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Role of Damping in Computational Modeling of Vibrations of Annular Plates

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Abstract: A modified computational model is introduced for analysis of axisymmetric vibrations of a non-homogeneous annular plate with variable thickness resting on elastic foundation. In this analysis the differential quadrature method (DQM) is employed to solve the governing equation and the results are compared with the values obtained by quintic spline technique under similar parametric conditions. The convergence and stability analysis of the method for the proposed model is presented. Accuracy of the results, obtained during implementation of the method, represent an excellent agreement with exact as well as existing numerical solutions which reflects the efficiency and versatility of the method, and ensure the flexibility of the mathematical model.

Keywords: Damping, Quadratic thickness, Non-homogeneity, Annular plate, Elastic foundation,Differential Quadrature Method

1. Introduction

Damping forces are small compared to elastic and inertia forces, however, have significant influence on vibration analysis of a system under certain special circumstance, therefore it attracts researchers constantly to refresh their thinking and re-modify their models. Either removal of energy by radiation or dissipation is the true cause of damping in any system. This is why the internal dissipation is expected and analyzed under frequency domain in many applications of vibration theory. Crandall [1] observed if the internal dissipation is due to a linear relaxation mechanism there will be a pronounced increase in loss factor when the oscillation frequency approaches the relaxation frequency. It is the prime point for predicting the damped free vibration of the actual physical system.

Analysis of plate vibration, established in 1787 with Chladni's work Leissa[2], made a way to study the effect of different aspects via mathematical modelling of plate vibrations along with consideration of different plate such as parameters such as orthotropy, variable thickness, non-homogeneity and elastic foundation, and has been continuing till today.

Due to immense use of circular and annular plates of variable thickness in aerospace industry, electronic and optical equipment's and missile technology, the study of vibrational behavior of such plates is the area of advanced research. While surveying the literature, author came across various models in account for non-homogeneity parameter [3-9] and finally assumed the variation in Young's modulus and density in distinct manner as considered earlier [10-11].Various models, approximating the supporting foundation, such as Winkler, Vlasov and Pasternak have been proposed[12-17], however, most of the studies have been carried out with Winkler model of foundation, and so as in this paper.

Present computational modeling, a simple and practical way, to calculate natural frequencies of normal modes of damped free vibrations of a non-homogeneous annular plate with arbitrary thickness variations for three different edge conditions along with elastic foundation which develops a new methodology, an ill-closed form solution, so that a designer can evaluate the natural frequencies as well as determine the damping coefficient comparing experimental and theoretical values. The author has contributed with consideration of viscous damping for rectangular plates [18-19] , circular plates[20-23], using quintic spline technique. After seeing trend of research for damped vibration analysis for different plates[24-27], this study is carried by ,using differential quadrature method as this method provides more accurate results. Authors [29-31] discused Nonlinear transient response of graphene platelets reinforced metal foams annular plate. Authors [32-40] presented vibration analysis of the related work.

The governing equation which provides the more realistic model after incorporating the damping effect is expressed as follows:

$$Dw_{rrrr}+\frac{2}{r}\left(D+rD_r\right)w_{rrr}+\frac{1}{r^2}\left\{-D+r(2+\upsilon)D_r+r^2D_{rr}\right\}w_{rr}+\frac{1}{r^3}\left\{D-rD_r+r^2D_{rr}\right\}w_r$$
$$+\rho hw_{tt}+Kw_t+K_f w=0 \tag{1}$$

Where, a is radius of annular plate, the flexural rigidity $=\frac{Eh^3}{12(1-\upsilon^2)}$, w the transverse deflection, t the time, K is damping constant and K_f is Winkler type elastic foundation constant. Under the assumption that Young's modulus and thickness h are function of space variable r only. Now equation (1) reduced to

