Generalized $(\sigma, \tau)$-Reverse Derivations in non ideal on Prime Rings

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Abstract: Let $R$ be a prime ring, $I$ be a non-zero ideal on $R$, and $\sigma, \tau$ be automorphisms of $R$. Suppose that $F$ is a generalized $(\sigma, \tau)$-reverse derivation on $R$ associated with $(\sigma, \tau)$-reverse derivation $d$ on $R$ respectively. In this paper, we studied the following identities in prime rings: (i) $F(xy) + d(x)F(y) = 0$; (ii) $F(xy) + d(x)F(y) + \sigma(xy) = 0$; (iii) $F(xy) + d(x)F(y) + \sigma(xy) = 0$; (iv) $F(xy) + d(x)F(y) + \sigma(xy) = 0$; (v) $F(xy) + d(x)F(y) + \sigma(xy) = 0$; (vi) $F(xy) + d(x)F(y) + \sigma(xy) = 0$; (vii) $F(xy) + d(x)F(y) + \sigma(xy) = 0$; (viii) $F(xy) + d(x)F(y) + \sigma(xy) = 0$; (ix) $F(xy) + F(x)F(y) = 0$; (x) $F(xy) + F(x)F(y) = 0$; for all $x, y$ in some suitable sub sets of $R$.

Keywords: Prime ring, Derivation, Reverse derivation, Generalized derivation, $(\sigma, \tau)$-derivation, Generalized $(\sigma, \tau)$-derivation, $(\sigma, \tau)$-reverse derivation, Generalized $(\sigma, \tau)$-reverse derivation.

Introduction:

In 1994, Yenigul and Argac in [8], obtained the some result for $\alpha$ derivation on prime rings. In 1999, Ashraf, Nadeem and Quadri in [3], extended the result for $(\theta, \varphi)$ derivation in prime and semiprime rings. Further Chirag Garg et al. in [5] studied on generalized $(\sigma, \tau)$-derivations in prime rings. The notion of reverse derivation has been introduced by Bresar and Vukman in [4] and the reverse derivations on semi prime rings have been studied by Samman and Alyamani in [7]. Aboubakr and Gonzalez in [1] studied the relationship between generalized reverse derivation and generalized derivation on an ideal in semi prime rings, and C. Jaya Subbareddy et.al in [6] is proved that in case $R$ is a prime ring with a non-zero right reverse derivation $d$ and $U$ is the left ideal of $R$ then $R$ is commutative. In 2011, the concepts of $(\theta, \varphi)$-reverse derivation, and generalized $(\theta, \varphi)$-reverse derivation has been introduced by Anwar Khaleel Faraj in [2]. In this paper, we inspire of Chirag Garg et al. in [5], we proved some results on generalized $(\sigma, \tau)$-reverse derivations in prime rings.

Preliminaries: Throughout this paper $R$ denote an associative ring with center $Z$. Recall that a ring $R$ is prime if $xRy = \{0\}$ implies $x = 0$ or $y = 0$. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$ and the symbol $(xoy)$ denotes the anticommutator $xy + yx$. An additive mapping $d: R \rightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. An additive mapping $d: R \rightarrow R$ is called a reverse derivation if $d(xy) = d(y)x + yd(x)$ for all $x, y \in R$. An additive mapping $d: R \rightarrow R$ is called a $(\sigma, \tau)$-derivation if $d(xy) = d(x)\sigma(y) + \tau(x)d(y)$ for all $x, y \in R$. An additive mapping
A derivation $d: R \to R$ is called a $(\sigma, \tau)$-reverse derivation if $d(xy) = d(x)\sigma(x) + \tau(y)d(x)$, for all $x, y \in R$. An additive mapping $F: R \to R$ is called a generalized derivation, if there exists a derivation $d: R \to R$ such that $F(xy) = F(x)\sigma(y) + xd(y)$, for all $x, y \in R$. An additive mapping $F: R \to R$ is called a generalized reverse derivation, if there exists a reverse derivation $d: R \to R$ such that $F(xy) = F(y)\sigma(x) + \tau(y)d(x)$, for all $x, y \in R$. An additive mapping $F: R \to R$ is said to be a generalized $(\sigma, \tau)$-derivation of $R$, if there exist automorphisms $\sigma$ and $\tau$ such that $F(xy) = F(x)\sigma(y) + \tau(y)d(x)$, for all $x, y \in R$.

