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# A Study on $W_1$- Curvature Tensor in $LP$-Kenmotsu Manifolds

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***Abstract:*** The objective of the present paper is to study the curvature properties of Lorentzian - Para Kenmotsu manifolds (abbreviated as "$\mathrm{LP}-$ Kenmotsu manifolds") satisfying the conditions of $\mathrm{W}_1$-flatness, $\xi-\mathrm{W}_1$ $-$flatness, $\mathrm{W}_1 \cdot \mathrm{Q}=0$, $\phi-\mathrm{W}_1-$ semisymmetric, $\mathrm{W}_1(\phi\mathrm{X}, \phi\mathrm{Y}, \phi\mathrm{Z}, \phi\mathrm{W})=0$ conditions and found some interesting results.

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***Keyword and Phrases:*** Almost contact manifolds, $\epsilon$- Kenmotsu manifolds, LP-Kenmotsu manifolds, $\phi$-symmmetric, $\phi$- semisymmetric, $\phi$ $-$semisymmetric, $\mathrm{W}_1$-curvature tensor, Einstein manifold, $\eta$ $-$Einstein manifold.

## 1. Introduction

The concept of Lorentzian paracontact, specifically Lorentzian para-Sasakian ($LP$ $-$Sasakian) manifolds, was first presented by K. Matsumoto [7] in 1989. Subsequently, numerous geometers, including Matsumoto and Mihai [8], Mihai and Rosca [6], Mihai, Shaikh and De [5], Venkatesha, Pradeep Kumar, and Bagewadi [15], Venkatesha, and Bagewadi [16,17], and obtained several outcomes from these manifolds. F. Ozen Zengin studied the nature of $LP-$Sasakian manifolds admitting the $M-$ projective curvature tensor and examined whether this manifold satisfies the condition $W(X,Y)\cdot R=0$. Moreover, he proved that in the $M-$ projective curvature tensor and examined whether this manifold satisfies the condition $W(X,Y)\cdot R=0$.

Moreover, he proved that in the $M$ $-$projectively flat $LP-$ Sasakian manifolds, the conditions $R\cdot R=0$ and $R(X,Y)\cdot S=0$ are satisfied and then he introduced the concept of $M$ $-$projectively flat space-time. A class of Vietually paracontact metric manifolds, called para-Kenmotsu (abbreviated P-Kenmotsu) and special para-Kenmotsu (abbreviated SP-Kenmotsu) manifolds was developed by Sinha and Sai Prasad [2] in 1995. These manifolds are comparable to P-Sasakian and SP-Sasakian manifolds. In 2018, Abdul Haseeb and Rajendra Prasad conducted research on $\phi$-semisymmetric $LP$ – Kenmotsu manifolds with a quarter-symmetric non-metric connection admitting Ricci solitons [13]. They also defined a class of Lorentzian almost paracontact metric manifolds, called Lorentzian para-Kenmotsu (abbreviated $LP$-Kenmotsu) manifolds [1]. Pokhariyal [3] explored these tensor fields' properties on a Sasakian manifolds in more detail. These notations were expanded to nearly para-contact structures by Matsumoto, Ianus, and Mihai in 1986. They also analyzed para-Sasakian manifolds that admitted these tensor fields [9], with De and Sarkar generalizing their results in 2009 [14].

Subsequently, Rajan, Gyanvendra Pratap Singh, Pawan Prajapati, Anand Kumar Mishra [20] studies $W_8$-curvature Tensor in Lorentzian $\alpha$ $-$Sasakian Manifold in 2020. The concept of $(\epsilon)$ $-$Kenmotsu manifold admitind $W_8$- Curvature tensor was studied by G.P. Singh, A.K. Mishra, Rajan, P. Prajapati, and A.P. Tiwari [25] in 2022. A Friedmann and J.A. Schouten [11] introduced the concept of semi-symmetric linear connection on a differentiable manifold in 1924. H.A. Hayden [13] first described and researched semi-symmetric metric connection in 1932. G.P. Singh, Rajan, A.K. Mishra, P. Prajapati [26] have studies the curvature properties of $W_8$- Curvature tensor in generalized Sasakian-space-forms 2023. The semi-symmetric metric connection in a Riemannian manifold was the subject of a symmetric study initiated by K. Yano [24] in 1970, which was later explained upon by a number of authers including S. Ahmad and S.I Hussain [29], M.M. Tripathi [22], C.OZgur et al. [18] and many others.