$$Eh^3w_{rrrr}+\frac{2}{r}\left[Eh^3+r\left(h^3E_r+3Eh^2h_r\right)\right]w_{rrr}+\frac{1}{r^2}\begin{bmatrix}-Eh^3+r(2+\upsilon)\left(h^3E_r+3Eh^2h_r\right)\\+r^2\left(h^3E_{rr}+6h^2E_rh_r+3E\left(2h\left(h_r\right)^2+h^2h_{rr}\right)\right)\end{bmatrix}w_{rr}$$
$$+\frac{1}{r^3}\left[Eh^3-r\left(h^3E_r+3Eh^2h_r\right)+r^2\upsilon\left(h^3E_r+6h^2E_rh_r+3E\left(2h\left(h_r\right)^2+h^2h_{rr}\right)\right)\right]w_r$$
$$+12\rho h(1-\upsilon^2)w_{tt}+12(1-\upsilon^2)Kw_t+12(1-\upsilon^2)K_f w=0. \tag{2}$$

Further to non-dimension Alize the problem, $x=\frac{r}{a},\overline{w}=\frac{w}{a},\overline{h}=\frac{h}{a}$ are assumed. In order to make the mode more flexible, certain assumptions are made as follows:

1. A tapered annular plate with quadratic thickness variation with the expression $\overline{h}=h_0(1+\alpha x+\beta x^2)$, such that $|\alpha|\le 1,|\beta|\le 1$ and $+\beta>-1$, is considered which helps to analyze vibrations in liner, parabolic and quadratic annular plates with the same model, where h_0 is the thickness at the center of the plate, α and β are taper parameters, and variable coefficients.
2. To introduce the non-homogeneity in the material, Young's modulus and density have been assumed to vary exponentially with $E=E_0e^{\mu x}$, $\rho=\rho_0e^{\eta x}$, in radial direction differently. where, μ and η are non-homogeneity parameters, and ρ_0 is the density and E_0 is Young's Modulus of plate material at inner edge, x=ε.
3. For harmonic motion, the solution can be assumed $\overline{w}(x,t)=W(x)e^{-\gamma t}\cos p\,t$, where p is the radian frequency and γ is decay constant.

Following the above assumptions, the equation (2) reduced to

$$W_{xxxx}+2/\{(1+\mu x)+3(\alpha+2\beta x)/(1+\alpha x+\beta x^2)\}w_{xxx}$$
$$+1/x^2[-1+(2+\upsilon)x\{\mu+3(\alpha+2\beta x)/(1+\alpha x+\beta x^2)\}+x^2$$
$$\{\mu^2+(6\beta+6(\alpha+2\beta x))/(1+\alpha x+\beta x^2)+6(\alpha+2\beta x)^2/(1+\alpha x+\beta x^2)^2\}]w_{xx}+1/x^3$$
$$[(1-x\mu+3(\alpha+2\beta x)/(1+\alpha x+\beta x^2)+x^2\upsilon\{(\mu^2+((6\beta+6(\alpha+2\beta x))/(1+\alpha x+\beta x^2)+6(\alpha+2\beta x)^2/(1+\alpha x+\beta x^2)^2]wx+12(1-\upsilon$$
$$^2)/E_0h_0^3(1+\alpha x+\beta x^2)^3\{a^2\rho_0h_0(1+\alpha x+\beta x^2)e^{(\eta-\mu)x}(\gamma^2-p^2)-aK\gamma e^{-\mu x}+aK_f e^{-\mu x}\}w=0. \tag{3}$$

Where, $\gamma=\frac{K}{2\rho_0e^{\eta x}ah_0(1+\alpha x+\beta x^2)}$, since Eq. (3) must be satisfied for all values of t.

For the parametric study, equation(3) (after simplification) can be expressed as

$$S_0 \frac{d^4W}{dx^4} + S_1 \frac{d^3W}{dx^3} + S_2 \frac{d^2W}{dx^2} + S_3 \frac{dW}{dx} + S_4 W = 0, \tag{4}$$

where,

$S_0 = 1,$
$S_1 = 2(1 + Qx)/x,$
$S_2 = Q^2 + R + \{(2 + \upsilon)Q/x\} - \{1/x^2\},$
$S_3 = \{(1 - Qx)/x^3\} + \{\upsilon(Q^2 + R)/x\},$
$S_4 = -\{\Omega^2 e^{(\eta-\mu)x}/P^2\} - \{D_k/e^{(\eta-\mu)x}P^4\} + \{(12(1 - \upsilon^2)E_f/e^{\mu x}P^3 h_0^3)\},$

