Tuijin Jishu/Journal of Propulsion Technology
ISSN: 1001-4055
Vol. 45 No. 2 (2024)

Characterization of Spider Graphs with Pair Sum Modulo Labeling

P.Amudha [1], J.Jayapriya [2]

[1] *Research Scholar, Department of Mathematics, Sathyabama Institute of Science and Technology, Chennai, Tamil Nadu, India*

[2] *Department of Mathematics, Sathyabama Institute of Science and Technology, Chennai, Tamil Nadu, India*

Abstract:- Labeling a graph involves assigning numbers to its vertices, edges, or both, depending on certain requirements. Numerous applications have made use of graph labeling. A mapping $\mathbf{f}:\mathbf{V}(\mathbf{G}) \rightarrow \{\pm\mathbf{1},\pm\mathbf{2},\pm\mathbf{3},\ldots,\pm|\tilde{V}|\}$ for a simple graph G‹(V,E) is referred to as a pair sum modulo (PSM) labeling if [f(u)+f(v)](mod |V|) is distinct for each edge uv. A PSM graph is a graph that allows PSM labelling. This article demonstrates there exists PSM labelling of spider graph with k legs of varying lengths (1,2,.,6).

Keywords*:* Labeling of graphs, Pair sum modulo labeling, Spider graph.

1. Introduction

In the ever-evolving landscape of science and technology, graph theory stands as a pivotal area of study. Its extensive applications span various disciplines, including computer science, chemistry, coding theory, biology, sociology, operations research, communication networks and algorithm design. One of the key contributions to this field was made by Rosa [3] in 1967 with the introduction of graph labeling, a concept that has since found numerous practical applications.

Graph labeling involves assigning numbers to the points, lines, or both, in a graph in accordance with specific rules. Over the past five decades, this intriguing area has inspired over 200 graph labeling methods, documented in more than 3000 research articles. Gallian [2] has been instrumental in keeping the academic community updated with his comprehensive surveys on the latest labeling methods.

Several notable researchers have contributed significantly to the development of graph labeling theories. Ponraj.R [7,8], for instance, has explored pair sum labeling for various graphs. Jayanthi.P [4] focused on edge pair sum labeling in some commonly used graphs, while A. Panpa [6] investigated the application of graceful labeling to spider graphs. More recently, Amudha P. and Jayapriya J. [1] introduced the concept of pair sum modulo (PSM) labeling.

A mapping $\varphi: V_g(G) \rightarrow \{\pm 1, \pm 2, \pm 3, \ldots, \pm|V_g|\}$ for a simple graph G(V_g,E_g) is referred to as a pair sum modulo (PSM) labeling if [φ(x)+ φ(y)](mod |V_g|) is distinct for each edge xy. A PSM graph is a graph that allows PSM labeling.

In this study, we focus exclusively on simple and finite graphs. Our objective is to demonstrate how spider graphs with L legs, each having size 1, 2, 3, 4, 5, and 6, satisfy PSM labeling. In this paper, a spider graph with L>0 legs of same length z is denoted by $S(L,z)$. Through this exploration, we aim to contribute to the broader understanding of graph labeling techniques and their applications.

2. Preliminaries

Definition 2.1

Spider graph S(L,z) is tree with one central vertex of degree >2 called a branch vertex and L legs, where each leg is a path connecting the branch and leaf.

Definition 2.2

The conditional job of designating numbers to vertices, edges, or both in a graph is called graph labeling.

Definition 2.3

A map $\varphi : V_g(G) \rightarrow \{\pm1, \pm2, \pm3, \dots, \pm|V_g|\}$ for a simple graph $G(V_g,E_g)$ is referred to as a pair sum modulo (PSM) labeling if $[\varphi(u_1)+ \varphi(u_2)](\text{mod } |V_g|)$ is distinct for each edge u_1u_2. A PSM graph is a graph that allows PSM labeling.