If $\nabla$ is assumed to be a linear connection and $M$ be an $n$ −dimensional differentiable manifold then the curvature tensor $R$ and torsion tensor $T$ of $\nabla$ are given by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

If Torsion tensor $T$ vanishes i.e, if $T = 0$, then the connection $\nabla$ is called to be symmetric else it is non-symmetric. The connection $\nabla$ is said to be metric connection if there exist a Riemannian metric $g$ in M such that $\nabla_g = 0$, otherwise it is non-metric. We know very well that the Levi-Civita connection is defined as;

A linear connection is Levi- Civita if it is symmetric as well as metric.

If torsion tensor $T$ of a linear connection $\nabla$ is of the form

$$T(X,Y) = \eta(Y)X - \eta(X)Y,$$

Then $\nabla$ is called semi-symmetric connection; where $\eta$ is 1- form.

The semi-symmetric metric connections are very cruicial in the study of Riemannian manifolds. The semi-symmetric metric connection is associated with a variety of physical issues. For instance, if a man moves over the surface of the earth always facing a specific location, such as, Jerusalem, Mekka, or the North pole, so this displacement is semi-symmetric and metric.

The paper is structured as follows in response to the studies mentioned above. We provide a brief overview of a $LP-$ Kenmotsu manifold and its features. We locate the $W_1$-flatness in $LP$ −Kenmotsu manifold in section 3. The analysis of the $\xi - W_1$ −flatness in $LP$ −Kenmotsu manifold is covered in section 4. We discover the $LP$ −Kenmotsu manifold satisfying the condition $W_1.Q = 0$ in section 5. $\phi - W_1$ semisymmetric $LP$ −Kenmotsu manifold in analyzed in section 6. A special condition i.e, $W_1(\phi X, \phi Y, \phi Z, \phi W) = 0$ condition which is given in the end of the section and present some interesting findings.

## 2. Preliminaries

An $n$ −dimensional differentiable manifold $M$ admitting a (1, 1) tensor field $\phi$, contravariant vector field $\xi$, a 1-form $\eta$ and the Lorentzian metric $g(X,Y)$ satisfying

$$\phi^2 X = X + \eta(X)\xi, \tag{2.1}$$

$$\eta(\xi) = -1, \tag{2.2}$$

$$g(\xi,\xi) = -1, \tag{2.3}$$

$$\eta(X) = g(X,\xi), \tag{2.4}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y) \tag{2.5}$$

For any vector fields $X, Y$ on $M$, then it is called Lorentzian almost paracontact manifold. In the Lorentzian paracontact manifold, the following relation holds:

$$\phi\xi = 0, \qquad \eta(\phi X) \tag{2.6}$$

Also, we have

$$\phi(X,Y) = \phi(Y,X), \tag{2.7}$$

Where $\phi(X,Y) = g(X, \phi Y)$

A Lorentzian almost paracontact manifold $M$ is called Lorentzian parasasakian manifold if

$$(\nabla_X \phi)(Y) = g(X,Y)\xi + \eta(Y)\phi X + 2\eta(X)\eta(Y)\xi, \tag{2.8}$$

where $\nabla$ is the Levi- Civita connection with respect to $g$ and for any vector fields $X, Y$ on $M$.

If $\xi$ is a killing vector field, the (para) contact structure is called $K$-(para) contact. In this case we have,

$$\nabla_X \xi = \phi X \tag{2.9}$$

Now, we define Lorentzian- para Kenmotsu manifold:

***Definition 2.1*:** A Lorentzian almost paracontact manifold $M$ is called Lorentzian para-Kenmotsu manifold if

$$(\nabla_X \phi)(Y) = -g(\phi X, Y)\xi - \eta(Y)\phi X. \tag{2.10}$$

In the Lorentzian-para Kenmotsu manifold, we have

$$\nabla_X \xi = -X - \eta(X)\xi, \tag{2.11}$$

$$(\nabla_X \eta)(Y) = -g(X, Y) - \eta(X)\eta(Y) \tag{2.12}$$

Additionally, the curvature tensor $R$, the Ricci tensor $S$ and the Ricci operator $Q$ in a Lorentzian para-Kenmotsu manifold $M$ with respect to the Livi-Civita connection satisfies [8]

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.13}$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X \tag{2.14}$$

$$R(\xi, X)\xi = X + \eta(X)\xi \tag{2.15}$$

$$g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) \tag{2.16}$$

$$\text{Or, } g(R(X, Y)Z, \xi) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y) \tag{2.17}$$

$$S(X, \xi) = -(n-1)\eta(X), \tag{2.18}$$

$$Q\xi = (n-1)\xi, \tag{2.19}$$

where $g(QX, Y) = S(X, Y)$.