$$\Omega^2 = 12\rho_0 a^2 p^2 (1-\upsilon^2)/E_0 h_0^2, \; E_f = \frac{aK_f}{E_0}, \; D_k = \frac{3(1-\upsilon^2)K^2}{E_0\rho_0 h_0^4}$$

$$P = 1 + \alpha x + \beta x^2, \; Q = \mu + \{3(\alpha + 2\beta x)/P\}, \; R = 3(2\beta - \alpha^2 - 2\beta^2 x^2 - 2\alpha\beta x)/P^2,$$

Once the value of eigenvalue parameter is obtained, then angular frequency p can also be calculated using the formula

$$p = \left[\frac{\Omega^2 E_0 h_0^2 e^{(\mu-\eta)x}(1+\alpha x+\beta x^2)^2}{12a^2(1-\upsilon^2)\rho_0} - \left(\frac{K}{2ah_0\rho_0 e^{\eta}(1+\alpha x+\beta x^2)} \right)^2 \right]^{1/2},$$

and the zero value of angular frequency can be achieved for keeping damping factor as

$$K = \left[\frac{\Omega^2 E_0 h_0^4 \rho_0 e^{(\mu+\eta)x}(1+\alpha x+\beta x^2)^4}{3(1-\upsilon^2)} \right]^{1/2}.$$

In many applications, damping, where it is light, resonant motion and this effect can be expressed in terms of loss factor at resonance.
A theoretical parametric study is carried out by using DQM method [28] in order to solve equation (4) which is a fourth-order linear differential equation with variable coefficients and its exact solution is not possible.
In recent years, quadratic thickness along with such kind of non homogeneity but without considering damping effect on circular and annular plates have been solved by using several kinds of numerical schemes including DQM. Due to the versatility of the method, in this paper, DQM is used to obtain non dimensional damped natural frequencies, also accuracy and efficiency of the proposed method are demonstrated by means of numerical as well as graphical illustrations.

2. Description of the Method

Let the computational domain [ε,1] is discretized by taking m grid points, x_m in the direction of where (m-2) interval grid points, chosen for collocation, are the zeros shifted Chebyshev polynomial of order (m-2) with orthogonality range (ε,1) given by

$$x_{k+1} = \frac{1}{2}\left[(1 + \varepsilon) + (1 - \varepsilon)\cos\left(\frac{2k-1}{m-2}\frac{\pi}{2}\right)\right], k = 1,2,\ldots,(m - 2). \tag{5}$$

It is assumed that the solution $W(x, t)$ of equation (4) at the grid x_i and at the time t is $W(xi, t) = \overline{w}(xi, t), i = 1,2,3, \ldots, m.$

2.1 Computation of the Weighting Coefficients

According to differential quadrature method, derivatives of W(x) is given by

$$\frac{d^n W(x_i)}{dx^n} = \sum_{j=1}^{m} c_{ij}^{(n)} W(x_j), n = 1,2,3,4 i = 1,2,\ldots,m \tag{6}$$

Where $c_{ij}^{(n)}$ are weighting coefficients at discrete point x_i.

Following Shu[28], the weighting coefficients in equation (6) are given by

$$c_{ij}^{(n)} = \frac{M^{(1)}(x_i)}{(x_i - x_j)M^{(1)}(x_j)}, i, j = 1,2, \ldots, m; i \neq j \quad (7)$$

where

$$M^{(1)}(x_i) = \prod_{\substack{j=1 \\ j \neq i}}^{m} (x_i - x_j), \quad (8)$$

and

$$c_{ij}^{(n)} = n\left(c_{ii}^{(n-1)} c_{ij}^{(1)} - \frac{c_{ij}^{(n-1)}}{(x_i - x_j)}\right), i, j = 1,2, \ldots, m; i \neq j; n = 2,3, \quad (9)$$

$$c_{ii}^{(n)} = -\sum_{\substack{j=1 \\ j \neq i}}^{m} c_{ij}^{(n)}, i = 1,2, \ldots, m; n = 1,2,3,4 \quad (10)$$