3. Main Results

Theorem 3.1. *PSM labeling can be achieved on a spider graph $S(L,z)$ with L>0, z=1.*

Proof. Let $G(V_g, E_g)$ can be thought of as a spider graph S(L,z), where z=1.

It is evident that $|V_g(G)|=L+1$ and $|E_g(G)|=L$. $V_g(G) = \{a_i ; 1 \leq i \leq L+1\}$ and

$E_g(G) = \{a_1a_i ; 2 \leq i \leq L+1\}$

Define $\varphi : V_g \rightarrow \{\pm1, \pm2, \pm3, \dots, \pm(L+1)\}$ by $\varphi(u_i) = \begin{Bmatrix} -1, i = 1 \\ i\text{-}1, 2 \leq i \leq L+1 \end{Bmatrix}$

The unique labels for each edge are found by using PSM labelling.

Hence $S(L, 1)$ is a PSM graph.

Theorem 3.2. *On S(L,2), PSM labeling is feasible.*

Proof. Let $G(V_g,E_g)$ denotes the spider graph S(L,2)

It can be observed that $|V_g| = 2L + 1$ and $|E_g| = 2L$.

Let $V_g = \{v_i ; 1 \leq i \leq 2L+1\}$ and

$E_g = \{v_1v_i ; 2 \leq i \leq L+1\} \cup \{v_{L+1+i}\, v_{i+1} ; 1 \leq i \leq L\}$

Define

$\varphi: V_g \rightarrow \{\pm1, \pm2, \pm3, \dots, \pm(2L+1)\}$ as follows:

(i): In the event that L is odd. Assuming L=2n+1, n=0,1,2,3,…

$\varphi(u_1) = 2L;$

$\varphi(v_i) = i - 1, 2 \leq i \leq L+1;$

$\varphi(v_i) = i - 1, L + 2 \leq i \leq \frac{3L+1}{2};$

$\varphi(v_i) = i - 6n - 2, \frac{3L+3}{2} \leq i \leq 2L;$

$\varphi(v_{2L+1}) = 2L + 1;$

(ii): If L is even, let L=2n, n=1,2,3,….

$\varphi(u_1) = 2L;$

$\varphi(v_i) = i - 1, 2 \leq i \leq L+1;$

$\varphi(v_i) = i - 1, L + 2 \leq i \leq \frac{3L+2}{2};$

$\varphi(v_i) = i - 6n, \frac{3L+4}{2} \leq i \leq 2L;$

Tuijin Jishu/Journal of Propulsion Technology
ISSN: 1001-4055
Vol. 45 No. 2 (2024)

$\varphi(v_{2L+1}) = 2L + 1;$

Through the PSM labeling method, we can assign a distinct integer $(\varphi(u_1)+\varphi(u_2))(\text{mod } 2L)$ for every edge u_1u_2.

Therefore, the spider graph S(L,2) can be classified as a PSM graph.

Theorem 3.3. *The spider graph $S(L,3)$ is a PSM graph for all L.*

Proof. Let $G(V_g,E_g)$ denotes the spider graph $S(L,3)$.

Clearly, $|V_g| = 3L + 1$, $|E_g| = 3L$

Let $V_g = \{v_i\ ;\ 1 \le i \le 3L+1\}$ and

$E_g = \{v_1v_i\ ;\ 2 \le i \le L+1\} \cup \{v_iv_{L+i}\ ;\ 2 \le i \le L+1\}$

$\cup\{v_iv_{i+L}\ ;\ L+2 \le i \le 2L+1\}$

Introduce a function $\varphi: V_g \to \{\pm 1, \pm 2, \pm 3, \ldots, \pm(3L+1)\}$ as follows:

$\varphi(u_1) = 3L;$

$\varphi(u_i) = i - 1, 2 \le i \le L+1\ ;$

$\varphi(v_i) = -2i + 2L + 2, L + 2 \le i \le 2L + 1;$

$\varphi(v_{3L+1}) = 3L + 1;$

$\varphi(v_{2L+2}) = -3L + 2;$

$\varphi(u_i) = i, 2L + 3 \le i \le 3L - 1;$

$\varphi(u_{3L}) = -3L\ ;$

By applying PSM labeling, unique labels for each edge are calculated, confirming that $S(L,3)$ is a PSM graph.