For any vector fields $X, Y$ and $Z$ on $M$ it yields to

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y) \tag{2.20}$$

Here we note that if $\epsilon = 1$ and the structure vector field $\xi$ is a space like, then an $\in$-Kenmotsu manifold same as usual Kenmotsu manifold.

***Definition 2.2:*** A Lorentzian almost paracontact manifold $M$ is said to be as $\eta$-Einstein manifold if its Ricci tensor $S$ is of the form

$$S(X, Y) = a\, g(X, Y) + b\eta(X)\eta(Y) \tag{2.22}$$

where $a$ and $b$ are scalar functions on $M$.

A Lorentzian almost paracontact manifold $M$ is said to be a generalized $\eta$-Einstein manifold if its Ricci tensor $S$ is of the form

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y) + c\phi(X, Y) \tag{2.23}$$

where $a, b$ and $c$ are scalar functions on $M$ and $\phi(X, Y) = g(\phi X, Y)$.

If $c = 0$, then the manifold reduces to an $\eta$- Einstein manifold.

Also, it is an Einstein manifold if $b$ and c both are 0.

## 3. $W_1$ −flat LP-Kenmotsu Manifold

In this section, we study $W_1$- flat in LP-Kenmotsu manifold:

***Definition3.1:*** An LP-Kenmotsu manifold is said to be $W_1 -$ flat if

$$W_1(X,Y)Z = 0, \tag{3.1}$$

for any vector fields $X, Y$ and $Z$ on $M$.

$W_1 -$ Curvature tensor [6] is defined as

$$W_1(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y] \tag{3.3}$$

$$\Rightarrow g(Y,Z)X - g(X,Z)Y + \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y] = 0 \tag{3.4}$$

Replacing $X$ by $\xi$ in (3.4) we get

$$g(Y,Z)\xi - g(\xi,Z)Y + \tfrac{1}{n-1}[S(Y,Z)\xi - S(\xi,Z)Y] = 0 \tag{3.5}$$

$$\Rightarrow g(Y,Z)\xi - \eta(Z)Y + \frac{1}{n-1}[S(Y,Z)\xi - (n-1)\eta(Z)Y] = 0$$

$$\Rightarrow \quad g(Y,Z)\xi - \eta(Z)Y + \frac{1}{n-1}[S(Y,Z)\xi - \eta(Z)Y] = 0$$

$$\Rightarrow \frac{1}{n-1}S(Y,Z)\xi = -g(Y,Z)\xi + 2\eta(Z)Y$$

$$\Rightarrow S(Y,Z)\xi = -(n-1)g(Y,Z)\xi + 2(n-1)\eta(Z)Y$$

Taking inner product with $\xi$, we get

$$-S(Y,Z) = (n-1)g(Y,Z) + 2(n-1)\eta(Y)\eta(Z)$$

$$S(Y,Z) = -(n-1)g(Y,Z) - 2(n-1)\eta(Y)\eta(Z) \tag{3.6}$$

Hence from the above discussion, we state the following theorem:

***Theorem 3.2:*** *If an $n$-dimensional $LP -$ Kenmotsu manifold satisfying $W_1$ −flat condition then the manifold is an $\eta$-Einstein manifold.*

## 4. $\xi$- $W_1$ flat LP-Kenmotsu Manifold

In this section, we study $\xi$-$W_1$- flat in LP-Kenmotsu manifold:

***Definition 4.1:*** An LP-Kenmotsu manifold is said to be $\xi - W_1$- flat if

$$W_1(X,Y).\xi = 0, \tag{4.1}$$

for every vector field $X, Y$ on $M$.