2.2 Execution of the Method

Discretizing the equation (4) at nodes $x_i, i = 3,4, \ldots, m-2$equation (4) reduces to ,

$$P_0 \frac{d^4 W(x_i)}{dx^4} + P_{1,i} \frac{d^3 W(x_i)}{dx^3} + P_{2,i} \frac{d^2 W(x_i)}{dx^2} + P_{3,i} \frac{dW(x_i)}{dx} + P_{4,i} W(x_i) = 0, \quad (11)$$

Substituting the expressions for first four derivatives at node x_i in equation (11) using relations (6) to (10) becomes

$$\sum_{j=1}^{m} (P_0 c_{ij}^{(4)} + P_{1,i} c_{ij}^{(3)} + P_{2,i} c_{ij}^{(2)} + P_{3,i} c_{ij}^{(1)}) W(x_j) + P_{4,i} W(x_i) = 0, i = 3,4, \ldots, (m-2). \quad (12)$$

The satisfaction of equation (12) at (m-4) nodal points $x_i, i = 3,4, \ldots, (m-2)$ provides a set of (m-4) equations in terms of unknowns $W_j, j = 1,2, \ldots, m$(where$W_j$stands for $W(x_j)$),which can be written in matrix form as

$$[B][W^*] = [0], \quad (13)$$

Where B and W* are matrices of order (m-4)*m and m*1, respectively.

2.3 Boundary Conditions

Finally, for complete specification of mathematical model discussed above, appropriate boundary conditions are employed to solve system of linear equations (13).

The following three cases of boundary conditions are considered by satisfying the relations, along with Eq.(13) which leads to three different cases for different boundaries:

Table 1. Different system of linear equations for diffrent boundary Conditions.

Boundary Condition	Mathematical expression	Corresponding different sets of equations
Clamped at both the edges	$W = \frac{dW}{dX} = 0$	$\begin{bmatrix} B \\ B^c \end{bmatrix} \{W^*\} = \{0\}$
Clamped at inner edge and simply supported at outer edge	$W = \frac{d^2W}{dX^2} + \frac{\upsilon}{X} \frac{dW}{dX} = 0$	$\begin{bmatrix} B \\ B^{ss} \end{bmatrix} \{W^*\} = \{0\}$
clamped at inner edge and free outer edge	$W = \frac{d^3W}{dX^3} + \frac{1}{X} \frac{d^2W}{dX^2} - \frac{1}{X^2} \frac{dW}{dX} = 0$	$\begin{bmatrix} B \\ B^f \end{bmatrix} \{W^*\} = \{0\}$

Finally, these systems (Table 1) can be solved using computational skills on different programming platforms. In this study a mathematical programme is developed to validate the mathematical model.

3. Results and Discussion

This section explaines the results of research as well as the comprehensive discussion on convergence and stability of used method and developed model, comparitive study with existing results. Results are presented in figures, graphs and tables. The complete discussion is being made in three sub-sections.

3.1. Convergence and Stability Analysis of the method

Stability of the method of solution depends upon the obtained mathematical model up to a great extent. However, if the differential equation is unstable, the stability of the system completely relies on the roots of the characteristic equation of "coefficient matrix" B, as the solutions can directly be obtained by using these eigenvalues. Thus, to prove the convergence and the stability of the solution up to four decimal places, a computer program was run to get appropriate values of non-dimensional frequency parameter. Figure 1 demonstrates that percentage error in the numerical values of non-dimensional frequency Ω up to fourth decimal places with the increase in the number of grid points which decides the suitable number for internal grid points for the domain. During numerical computation, it is found that the convergence of results depends on the value of n and becomes stable which leads to reliable results. A comparative study for evaluation of non-dimensional frequency Ω for a uniform homogeneous annular plate is demonstrated table 2.