Illustration 3.1. PSM labeling of S(L,z) with L=5, z=3 is depicted in figure-1.

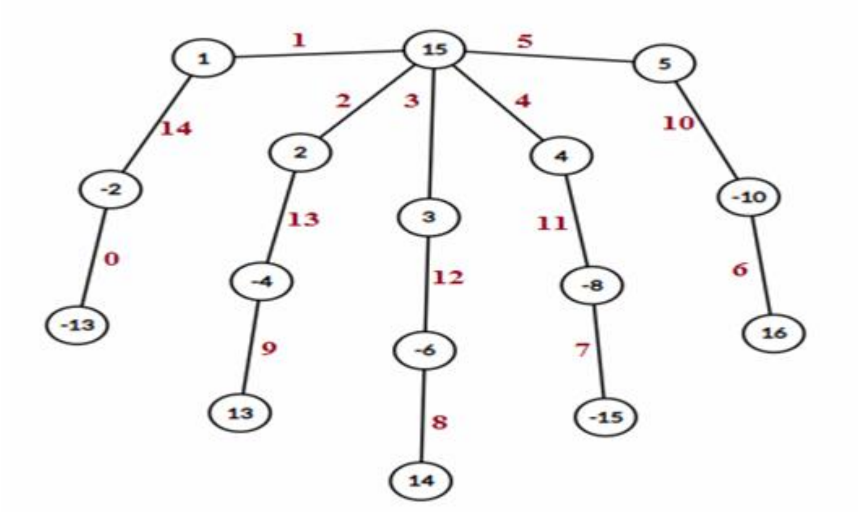

Fig. 1. PSM labeling of S(5,3)

Theorem 3.4. *The spider graph $S(L,4)$ is a PSM graph.*

Proof. Let $G(V_g,E_g)$ denotes $S(L,4)$.

Clearly, $|V_g| = 4L + 1$ and $|E_g| = 4L$.

Take $V_g = \{v_i\ ;\ 1 \le i \le 4L+1\}$ and

$E_g = \{v_1v_i\,; 2 \le i \le L{+}1\} \cup \{v_iv_{L+i}\,; 2 \le i \le L{+}1\} \cup \{v_iv_{i+L}; L + 2 \le i \le 2L{+}1\} \cup \{v_iv_{i+L}; 2L + 2 \le i \le 3L{+}1\}$

To establish a function $\varphi: V_g \to \{\pm 1, \pm 2, \ldots \pm (4L + 1)\}$ the following scenarios can be taken into account:

(i): When L is odd

If L=1

$\varphi(u_1) = -1\,; \varphi(u) = 5; \varphi(v_i) = i\text{-}2, 3 \le i \le 5$.