Now, (4.1) turns into

$$R(X,Y).\xi + \tfrac{1}{n-1}[S(Y,\xi)X - S(X,\xi)Y] = 0$$

$$\eta(Y)X - \eta(X)Y + \frac{1}{n-1}[(n-1)\eta(Y)X - (n-1)\eta(X)Y] = 0$$

$$\eta(Y)X - \eta(X)Y + \eta(Y)X - \eta(X)Y = 0$$

$$2\eta(Y)X - 2\eta(X)Y = 0$$

$$\eta(Y)X - \eta(X)Y = 0$$

Putting $X = \xi$ and $Y = QY$, we get

$$\eta(QY)\xi - \eta(\xi)QY = 0$$

$$\eta((n-1)Y\xi + QY = 0$$

$$QY = -(n-1)\eta(Y)\xi$$

$$QY = -(n-1)\eta(Y)\xi$$

$$g(QY, Z) = -(n-1)\eta(Y)\xi$$

$$S(Y, Z) = -(n-1)\eta(Y)\eta(Z)$$

Hence from the above discussion, we state the following theorem:

***Theorem 4.2:*** *If a $n$ −dimensional LP-Kenmotsu manifold satisfying $\xi - W_1$-flat condition then manifold is a special type of $\eta$ Einstein manifold.*

## 5. LP-Kenmotsu Manifolds Satisfying $W_1 . Q = 0$

In this section, we study LP-Kenmotsu Manifold satisfying $W_1 . Q = 0$. Then, we get

$$W_1(X, Y)QZ - Q(W_1(X, Y)Z) = 0. \tag{5.1}$$

Putting $Y = \xi$,we get

$$W_1(X, \xi)QZ - Q(W_1(X, \xi)Z) = 0. \tag{5.2}$$

Using (3.2), we get

$$\Rightarrow R(X, \xi)QZ + \frac{1}{n-1}[S(\xi, QZ)X - S(X, QZ)\xi] - Q\left[R(X, \xi)Z + \frac{1}{n-1}S(\xi, Z)X - S(X, Z)\xi\right] = 0.$$

$$g(\xi, QZ)X - g(X, QZ)\xi + \frac{1}{n-1}[(n-1)\eta(QZ)X - S(X, QZ)\xi]$$
$$- Q\left[g(\xi, Z)X - g(X, Z)\xi + \frac{1}{n-1}(n-1)\eta(Z)X - S(X, Z)\xi\right] = 0. \tag{5.3}$$

$$\text{Or, } \eta(QZ)X - g(X, QZ)\xi + \eta(QZ)X - \tfrac{1}{n-1}S(X, QZ)\xi - Q\left[\eta(Z)X - g(X, Z)\xi + \eta(Z)X - \tfrac{1}{n-1}(S(X, Z)\xi)\right] = 0$$

$$\Rightarrow 2\eta(QZ)X - S(Z, X)\xi - S(X, Z)\xi - Q[2\eta(Z)X - g(X, Z)\xi - \frac{1}{n-1}S(X, Z)\xi] = 0$$

$$\text{Or, } 2[(n-1)\eta(z)X - S(X, Z)\xi] - 2(n-1)\eta(Z)X + (n-1)g(X, Z)Q\xi + S(X, Y)\xi = 0.$$

$$\text{Or, } \quad -S(X, Z)\xi + (n-1)g(X, Z)\xi = 0.$$

$$\text{Or, } \quad S(X, Z)\xi = (n-1)g(X, Z)\xi.$$

Taking inner product with respect to $\xi$, we have

$$S(X, Z) = (n-1)g(X, Z).$$

Hence we can state the following theorem:

***Theorem 5.1:*** *If a $n$-dimensional $LP -$ Kenmotsu manifold satisfying $W_1 \cdot Q = 0$ condition then manifold is Einstein manifold.*

## 6. $\phi - W_1$ semisymmetric $LP$ −Kenmotsu Manifolds

In this section, we study $\phi - W_1$ semisymmetric LP – Kenmotsu manifolds. Let $W_1$ is $\phi -$ semisymmetric, then we get

$$W_1(X,Y) \cdot \phi = 0 \tag{6.1}$$

By the definition of $\phi -$ semisymmetric, we have

$$W_1(X,Y) \cdot \phi Z - \phi W_1(X,Y) \cdot Z = 0. \tag{6.2}$$

Where, $X, X$ and $Z$ are arbitrary vector fields in $M$.