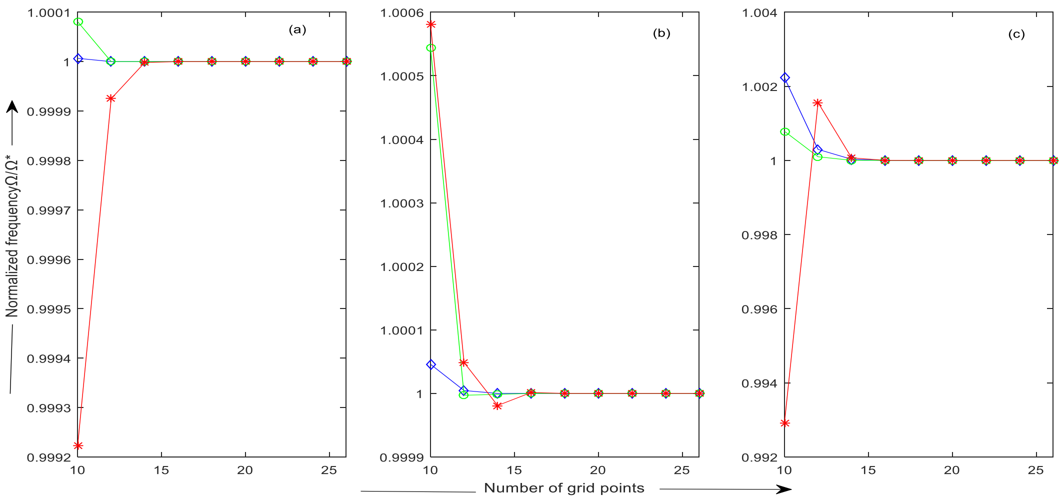

Figure 1. Stability analysis: Convergence of the normalized frequency parameter Ω/ Ω* when (a) C-C Plate (b) C-SS Plate (c) C-F Plate vibrates in ◊first mode, ○ second mode and *third mode for υ=0.3,α=-0.5,β=0.0,η=-0.5,μ=-0.5,Dk =0.0, Ef =0.0, and Ω* -obtained using 26 grid points.

Table 2 Comparison of dimensionless frequency parameter keeping:μ=0.0, η=0.0, α=0.0, β=0.0 and ε=0.3.

	Poisson Ratio υ=0.3						Poisson Ratio υ=0.33		
	C-C			C-S			C-F		
	Mode I	Mode II	Mode III	Mode I	Mode II	Mode III	Mode I	Mode II	Mode III
Using DQM	45.3462	125.3621	246.1573	29.9777	100.4228	211.1294	6.6604	42.6142	123.4661
Using Quintic Spline	45.3461	125.3631	246.1626	29.9776	100.4235	211.1336	6.6603	42.6142	123.4671
Exact Values taken by Leissa	45.2	125		29.9	100				

3.2. Stability Analysis of the model

In order to prove the stability of the obtained ordinary differential equation (mathematical model), it is adequate to get non-positive values of oscillatory component in the assumed solution, or in other words $Value\ of(-\gamma) < 0$ at each node. A particular case of aluminum material has been considered to show the authenticity/stability of the mathematical model (Figure 2-5). In the present example the material properties E=70 GPa and ρ =2,702 kg/m3 considered to illustrate the deflection of plates with different boundary conditions while vibrating in first two modes, with or without considering the damping.

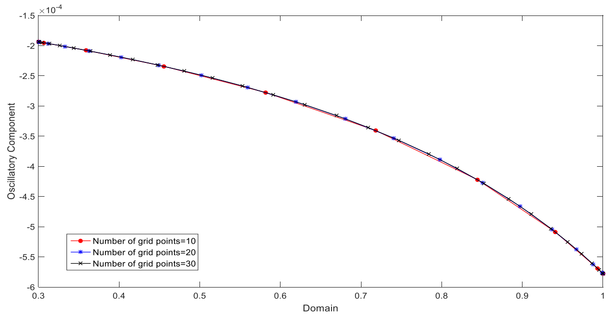

Figure 2. Stability analysis: Non-positive values of oscillatory component (decay constant) corresponding to different grid sizes.

