in 1995. These manifolds are comparable to P-Sasakian and SP-Sasakian manifolds. In 2018,

Abdul Haseeb and Rajendra Prasad conducted research on φ-semisymmetric LP - Kenmotsu manifolds with a quarter-symmetric non-metric connection admitting Ricci solitons [13]. They also defined a class of Lorentzian almost paracontact metric manifolds, called Lorentzian para-Kenmotsu (abbreviated LP-Kenmotsu) manifolds [1]. Pokhariyal [3] explored these tensor field's properties on a Sasakian manifold in more detail. These notions were expanded to nearly para-contact structures by Matsumoto, Ianus, and Mihai in 1986. They also analyzed para-Sasakian manifolds that admitted these tensor fields [9], with De and Sarkar generalizing their results in 2009 [14].

A. Friedmann and J. A. Schouten [11] introduced the concept of semisymmetric linear connection on a differentiable manifold in 1924. H. A. Hayden [13] first described and researched semi-symmetric metric connection in 1932. The semi-symmetric metric connection in a Riemannian manifold was the subject of a symmetric study initiated by K. Yano[23] in 1970, which was later explained upon by a number of authers including S. Ahmad and S. I. Hussain [26], M. M. Tripathi [21], C.Ozgur et al. [18] and many others.

If ∇ is assumed to be a linear connection and M be an n-dimensional differentiable manifold then the curvature tensor R and torsion tensor T of ∇ are given by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

If Torsion tensor T vanishes i.e. if $T = 0$, then the connection ∇ is called to be symmetric else it is non-symmetric. The connection ∇ is said to be metric connection if there exist a Riemannian metric g in M such that $\nabla g = 0$, otherwise it is non-metric. We know very well that the Levi-Civita connection is defined as;

A linear connection is Levi-Civita if it is symmetric as well as metric.

If torsion tensor T of a linear connection ∇ is of the form

$T(X,Y) = \eta(Y)X - \eta(X)Y,$

then ∇ is called semi-symmetric connection; where η is 1-form.

The semi-symmetric metric connections are very crucial in the study of Riemannian manifolds. The semi-symmetric metric connection is associated with a variety of physical issues. For instance, if a man moves over the surface of the earth always facing a specific location, such as, Jerusalem, Mekka, or the North pole, so this displacement is semi-symmetric and metric.

The paper is structured as follows in response to the studies mentioned above. We provide a brief overview of LP-Kenmotsu manifold and its features. We locate the W_8 flatness in LP- Kenmotsu manifold in section 3. The analysis of the $\varphi - W_8$-semisymmetric condition in LP-Kenmotsu manifold with regard to the semi-symmetric metric connection is covered in section 4. We discover that the LP-Kenmotsu manifold satisfying the condition $W_8 \cdot Q = 0$ in section 5 and present some interesting findings.

2 Preliminaries

An $n-$ dimensional differentiable manifold M admitting a (1,1) tensor field φ, contravariant vector field ξ, a 1-form η and the Lorentzian metric $g(X,Y)$ satisfying

$$\varphi^2 X = X + \eta(X)\xi, \tag{2.1}$$

$$\eta(\xi) = -1, \tag{2.2}$$

$$g(\xi,\xi) = -1, \tag{2.3}$$

$$\eta(X) = g(X,\xi), \tag{2.4}$$

$$g(\varphi X,\varphi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.5}$$

for any vector fields X,Y on M, then it is called Lorentzian almost paracontact manifold. In the Lorentzian paracontact manifold, the following relation hold:

$$\varphi\xi = 0, \qquad \eta(\varphi X) = 0. \tag{2.6}$$

Also, we have

$$\Phi(X,Y) = \Phi(Y,X), \tag{2.7}$$

where $\Phi(X,Y) = g(X,\varphi Y)$.

A Lorentzian almost paracontact manifold M is called Lorentzian parasasakian manifold if

$$(\nabla_X\varphi)(Y) = g(X,Y)\xi + \eta(Y)\varphi X + 2\eta(X)\eta(Y)\xi, \tag{2.8}$$

where ∇ is the Levi- Civita connection with respect to g and for any vector fields X,Y on M.