Throughout this paper, we shall make use of the basic commutator identities:

$[x, yz] = y[x, z] + [x, y]z; \quad [xy, z] = [x, z]y + x[y, z].$

**Lemma 1:** [3, Lemma 2] Let $R$ be a 2-torsion free prime ring and $U$ be a non-zero square-closed Lie ideal of $R$. If $[\sigma(x), \beta(y)] = 0$, for all $x, y \in U$, where $\alpha, \beta$ are automorphisms on $R$, then $U \subseteq Z$.

**Lemma 2:** Let $R$ be a prime ring and $I$ a nonzero lie ideal of $R$. If $d$ is a non zero $(\sigma, \tau)$-reverse derivation of $R$ such that $d(I) = 0$, then $I \subseteq Z$.

**Proof:** We have $d(1) = 0$, for all $u \in I$. (1)

We replacing $u$ by $[u, r]$ in equation (1), we get

$d([u, r]) = 0$

$d(u - r) = 0$

$d(r)\sigma(u) + \tau(r)d(u) - d(u)\sigma(r) - \tau(u)d(r) = 0$, for all $u \in I, r \in R$.

Using equation (1) in the above equation, we get

$d(r)\sigma(u) - \tau(u)d(r) = 0$, for all $u \in I, r \in R$. (2)

We replacing $r$ by $r\nu$ in the above equation, we get

$d(r\nu)\sigma(u) - \tau(u)d(r\nu) = 0$

$d(v)\sigma(r)\sigma(u) + \tau(v)d(r)\sigma(u) - \tau(u)d(v)\sigma(r) - \tau(u)\tau(v)d(r) = 0$, for all $u, v \in I, r \in R$.

Using equation (1) in the above equation, we get

$\tau(v)d(r)\sigma(u) - \tau(u)\tau(v)d(r) = 0$, for all $u, v \in I, r \in R$. (3)

Left multiplying equation (2) by $\tau(v)$, we get

$\tau(v)d(r)\sigma(u) - \tau(u)\tau(v)d(r) = 0$, for all $u, v \in I, r \in R$. (4)

We subtracting equation (4) from equation (3), we get

$\tau[u, v]d(r) = 0$, for all $u, v \in I, r \in R$. (5)
We replacing \( v \) by \( sv, s \in R \) in equation (5), we get
\[
\tau[u, sv]d(r) = 0
\]
\[
\tau(s)\tau[u, v]d(r) + \tau[u, s]\tau(v)d(r) = 0
\]
Using equation (5) in the above equation, we get
\[
\tau[u, s]\tau(v)d(r) = 0, \text{ for all } u, v \in l, r, s \in R.
\]
We replacing \( v \) by \( tv, t \in R \) in the above equation, we get
\[
\tau[u, s]\tau(tv)d(r) = 0, \text{ for all } u, v \in l, r, s, t \in R.
\]
\[
\tau[u, s]R\tau(tv)d(r) = 0, \text{ for all } u, v \in l, s \in R. \text{ Since } R \text{ is a prime ring and } I \text{ is a nonzero lie ideal of } R.
\]
we get either \( \tau[u, s] = 0 \) or \( d(r) = 0 \). If \( d(r) = 0 \), is contradiction to our assumption. So we get \( [u, s] = 0 \), for all \( u \in l, s \in R \). Then \( I \subseteq Z \).

**Theorem 1:** Let \( R \) be a prime ring and \( I \) be a non-zero ideal on \( R \). Suppose that \( F \) is a generalized \( (\sigma, \tau) \)-reverse derivation on \( R \) with \( \sigma(I) \neq 0 \) and \( \sigma(\tau) \neq 0 \). If \( F(xy) + d(x)F(y) = 0 \), for all \( x, y \in I \), then \( I \subseteq Z \).