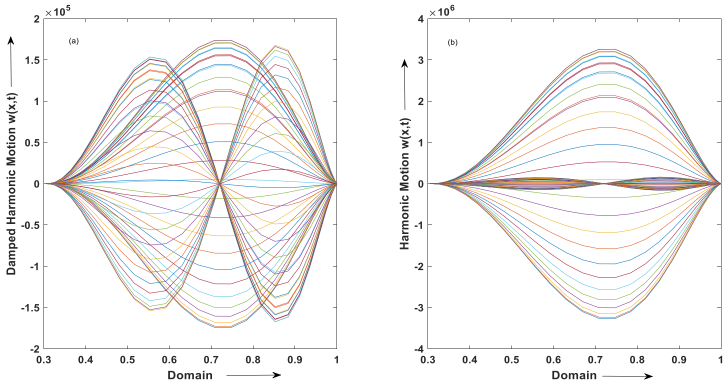

Figure 3. Deflection under first two modes for C-C aluminum material (a) with damping parameter (b) without damping parameter.

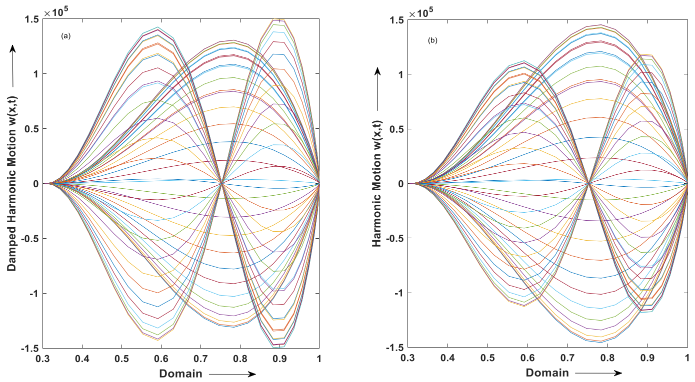

Figure 4. Deflection under first two modes for C-SS aluminum material (a) with damping parameter (b) without damping parameter.

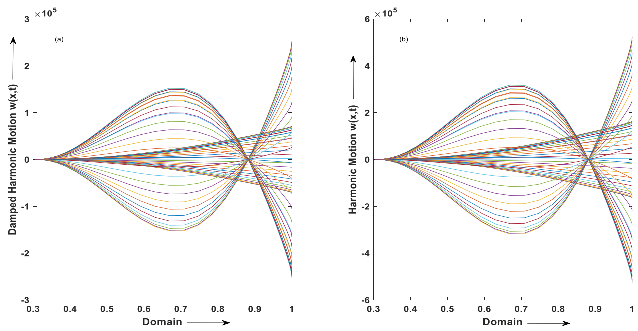

Figure 5. Deflection under first two modes for C-F aluminum material (a) with damping parameter (b) without damping parameter.

3.3. Parametric Study

After analyzing the stability and accuracy of the scheme, primarily the effect of damping parameter over the natural frequencies for the first three modes is observed as well as compared, since this factor is introduced and brings novelty using another advanced method in the paper. Table 3 demonstrates the comparison of numerical results with those obtained by another method called Quintic Spline Technique [17] which shows that present results are comparatively computationally cheaper and better. Simultaneously Table 3 also depicts that the parametric values of non-dimensional frequency are decreasing with the increasing value of damping parameter. Same has been depicted through figure 7. Just to show the flexibility of the model a few numerical studies are demonstrated with respect to parabolic variation, as this model provides the liberty to choose linear, parabolic as well as quadratic variation in thickness. From figure 7 and 8, increasing behavior of Ω has been observed under the effect of elastic foundation for first three modes, with or without damping effect, respectively. After analyzing the normalized displacement behavior, from figure 9, values of Ω for Uniform plate is also analyzed , it shows that frequencies for uniform plates are smaller than the parabolic plates with thinner outer edge and greater than the parabolic plate with thicker edge irrespective of the value of other plate parameters.