If L>1,

$\varphi(u_1) = 4L;$

$\varphi(u_i) = i - 1, 2 \le i \le \frac{5L+1}{2};$

$\varphi(v_i) = i{+}L{-}1, \frac{5L+3}{2} \le i \le 3L;$

$\varphi(v_{3L+1}) = \text{-}4L\,;$

$\varphi(v_{4L+1}) = 3L + 1;$

$\varphi(v_i) = -(i\text{-}6L), 3L + 2 \le i \le \frac{7L+1}{2};$

$\varphi(v_i) = i\text{-}5L\,; \frac{7L+3}{2} \le i \le 4L\,;$

(ii) When L is even

If L = 2,

$$\varphi(u_1) = 4L; \varphi(v_i) = i - 1, 2 \le i \le L{+}1;$$

$\varphi(u_i) = i - 1, L + 2 \le i \le 2L{+}1\,;$

$\varphi(v_{4L+1}) = -2;$

$\varphi(v_i) = i - 1, 2L + 2 \le i \le \frac{5L+2}{2}\,;$

$\varphi(v_i) = i{+}L{-}2, \frac{5L+4}{2} \le i \le 3L + 1;$

$\varphi(v_{3L+2}) = \text{-}6;$

If L=4,

$\varphi(u_1) = 4k; \varphi(v_i) = i - 1, 2 \le i \le L{+}1;$

$\varphi(v_i) = i - 1, L + 2 \le i \le 2L{+}1;$

$\varphi(v_i) = i - 1, 2L + 2 \le i \le \frac{5L+2}{2}\,;$

$\varphi(v_i) = i{+}L{-}2, \frac{5L+4}{2} \le i \le 3L + 1;$

$\varphi(v_{3L+2}) = \text{-}4L;$

$\varphi(u_i) = 4\text{-}i, 3L + 3 \le i \le \frac{7L+2}{2};$

$\varphi(v_i) = i\text{-}L + 1; \frac{7L+4}{2} \le i \le 4L;$

$\varphi(v_{4L+1}) = -2.$

The PSM labeling of S(4,4) is shown in figure 2.

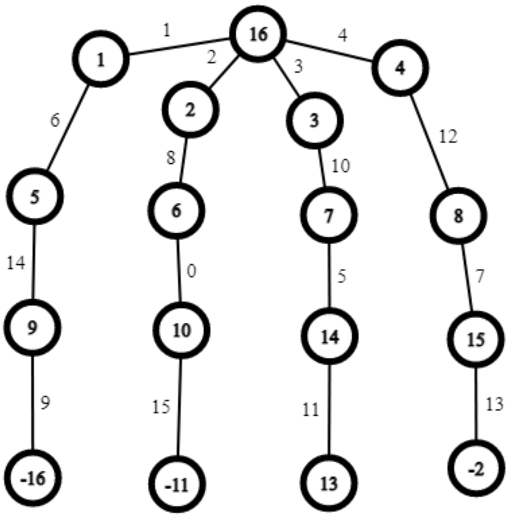

Fig. 2. PSM labeling S(4,4)

If L>4,

$\varphi(v_1) = 4L;$

$\varphi(v_i)$=$i - 1, 2 \leq i \leq L+1;$

$\varphi(v_i)$=$i - 1, L + 2 \leq i \leq 2L+1;$

$\varphi(v_i)$=$i - 1, 2L + 2 \leq i \leq \dfrac{5L+2}{2};$

$\varphi(v_i)$=$i+L-2, \dfrac{5L+4}{2} \leq i \leq 3L + 1;$

$\varphi(v_{3L+2}) = -4L;$

$\varphi(v_i) = -i+4, 3L + 3 \leq i \leq \dfrac{7L+2}{2};$

$\varphi(v_i) = i-L + 1 ; \dfrac{7L+4}{2} \leq i \leq 4L + 1;$

By applying PSM labeling, it has been found that the edge labels for all the cases mentioned above are distinct.

Therefore, S(L,4) is a PSM graph.

Theorem 3.5. *PSM labeling can be achieved on S(L,5).*

Proof. Let $G(V_g,E_g)$ denotes S(L,5).

It is evident that $|V_g| = 5L + 1$ and $|E_g| = 5L$.

Let

$V_g = \{u_i ; 1 \leq i \leq 5L + 1\}$ and

$$E_g = \{u_1u_i ; 2 \leq i \leq L+1\} \cup \{u_iu_{L+i}; 2 \leq i \leq L+1\} \cup \{u_iu_{i+L}; L + 2 \leq i \leq 2L+1\}$$
$$\cup \{u_iu_{i+L}; 2L + 2 \leq i \leq 3L+1\} \cup \{u_iu_{i+L} ; 3L + 2 \leq i \leq 4L+1\}$$

Create a map

$\varphi : V_g \to \{\pm 1, \pm 2, .., \pm (5L + 1)\}$ by

for L=1, $\varphi(u_1) = 5$; $\varphi(v_i)=i - 1, 2 \le i \le 4$; $\varphi(u_5) = -1$;

$\varphi(u_6) = -5$;

By utilizing PSM labeling, the separate labels 0,1,2,3,4 for all the edges are determined.