**Proof:** We have \( F(xy) + d(x)F(y) = 0 \), for all \( x, y \in I \). (6)

We replacing \( y \) by \( xy \) in equation (6), we obtain
\[
F(xxy) + d(x)F(xy) = 0, \text{ for all } x, y \in I
\]
\[
F(xy)\sigma(x) + \tau(xy)d(x) + d(x)(F(y)\sigma(x) + \tau(y)d(x)) = 0
\]
\[
(F(xy) + d(x)F(y))\sigma(x) + \tau(xy)d(x) + d(x)(\tau(y)d(x) = 0, \text{ for all } x, y \in I.
\]
Using equation (6), it reduces to
\[
\tau(xy)d(x) + d(x)\tau(y)d(x) = 0, \text{ for all } x, y \in I. \quad (7)
\]

We replacing \( y \) by \( xy \) in equation (7), we get
\[
\tau(xxy)d(x) + d(x)\tau(xy)d(x) = 0, \text{ for all } x, y \in I. \quad (8)
\]
Left multiplying equation (7) by \( \tau(x) \), we get
\[
\tau(x)\tau(xy)d(x) + \tau(x)d(x)\tau(y)d(x) = 0, \text{ for all } x, y \in I. \quad (9)
\]
We subtracting equation (9) from equation (8), we get
\[
d(x)\tau(x)\tau(y)d(x) - \tau(x)d(x)\tau(y)d(x) = 0
\]
\[
[d(x), \tau(x)]\tau(y)d(x) = 0, \text{ for all } x, y \in I. \quad (10)
\]
We replacing \( y \) by \( sy, s \in R \) in equation (10), we get
\[
[d(x), \tau(x)]\tau(sy)d(x) = 0
\]
\[
[d(x), \tau(x)]R\tau(y)d(x) = 0, \text{ for all } x, y \in I, s \in R. \quad (11)
\]
Since $R$ is prime, we get either $[d(x), \tau(x)] = 0$, for all $x \in I$ or $\tau(y)d(x) = 0$, for all $x, y \in I$. Since $\tau$ is an automorphism of $R$ and $\tau(I) \neq 0$, we have either $[d(x), \tau(x)] = 0$, for all $x \in I$ or $d(x) = 0$, for all $x \in I$

Now let $A = \{x \in I/ [d(x), \tau(x)] = 0\}$ and $B = \{x \in I/ d(x) = 0\}$. Clearly, $A$ and $B$ are additive proper subgroups of $I$ whose union is $I$. Since a group cannot be the set theoretic union of two proper subgroups. Hence either $A = I$ or $B = I$.

If $B = I$, then $d(x) = 0$, for all $x \in I$, by lemma 2 implies that $I \subseteq Z$.

On the other hand if $A = I$, then $[d(x), \tau(x)] = 0$, for all $x \in I$.

If $[d(x), \tau(x)] = 0$, for all $x \in I$. (12)

We replacing $x$ by $x + y$ in equation (12), we get

$$[d(x + y), \tau(x + y)] = 0$$

$$[d(x), \tau(x)] + [d(x), \tau(y)] + [d(y), \tau(x)] + [d(y), \tau(y)] = 0, \text{ for all } x, y \in I.$$  

Using equation (12) in the above equation, we get

$$[d(x), \tau(y)] + [d(y), \tau(x)] = 0, \text{ for all } x, y \in I.$$  

We replacing $y$ by $yx$ in equation (13), we get

$$[d(x), \tau(yx)] + [d(yx), \tau(x)] = 0$$

$$[d(x), \tau(y)][\tau(x) + \tau(y)[d(x), \tau(x)] + [d(x), \tau(y)] + [\tau(x), \tau(x)] = 0$$

$$[d(x), \tau(y)][\tau(x) + \tau(y)[d(x), \tau(x)] + [d(x), \tau(y)] + [\tau(x), \tau(x)] = 0$$

$$[d(x), \tau(y)]\tau(x) + \tau(y)[d(x), \tau(x)] + [d(x), \tau(x)]\tau(x) + [\tau(x), \tau(x)]\tau(x) = 0$$

$$[d(x), \tau(y)]\tau(x) + \tau(y)[d(x), \tau(x)] + [d(x), \tau(x)]\tau(x) + [\tau(x), \tau(x)]\tau(x) = 0$$

$$, \text{ for all } x, y \in I.$$  

Using equation (12) in the above equation, we get

$$[d(x), \tau(y)]\tau(x) + d(x)[\tau(y), \tau(x)] + \tau(x)[d(y), \tau(x)] = 0, \text{ for all } x, y \in I.$$  

Right multiplying equation (13) by $\tau(x)$, we get

$$[d(x), \tau(y)]\tau(x) + [d(y), \tau(x)]\tau(x) = 0, \text{ for all } x, y \in I.$$  

We subtracting equation (15) from equation (14), we get

$$d(x)[\tau(y), \tau(x)] + \tau(x)[d(y), \tau(x)] - [d(y), \tau(x)]\tau(x) = 0, \text{ for all } x, y \in I.$$  