Using equation (3.2) in (6.2), we have

$$R(X,Y)\phi Z + \tfrac{1}{n-1}[S(Y,\phi Z)X - S(X,\phi Z)Y] - \phi R(X,Y)Z - \tfrac{1}{n-1}[S(Y,Z)\phi X - S(X,Z)\phi Y] = 0. \tag{6.3}$$

Putting $X = \xi$ in above equation, we get

$$R(\xi,Y)\phi Z + \tfrac{1}{n-1}[S(Y,\phi Z)\xi - S(\xi,\phi Z)Y] - \phi R(\xi,Y)Z - \tfrac{1}{n-1}[S(Y,Z)\phi\xi - S(\xi,Z)\phi Y] = 0. \tag{6.4}$$

Using equations (2.11) and (2.15) in above and on simplification, we get

$$g(X,\phi Z)\xi + \frac{1}{n-1}S(Y,\phi Z)\xi + \eta(Z)\phi Y + \frac{1}{n-1}S(\xi,\phi Z)\phi Y = 0.$$

$$S(Y,\phi Z)\xi = -(n-1)g(Y,\phi Z)\xi - 2(n-1)\eta(Z)\phi Y.$$

Taking inner product with $\xi$, we get

$$S(Y,\phi Z) = -(n-1)g(Y,\phi Z).$$

Replacing $Z$ by $\phi Z$, we get

$$S(Y,\phi^2 Z) = -(n-1)g(Y,\phi^2 Z)$$

Or, $$S(Y, Z + \eta(Z)\xi) = -(n-1)[g(Y,Z) + \eta(Z)\eta(Y)]$$

$$S(Y,Z) + \eta(Y)S(Y,\xi) = -(n-1)[g(Y,Z) + \eta(Z)\eta(Y)]$$

Using (2.18) in above, we get

$$S(Y,Z) = -(n-1)g(Y,Z) - 2(n-1)\eta(Y)\eta(Z)$$

Hence we can state the following theorem:

***Theorem 6.1:*** *An $n$-dimensional LP $-$ Kenmotsu manifold is said to be $\phi -$ $W_1$ semisymmetric if and only if it is an $\eta -$ Einstein manifold.*

## 7. $W_1$ curvature tensor satisfying $W_1(\phi X, \phi Y, \phi Z, \phi W) = 0$ condition

In this section we study a special property of $W_1$ curvature tensor and found interesting result. We have

$$W_1(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y]$$

Replacing $X$ by $\phi X$, $y$ by $\phi Y$, $Z$ by $\phi Z$ then we get

$$W_1(\phi X,\phi Y)\phi Z = R(\phi X,\phi Y)\phi Z + \frac{1}{n-1}[S(\phi Y,\phi Z)\phi X - S(\phi X,\phi Z)\phi Y]$$

Using equation (3.2), we get

$$W_1(\phi X,\phi Y)\phi Z = g(\phi Y,\phi Z)\phi X - g(\phi X,\phi Z)\phi Y + \frac{1}{n-1}[S(\phi Y,\phi Z)\phi X - S(\phi X,\phi Z)\phi Y] \tag{7.1}$$

$$W_1(\phi X,\phi Y)\phi Z = [g(Y,Z) + \eta(Y)\eta(Z)]\phi X - [g(X,Z)\eta(X)\eta(Z)]\phi Y + \frac{1}{n-1}[S(Y,Z)\phi X + (n-1)\eta(Y)\eta(Z)\phi X - S(X,Z)\phi Y] \quad - (n-1)\eta(X)\eta(Z)\phi Y$$

On simplification, we get

$$W_1(\phi X,\phi Y)\phi Z = g(Y,Z)\phi X - g(X.Z)\phi Y + 2[\eta(Y)\eta(Z)\phi X - \eta(X)\eta(Z)\phi Y + \frac{1}{n-1}[S(Y,Z)\phi X - S(X,Z)\phi Y]$$

Taking inner product with $\phi W$, we have

$$W_1(\phi X,\phi Y,\phi Z,\phi W) = g(Y,Z)g(\phi X,\phi W) - g(X.Z)g(\phi Y,\phi W) + 2[\eta(Y)\eta(Z)g(\phi X,\phi W) - \eta(X)\eta(Z)g(\phi Y,\phi W) + \frac{1}{n-1}[S(Y,Z)g(\phi X,\phi W) - S(X,Z)g(\phi Y,\phi W)]$$

Now,

$$g(Y,Z)g(\phi X,\phi W) - g(X.Z)g(\phi Y,\phi W) + 2[\eta(Y)\eta(Z)g(\phi X,\phi W) - \eta(X)\eta(Z)g(\phi Y,\phi W) + \frac{1}{n-1}[S(Y,Z)g(\phi X,\phi W) - S(X,Z)g(\phi Y,\phi W)] = 0$$

Let $\{e_1, e_2, \ldots, e_n\}$ be the set of orthonormal basis defined on manifold $M$.