If L>1,

$$\varphi(u_1) = -5L;\ \varphi(v_i)=i - 1, 2 \le i \le L+1;$$

$\varphi(v_i) = i-1, L + 2 \le i \le 3L + 1$;

$\varphi(v_i) = i-4L-2, 3L + 2 \le i \le 4L + 1$;

$\varphi(v_i)=i+1, 4L + 2 \le i \le 5L$;

$\varphi(v_{5L+1}) = 3L + 2$;

By applying the PSM labeling definition, unique labels for all lines are identified.

Thus, the spider graph S(L,5) is a PSM graph.

Illustration 3.2

PSM labeling of S(5,5) is depicted in figure-3.

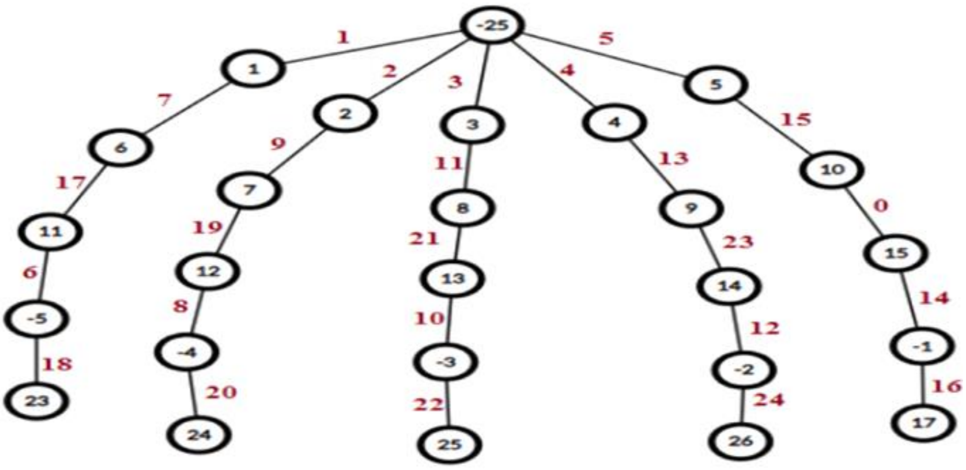

Fig. 3. PSM labeling of S(5,5)

Theorem 3.6. *The spider graph S(L,6) is a PSM graph.*

Proof. Let $G(V_g,E_g)$ denotes S(L,6).

Then $|V_g| = 6L + 1$ and $|E_g| = 6L$.

Let $V_g = \{v_i\,; 0 \le i \le 6L\}$ and

$$E_g = \{v_0v_i; 1 \le i \le L\} \bigcup_{j=0}^{5} \{v_{Lj+i}v_{(j+1)L+i}; 1 \le i \le L\}.$$

Define a vertex labeling function

$\varphi: V_g \to \{\pm 1, \pm 2, .., \pm (6L + 1)\}$ by

Case(i): If L is odd,

$\varphi(v_0) = 6L$;

$\varphi(v_i) = i, 1 \leq i \leq \frac{7L-1}{2};$

$\varphi(v_i) = i + L, \frac{7L+1}{2} \leq i \leq \frac{9L-1}{2};$

$\varphi(v_j) = j - 6L, \frac{9L+1}{2} \leq j \leq \frac{11L-1}{2};$

$\varphi(v_i) = i - 11L, \frac{11L+1}{2} \leq i \leq 6L;$

Case(ii): When L is even,

$\varphi(v_0) = 6L;$

$\varphi(u_i) = i, 1 \leq i \leq \frac{7L}{2};$

$\varphi(u_i) = i + L\ \frac{7L+2}{2} \leq i \leq \frac{9L}{2};$

$\varphi(u_i) = i - 6L, \frac{9L+2}{2} \leq i \leq \frac{11L}{2};$

$\varphi(u_i) = i - 11L, \frac{11L+2}{2} \leq i \leq 6L;$

By applying the definition of PSM labeling, it is observed that the edge labels are distinct, confirming that a spider graph S(L,6) is a PSM graph.