If ξ is a killing vector field, the (para) contact structure is called $K-$ (para) contact. In this case we have,

$$\nabla_X\xi = \varphi X \tag{2.9}$$

Now, we define Lorentzian-para Kenmotsu manifold:

Definition 2.1: A Lorentzian almost paracontact manifold M is called Lorentzian para-Kenmotsu manifold if

$$(\nabla_X\varphi)(Y) = -g(\varphi X,Y)\xi - \eta(Y)\varphi X. \tag{2.10}$$

In the Lorentzian-para Kenmotsu manifold, we have

$$\nabla_X\xi = -X - \eta(X)\xi, \tag{2.11}$$

$$(\nabla_X\eta)(Y) = -g(X,Y) - \eta(X)\eta(Y). \qquad (2.12)$$

Additionally, the curvature tensor R, the Ricci tensor S and the Ricci operator Q in a Lorentzian para-Kenmotsu manifold M with respect to the

Livi-Civita connection satisfies [8]

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \qquad (2.13)$$

$$R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X, \qquad (2.14)$$

$$R(\xi,X)\xi = X + \eta(X)\xi, \qquad (2.15)$$

$$g(R(X,Y)Z,\xi) = \eta(R(X,Y)Z) \qquad (2.16)$$

$$= g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \qquad (2.17)$$

$$S(X,\xi) = -(n-1)\eta(X), \qquad (2.18)$$

$$Q\xi = (n-1)\xi, \qquad (2.19)$$

(2.20)

where $g(QX,Y) = S(X,Y)$. For any vector fields X,Y and Z on M it yields $S(\varphi X,\varphi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y)$. (2.21)

Definition 2.2: A Lorentzian almost paracontact manifold M is said to be an $\eta-$ Einstein manifold if its Ricci tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \qquad (2.22)$$

where a and b are scalar functions on M.

A Lorentzian almost paracontact manifold M is said to be a generalized $\eta-$ Einstein manifold if its Ricci tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) + c\Phi(X,Y), \qquad (2.23)$$

where a,b and c are scalar functions on M and $\Phi(X,Y) = g(\varphi X,Y)$. If $c = 0$, then the manifold reduces to an $\eta-$ Einstein manifold. Also, it is an Einstein manifold if b and c both are 0.

3 $\xi - W_8$-flat in - Kenmotsu Manifold

In this section, we study $\xi - W_8-$ flat in LP - Kenmotsu manifold:

Definition 3.1: An LP- Kenmotsu manifold is said to be $\xi - W_8-$ flat if

$$W_8(X,Y)\xi = 0, \qquad (3.1)$$

for any vector fields X,Y on M. W_8-curvature tensor [6] is defined as

$$W_8(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[S(X,Y)Z - S(Y,Z)X], \qquad (3.2)$$

where R and S are the curvature tensor and Ricci tensor of the manifold respectively.

putting $Z = \xi$ in (3.2), we get

$$W_8(X,Y)\xi = R(X,Y)\xi + \frac{1}{n-1}[S(X,Y)\xi - S(Y,\xi)X]. \qquad (3.3)$$

By using (3.1) in (3.3), we get

$$R(X,Y)\xi + \frac{1}{n-1}[S(X,Y)\xi - S(Y,\xi)X] = 0. \qquad (3.4)$$

By virtue of (2.13), (2.18) in (3.4) and on simplification, we obtained

$$\{\eta(X)Y - \eta(Y)X\} + \frac{1}{n-1}[S(X,Y)\xi + (n-1)\eta(Y)X] = 0 .$$

By taking inner product with ξ in (3.5) and on simplification, we have

$S(X,Y) = (n-1)\eta(Y)\eta(X)$.

Hence from the above discussion, we state that the following theorem:

Theorem 3.2: If an LP- Kenmotsu manifold satisfying $\xi - W_8-$flat condition then the manifold is a special type of η-Einstein manifold.

4 $\varphi - W_8$- semisymmetric Condition in LP Kenmotsu Manifold

In this section, we study $\varphi - W_8-$ semisymmetric condition in an Kenmotsu manifold:

Definition 4.1 : An -Kenmotsu manifold is said to be $\varphi - W_8-$ semisymmetric if

$$W_8(X,Y) \cdot \varphi = 0, \tag{4.1}$$

for every vector field X,Y on M.

Now, (4.1) turns into

$$R(X,Y)\cdot\phi Z - \phi R(X,Y)Z + \frac{1}{n-1}[S(Y,Z)\phi X - S(Y,\phi Z)X] = 0 .$$

Putting $X = \xi$, we get

$$R(\xi,Y)\cdot\phi Z - \phi R(\xi,Y)Z + \frac{1}{n-1}[S(Y,Z)\phi\xi - S(Y,\phi Z)\xi] = 0 .$$