Let $X = W = \xi, 1 \leq i \leq n$ and taking summation, we get

$$g(Y,Z)(\phi e_i,\phi e_i) - g(e_i.Z)g(\phi Y,\phi e_i) + 2[\eta(Y)\eta(Z)g(\phi e_i,\phi e_i) - \eta(X)\eta(Z)g(\phi Y,\phi e_i) + \frac{1}{n-1}[S(Y,Z)g(\phi e_i,\phi e_i) - S(e_i,Z)g(\phi Y,\phi e_i)] = 0$$

Since, $g(\phi e_i\phi, e_i) = g(e_i,e_i) - \eta(e_i)\eta(e_i)$

= (n-1).

Therefore,

$$(n-1)g(Y,Z) - g(e_i.Z)g(\phi Y,\phi e_i) + 2[(n-1)\eta(Y)\eta(Z) - \eta(X)\eta(Z)g(\phi Y,\phi e_i) + \frac{1}{n-1}[(n-1)S(Y,Z) - S(e_i,Z)g(\phi Y,\phi e_i)] = 0 \tag{7.2}$$

Again,

$$g(\phi Y,\phi e_i) = g(Y,e_i) + \eta(Y)\eta(e_i).$$

Since, we have

$$g(e_i,Z)g(\phi Y,\phi e_i) = g(e_i,Z)g(Y,e_i) + g(e_i,Z)\eta(Y)\eta(e_i),$$
$$= g(Y,Z) + \eta(Y)\eta(Z),$$
$$= g(\phi Y,\phi Z).$$

And,

$$\eta(e_i)g(\phi Y,\phi e_i) = \eta(e_i)g(Y,e_i) + \eta(e_i)\eta(Y)\eta(e_i),$$
$$= \eta(Y) - \eta(Y),$$

$$= 0.$$

Also,

$$S(e_i, Z)g(\phi Y, \phi e_i) = S(e_i, Z)g(Y, e_i) + S(e_i, Z)\eta(Y)\eta(e_i),$$
$$= S(Y, Z) + S(\xi, Z)\eta(Y)$$
$$= S(Y, Z) + (n-1)\eta(Y)\eta(Z).$$

Using above in (7.2), we get

$$(n-1)g(Y,Z) - g(Y.Z) - \eta(Y)\eta(Z) + 2(n-1)\eta(Y)\eta(Z) + \frac{1}{n-1}[(n-1)S(Y,Z) - S(Y,Z) - (n-1)\eta(Y)\eta(Z)] = 0.$$

$$(n-2)g(Y,Z) + (2n-3)\eta(Y)\eta(Z) + \frac{n-2}{n-1}S(Y,Z) - \eta(Y)\eta(Z) = 0.$$

$$(n-2)g(Y,Z) + 2(n-2)\eta(Y)\eta(Z) + \frac{n-2}{n-1}S(Y,Z) = 0.$$

$$(n-2)[g(Y,Z) + 2\eta(Y)\eta(Z) + \frac{1}{n-1}S(Y,Z)] = 0$$

That shows that either $n = 2$ or,

$$g(Y,Z) + 2\eta(Y)\eta(Z) + \frac{1}{n-1}S(Y,Z) = 0,$$

Or,

$$S(Y,Z) = (1-n)g(Y,Z) + 2(1-n)\eta(Y)\eta(Z).$$

Hence, we state the following theorem

***Theorem 7.1:*** *Let $M$ be an $n-$ dimensional $LP-$ Kenmotsu manifold then $M$ is said to satisfy $W_1(\phi X, \phi Y, \phi Z, \phi W) = 0$, if and only if either $n = 2$ or $M$ is an $\eta-$ Einstein manifold.*

## 8. Conclusions

In this paper, we proposed that $W_1 -$ flat in $LP -$ Kenmotsu manifold is an $\eta -$ Einstein manifold. Again, we have found that $\xi - W_1 -$ flat $LP -$ Kenmotsu manifold is an $\eta -$ Einstein manifold. Next, we deal with $LP -$ Kenmotsu manifold satisfying $W_1 \cdot Q = 0$ condition and found to be an Einstein manifold. Again, we discuss the $LP -$ Kenmotsu manifold regard with $\phi - W_1 -$semisymmetric condition and it comes out to be an $\eta -$ Einstein manifold. Further, we discudded a $LP -$ Kenmotsu manifold satisfying $W_1(\phi X, \phi Y, \phi Z, \phi W) = 0$ condition and then it comes out to be an $\eta -$ Einstein manifold.

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**Conflicts of Interest:** We declare that there is no con ict of interest be tween the authors.

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