Illustration 3.3. The illustration of PSM labeling of S(5,6) is depicted in the figure-4,

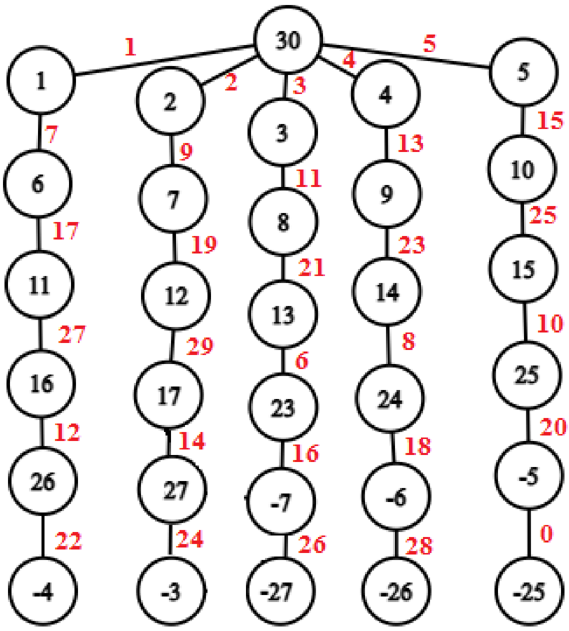

Fig. 4. PSM labeling of S(L,z) with L=5, z=6

4. Conclusion

In this paper, we have explored the pair sum modulo labeling of spider graphs with L legs of varying lengths. Through rigorous analysis and proof, we have demonstrated that spider graphs with L>0 legs, where each leg can be of size 1, 2, 3, 4, 5, or 6, indeed satisfy the pair sum modulo labeling property. The methods employed to prove the PSM labeling of these spider graphs can be adapted and extended to study other graph structures. The systematic approach to labeling vertices and ensuring unique pair sums provides a template for future research in graph theory. Our results add to the growing body of knowledge in graph labeling, particularly in the context of spider graphs. Understanding the labeling properties of these graphs can have implications for network design, information dissemination, and other applications where graph structures are utilized. Future work can extend these findings by exploring pair sum modulo labeling for spider graphs with legs of sizes beyond those considered in this paper. Additionally, investigating other variations and generalizations of spider graphs could reveal new insights and further applications of pair sum modulo labeling in different domains.

In conclusion, this study confirms that spider graphs with $L>0$ legs of sizes 1 through 6 satisfy the pair sum modulo labeling property. This contributes a meaningful advancement to the field of graph theory, offering new avenues for research and practical applications in mathematical and computational contexts.

References

[1] Amudha.P and Jayapriya.J, “Pair Sum Modulo Labeling of Graph”, Advances in Mathematics: Scientific Journal, vol. 10, no.2, pp. 723–727, 2021.
[2] Gallian.J.A, “A dynamic survey of graph labeling”, Electronic Journal of Combinatorics, 2019, DS6.
[3] Harary.F, “Graph Theory”, Narosa Publishing House, New-Delhi, 1998.
[4] Jeyanthi.J and Saratha Devi.T, “Edge Pair Sum Labeling”, Journal of Scientific Research, Vol.5 (3), pp. 457-467, 2013.
[5] Malathi.S, “Vertex labeling of spider graph”, Advances in Mathematics: Scientific Journal, vol. 8, no.3, pp. 580–585, 2019.
[6] A. Panpa, T. Poomsa-ard, "On Graceful Spider Graphs with at Most Four Legs of Lengths Greater than One", Journal of Applied Mathematics, vol. 2016, Article ID 5327026, 5 pages, 2016. https://doi.org/10.1155/2016/5327026
[7] Ponraj.R, Vijaya Xavier Parthipan.J, “Pair sum labeling of graphs”, The Journal of Indian Academy of Mathematics, Vol. 32, No.2, pp. 587-595, 2010.