From equation (2.14), we get

$$-\eta(\phi Z)Y + g(Y,\phi Z)\xi - \phi[-\eta(Z)Y + g(Y,Z)\xi] + \frac{1}{n-1}[-S(Y,\phi Z)\xi] = 0$$

$$or,\ -\eta(\phi Z)Y + g(Y,\phi Z)\xi + \phi\eta(Z)Y - \phi g(Y,Z)\xi - \frac{1}{n-1}[S(Y,\phi Z)\xi] = 0.$$

$$or, g(Y,\phi Z)\xi + \phi\eta(Z)Y - \frac{1}{n-1}S(Y,\phi Z)\xi = 0. .$$

Interchanging Z and φZ, we get

$$g(Y,Z)\xi + \phi\eta(\phi Z)Y - \frac{1}{n-1}S(Y,Z)\xi = 0$$

$$or, g(Y,Z)\xi = \frac{1}{n-1}S(Y,Z)\xi.$$

$$or, S(Y,Z)\xi = (n-1)g(Y,Z)\xi. .$$

Taking inner product with ξ, we get

$S(Y,Z) = -(n-1)g(Y,Z)$.

Hence from the above discussion, we state the following theorem:

Theorem 4.2 : If an -Kenmotsu manifold satisfying $\varphi - W_8-$ semisymmetric condition then manifold is an Einstein manifold.

5 LP - Kenmotsu Manifolds satisfying $W_8 \cdot Q$ =0

In this section, we study LP-Kenmotsu Manifolds satisfying $W_8 \cdot Q = 0$. Then we have

$W_8((X,Y)Q)Z - Q(W_8(X,Y)Z) = 0.$

Putting Y= ξ in above, we get

$W_8((X,\xi)Q)Z - Q(W_8(X,\xi)Z) = 0.$

Using $W_8(X,Y)Z = R(X,Y)Z + \frac{1}{(n-1)}[S(X,Y)Z - S(Y,Z)X$], we get

$$R(X,\xi)QZ + \frac{1}{(n-1)}[S(X,\xi)QZ - S(\xi,QZ)X] - Q[R(X,\xi)Z + \frac{1}{(n-1)}\{S(X,\xi)Z - S(\xi,Z)X\}] = 0. \quad (5.1)$$

We have,

$R(\xi,X)Y = -\eta(Y)X + g(X,Y)\xi$

$S(X,Y) = -(n-1)\eta(X)$

Solving (5.1), we get

$$-g(X,QZ)\xi + \eta(QZ)X + \frac{1}{(n-1)}[-(n-1)\eta(X)QZ + (n-1)\eta(QZ)X] - Q[-g(X,Z)\xi + \eta(Z)X + \frac{1}{(n-1)}[-(n-1)\eta(X)Z + (n-1)\eta(Z)X] = 0.$$

or, $\eta(QZ)X - g(X,QZ)\xi - \eta(X)QZ + \eta(QZ)X -$

$$\eta(Z)QX + g(X,Z)Q\xi + \eta(X)QZ - \eta(Z)QX = 0. \quad (5.2)$$

Solving equation (5.2) and by by the vertue of (2.19), we have *or,* $2\eta(QZ)X - 2\eta(Z)QX - g(X,QZ)\xi + (n-1)g(X,Z) = 0.$

or, $g(X,QZ)\xi = 2\eta(Z)QX - 2\eta(QZ)X - (n-1)g(X,Z)\xi.$

Since, $S(X,Y) = g(QX,Y)$, then we have

$$S(X,Y)\xi = 2\eta(Z)QX - 2\eta(QZ)X - (n-1)g(X,Z)\xi. \quad (5.3)$$

Using $QX = (n-1)Xin(5.3)$, we have

$S(X,Z)\xi = -(n-1)g(X,Z)\xi$

Taking inner product with ξ, we get

$S(X,Z) = (n-1)g(X,Z).$

Hence from the above discussion, we state the following theorem:

Theorem 5.1: An LP-Kenmotsu manifold satisfying $W_8 \cdot Q = 0$, is an

Einstein manifold.

6 Conclusions

In this paper, we proposed that a W_8 - flat LP - Kenmotsu manifold is a special type of η Einstein manifold. Next, we deal with $\varphi - W_8$- semisymmetric condition in LP -Kenmotsu manifold and found it to be Einstein manifold. Again, we discussed the LP - Kenmotsu manifolds satisfying $W_8 \cdot Q = 0$ condition and it comes out to be an Einstein manifold.

7 Statements and Declarations

Fundings: The authors declare that no funds, grants or other support were received during the preperation of this manuscripts.

Informed Consent statement: Not applicable.

Data Availability statement: Data from the previous studies have been used and they are cited at the relevant places according as the reference list of the paper.

Conflicts of Interest: We declare that there is no conflict of interest between the authors.

8 Acknowledgement

The authors are very thankful to the anonymous refree and editor for his/her valuable comments and suggestions for improvement of the quality and presentation of this paper.

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