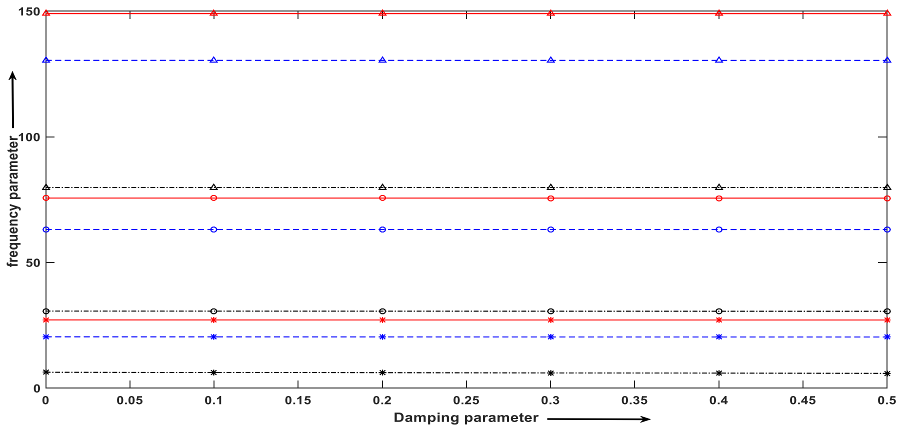

Figure 6. Effect of damping parameter: C-C Plate, ▬;C-SS Plate,---;C-F Plate, -.-.; vibrate in *, I-mode; ○, II-mode; and Δ, III- mode for υ=0.3,α=-0.5,β=-0.1,η=-0.5,μ=-0.5, Ef =0.0, and Ω* -obtained using 26 grid points.

Table 3. Comparison of dimension less Ω with the variation of damping parameter D_k for all three boundary conditions, keeping μ=-0.5, η=-0.5, α=-0.5, β=-0.1, and ε=0.3.

		Mode	Dk=0	Dk=0.1	Dk=0.2	Dk=0.3	Dk=0.4	Dk=0.5
Clamped-Clamped edge Condition	*Using QST at N=320*	I	27.1716	27.1705	27.1675	27.1623	27.1551	27.1459
		II	75.6216	75.6212	75.62	75.618	75.6152	75.6116
		III	149.02	149.02	149.019	149.018	149.017	149.015
	Using DQM at N=30	I	27.1718	27.1616	27.1513	27.141	27.1308	27.1205
		II	75.6216	75.6176	75.6136	75.6096	75.6056	75.6016
		III	149.018	149.016	149.013	149.011	149.009	149.007
Clamped-Simply Supported edge Conditions	*Using QST at N=320*	I	20.4357	20.4341	20.4294	20.4216	20.4107	20.3967
		II	63.1054	63.1049	63.1033	63.1008	63.0972	63.0926
		III	130.445	130.445	130.444	130.443	130.441	130.439
	Using DQM at N=30	I	20.4359	20.4203	20.4047	20.3891	20.3734	20.3578
		II	63.1054	63.1002	63.0951	63.09	63.0848	63.0797

		III	130.443	130.44	130.438	130.435	130.433	130.43
Clamped-Free edge Conditions	*Using QST at N=320*	I	6.4782	6.47	6.4455	6.4044	6.3464	6.2711
		II	30.723	30.7217	30.7178	30.7114	30.7024	30.6909
		III	79.874	79.8735	79.8722	79.8699	79.8668	79.8628
	Using DQM at N=30	I	6.3271	6.2429	6.1575	6.0709	5.9831	5.8939
		II	30.6597	30.6469	30.634	30.6212	30.6083	30.5955
		III	79.838	79.8335	79.829	79.8245	79.82	79.8155