We replacing $d(y)$ by $\tau(x)$ in the above equation, we get

$$d(x)[\tau(y), \tau(x)] = 0, \text{ for all } x, y \in I.$$  

We replacing $y$ by $ys$ in equation (16), we get
Using equation (16) in the above equation, we get
\[ d(x)\sigma(y)\sigma(s), \tau(x) ] = 0, \text{ for all } x, y, s \in I. \]

We replacing \( y \) by \( yv, v \in R \) in the above equation, we get
\[ d(x)\sigma(yv)\sigma(s), \tau(x) ] = 0, \text{ for all } x, y, s \in I, v \in R. \]

\[ d(x)\sigma(y)R[\sigma(s), \tau(x) ] = 0, \text{ for all } x, y, s \in I. \]

Since \( R \) is prime, we get either \( d(x)\sigma(y) = 0, \text{ for all } x, y \in I \) or \([\sigma(s), \tau(x) ] = 0, \text{ for all } x, s \in I. \)

Since \( \sigma \) is an automorphism of \( R \) and \( \sigma(I) \neq 0 \), we have either \([\sigma(x), \tau(y) ] = 0, \text{ for all } x, y \in I \) or \( d(x) = 0, \text{ for all } x \in I. \)

If \( d(x) = 0, \text{ for all } x \in I \), by lemma 2 implies that \( I \subseteq Z \). If \([\sigma(x), \tau(y) ] = 0, \text{ for all } x, y \in I \), by lemma 1 implies that \( I \subseteq Z \).

**Theorem 2:** Let \( R \) be a prime ring and \( I \) be a non-zero ideal on \( R \). Suppose that \( F \) is a generalized \((\sigma, \tau)\)-reverse derivation on \( R \) associated with \((\sigma, \tau)\)-reverse derivation \( d \) on \( R \) respectively, \( \sigma(I) \neq 0 \). If \( G(xy) + d(x)F(y) + \sigma(xy) = 0, \text{ for all } x, y \in I, \) then \( I \subseteq Z \).

**Proof:** We replacing \( F \) by \( F + \sigma \) in theorem 1, we get the required result.

**Theorem 3:** Let \( R \) be a prime ring and \( I \) be a non-zero ideal on \( R \). Suppose that \( F \) is a generalized \((\sigma, \tau)\)-reverse derivation on \( R \) associated with \((\sigma, \tau)\)-reverse derivation \( d \) on \( R \) respectively, \( \sigma(I) \neq 0 \). If \( F(xy) + d(x)F(y) + \sigma(xy) = 0, \text{ for all } x, y \in I, \) then \( I \subseteq Z \).

**Proof:** We have \( F(xy) + d(x)F(y) + \sigma(xy) = 0, \text{ for all } x, y \in I. \)

We replacing \( y \) by \( xy \) in equation (17), we obtain
\[ F(xyy) + d(x)F(xy) + \sigma(xyy) = 0, \text{ for all } x, y \in I \]
\[ F(xy)s + \tau(xy)d(x) + d(xy)(F(y)s + \tau(y)d(x)) + \sigma(xyy) = 0 \]
\[ (F(xy) + d(x)F(y))\sigma(x) + \tau(xy)d(x) + d(xy)\tau(y)d(x) + \sigma(xyy) = 0, \text{ for all } x, y \in I. \]

Using equation (17), it reduces to
\[ \tau(xy)d(x) + d(x)\tau(y)d(x) + \sigma(xyy) - \sigma(yxx)\sigma(x) = 0 \]
\[ \tau(xy)d(x) + d(x)\tau(y)d(x) + \sigma[x, y]\sigma(x) = 0, \text{ for all } x, y \in I. \]

We replacing \( y \) by \( xy \) in equation (18), we get
\[ \tau(xxy)d(x) + d(x)\tau(xy)yd(x) + \sigma[x, xy]\sigma(x) = 0 \]
\[ \tau(xxy)d(x) + d(x)\tau(xy)d(x) + \sigma(x)\sigma[x, y]\sigma(x) = 0, \text{ for all } x, y \in I. \]
\( \tau(x) \tau(x y) d(x) + \tau(x) d(x) \tau(y) d(x) + \tau(x) \sigma[x, y] \sigma(x) = 0 \), for all \( x, y, z \in I \).  \( (20) \)

We subtracting equation (20) from equation (19), we get
\[
d(x) \tau(x) \tau(y) d(x) - \tau(x) d(x) \tau(y) d(x) + \sigma(x) \sigma[x, y] \sigma(x) - \tau(x) \sigma[x, y] \sigma(x) = 0
\]
\[
[d(x), \tau(x)] \tau(y) d(x) + (\sigma(x) - \tau(x)) \sigma[x, y] \sigma(x) = 0 \), for all \( x, y \in I \).
\]

We replacing \( \tau(x) \) by \( \sigma(x) \) in the above equation, we get
\[
[d(x), \sigma(x)] \tau(y) d(x) = 0 \), for all \( x, y \in I \).
\( (21) \)

We replacing \( \sigma(x) \) by \( \tau(x) \) and \( y \) by \( sy \), \( s \in R \) in equation (21), we get
\[
[d(x), \tau(x)] \tau(sy) d(x) = 0
\]
\[
[d(x), \tau(x)] R \tau(y) d(x) = 0 \), for all \( x, y \in I, s \in R \).
\( (22) \)

The equation (22) is same as equation (11) in theorem 1. Thus, by same argument of theorem 1, we can conclude the result \( I \subseteq Z \).

**Theorem 4:** Let \( R \) be a prime ring and \( I \) be a non-zero ideal on \( R \). Suppose that \( F \) is a generalized \( (\sigma, \tau) \)-reverse derivation on \( R \) associated with \( (\sigma, \tau) \)-reverse derivation \( d \) on \( R \) respectively, \( \tau(I) \neq 0 \) and \( \sigma(I) \neq 0 \). If \( F(xy) + d(x) F(y) + \sigma(x \sigma y) = 0 \), for all \( x, y \in I \), then \( I \subseteq Z \).

**Proof:** We replacing \( F \) by \( F + \sigma \) in theorem 3, we get the required result.

**Theorem 5:** Let \( R \) be a prime ring and \( I \) be a non-zero ideal on \( R \). Suppose that \( F \) is a generalized \( (\sigma, \tau) \)-reverse derivation on \( R \) associated with \( (\sigma, \tau) \)-reverse derivation \( d \) on \( R \) respectively, \( \tau(I) \neq 0 \) and \( \sigma(I) \neq 0 \). If \( F(xy) + d(x) F(y) = 0 \), for all \( x, y \in I \), then \( I \subseteq Z \).

**Proof:** We have \( F(xy) + d(y) F(x) = 0 \), for all \( x, y \in I \).
\( (23) \)

We replacing \( x \) by \( xw \) in equation (23), we obtain
\[
F(xwy) + d(y) F(xw) = 0
\]
\[
F(wy) \sigma(x) + \tau(wy) d(x) + d(y) (F(w) \sigma(x) + \tau(w) d(x)) = 0
\]
\[
(F(wy) + d(y) F(w)) \sigma(x) + \tau(wy) d(x) + d(y) \tau(w) d(x) = 0 \), for all \( x, y, w \in I \).

Using equation (23), it reduces to
\[
\tau(wy) g(x) + d(y) \tau(w) d(x) = 0 \), for all \( x, y, w \in I \).
\( (24) \)

We replacing \( y \) by \( zy \) in equation (24), we get
\[
\tau(wzy) d(x) + d(zy) \tau(w) d(x) = 0
\]
\[
\tau(wzy) d(x) + d(y) \sigma(z) \tau(w) d(x) + \tau(y) d(z) \tau(w) d(x) = 0 \), for all \( x, y, z, w \in I \).
\( (25) \)

We replacing \( y \) by \( z \) in equation (24), we get
\[
\tau(wz) d(x) + d(z) \tau(w) d(x) = 0 \), for all \( x, z, w \in I \).
\( (26) \)
Left multiplying equation (26) by \( \tau(y) \), we get

\[
\tau(y)\tau(wz)d(x) + \tau(y)d(z)\tau(w)d(x) = 0 \quad \text{for all} \ x, y, z \in I. \quad (27)
\]

We subtracting equation (27) from equation (25), we get

\[
\begin{align*}
\tau(wzy) - \tau(zyw)d(x) + d(y)\sigma(z)\tau(w)d(x) &= 0 \\
\tau(wz, y)d(x) + d(y)\sigma(z)\tau(w)d(x) &= 0
\end{align*}
\]

for all \( x, y, z, w \in I \).