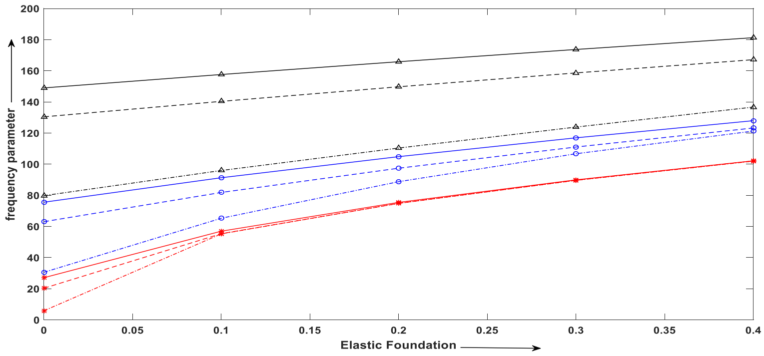

Figure 7. Effect of Winkler type foundation: C-C Plate, ▬;C-SS Plate,---;C-F Plate, -.-.; vibrate in *, I-mode; ○, II-mode; and △, III-mode for υ=0.3,α=-0.5,β=-0.1,η=-0.5,μ=-0.5, Dk =0.5, and Ω* -obtained using 26 grid points.

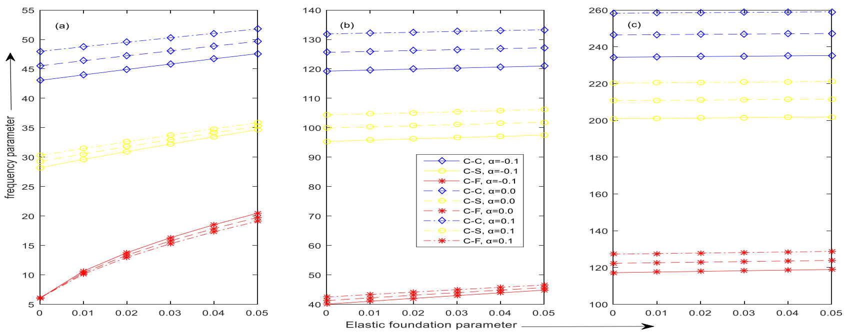

Figure 8. Parabolic plates vibrating under (a) I-mode (b) II- mode (c) III- mode keeping elastic foundation and non-homogeneity parameter under consideration for:μ=0.5, η =0.5, with edge conditions: "◊ C-C plate ; ○ C-S plate; *C-F edge;" for different value of taper constant :"—, α =-0.1; ---,α =0.0.;-.- α =0.1" .

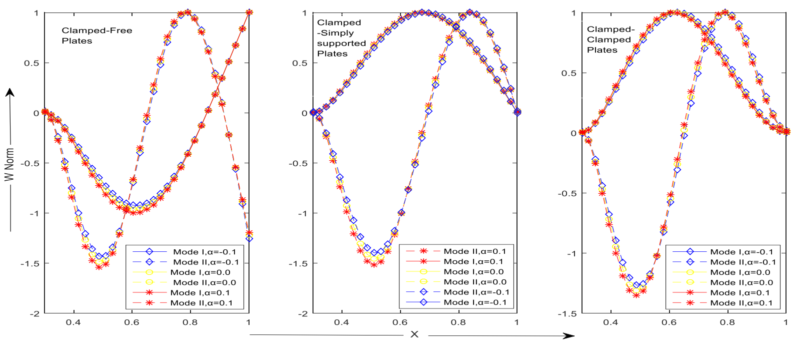

Figure 9. Normalized displacement for parabolic C-F, C-S, C-C Plates, for non-homogeneity parameter, μ=0.5, density parameter, η=0.5, for "—, first mode;——,Second mode, .-.-.,Third mode;" with different value of taper constant : " ◊,, α=-0.1; o, α=0.0; *, α=0.1.

4. Conclusion

Computational model for natural frequencies of a non-homogeneous annular plate with varying thickness under the effect of Winkler type elastic is modified by considering the effect of damping and then solved by differential quadrature method, which produces highly accurate as well as reasonably stable results. Study of damping parameter encourages us to incorporate this factor in the study of vibration analysis of more complex structure in terms of material as well as physical properties. Hence the present study will be a benchmark to evaluate more accurate results even for highly complex combination of different parameters of vibrating plates of different types.

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