We replacing \( z \) by \( y \) and \( w \) by \( y \) in the above equation, we get

\[
d(y)\sigma(y)\tau(y)d(x) = 0 \quad \text{for all} \ x, y \in I. \quad (28)
\]

We replacing \( x \) by \( s x \) in equation (28), we get

\[
d(y)\sigma(y)\tau(y)d(sx) = 0
\]

\[
d(y)\sigma(y)\tau(y)d(x)\sigma(s) + d(y)\sigma(y)\tau(y)\tau(x)d(s) = 0 \quad \text{for all} \ x, y, s \in I.
\]

Using equation (28) in the above equation, we get

\[
d(y)\sigma(y)\tau(y)d(s) = 0 \quad \text{for all} \ x, y, s \in I.
\]

We replacing \( x \) by \( r x, r \in R \) in the above equation, we get

\[
d(y)\sigma(y)\tau(y)\tau(rx)d(s) = 0
\]

\[
d(y)\sigma(y)\tau(y)R\tau(x)d(s) = 0 \quad \text{for all} \ x, y, s \in I.
\]

Since \( R \) is prime, we get either \( d(y)\sigma(y)\tau(y) = 0 \), for all \( y \in I \) or \( \tau(x)d(s) = 0 \), for all \( x, s \in I \).

Since \( \tau \) is an automorphism of \( R \) and \( \tau(I) \neq 0 \), we have either \( d(x)\sigma(x) = 0 \), for all \( x \in I \) or \( d(x) = 0 \), for all \( x \in I \). If \( d(x) = 0 \), for all \( x \in I \), by lemma 2 implies that \( I \subseteq Z \).

If \( d(x)\sigma(x) = 0 \), for all \( x \in I \). Since \( \sigma \) is an automorphism of \( R \) and \( \sigma(I) \neq 0 \) then \( d(x) = 0 \), for all \( x \in I \), by lemma 2 implies that \( I \subseteq Z \).

**Theorem 6:** Let \( R \) be a prime ring and \( I \) be a non-zero ideal on \( R \). Suppose that \( F \) is a generalized \((\sigma, \tau)\)-reverse derivation on \( R \) associated with \((\sigma, \tau)\)-reverse derivation \( d \) on \( R \) respectively, \( \tau(I) \neq 0 \) and \( \sigma(I) \neq 0 \). If \( F(xy) + d(y)F(x) + \sigma(xy) = 0 \), for all \( x, y \in I \), then \( I \subseteq Z \).

**Proof:** We replacing \( F \) by \( F + \sigma \) in theorem 5, we get the required result.

**Theorem 7:** Let \( R \) be a prime ring and \( I \) be a non-zero ideal on \( R \). Suppose that \( F \) is a generalized \((\sigma, \tau)\)-reverse derivation on \( R \) associated with \((\sigma, \tau)\)-reverse derivation \( d \) on \( R \) respectively, \( \tau(I) \neq 0 \) and \( \sigma(I) \neq 0 \). If \( F(xy) + d(y)F(x) + \sigma(yx) = 0 \), for all \( x, y \in I \), then \( I \subseteq Z \).

**Proof:** We have \( F(xy) + d(y)F(x) + \sigma(yx) = 0 \), for all \( x, y \in I \).

We replacing \( x \) by \( xw \) in equation (29), we obtain
Using equation (29), it reduces to
\[ \tau(\text{r}(w)d(x) + d(y)\tau(w)d(x) + \sigma(\text{r}(z)) = 0, \text{ for all } \alpha, \beta, \gamma. \] (30)

We replacing \( y \) by \( z \) and \( w \) by \( y \) in equation (30), we get
\[ \tau(\text{r}(w)d(x) + d(y)\tau(w)d(x) + \sigma(z)\sigma[x, w] = 0, \text{ for all } x, y, w \in I. \] (31)

We replacing \( y \) by \( z \) in equation (30), we get
\[ \tau(\text{r}(w)d(x) + d(z)\tau(w)d(x) + \sigma(z)\sigma[x, w] = 0, \text{ for all } x, z, w \in I. \] (32)

Left multiplying equation (32) by \( \tau(y) \), we get
\[ \tau(y)\tau(\text{r}(w)d(x) + d(z)\tau(w)d(x) + \tau(\text{r}(z))\sigma[x, w] = 0, \text{ for all } x, y, z, w \in I. \] (33)

We subtracting equation (33) from equation (31), we get
\[ \tau(\text{r}(w)d(x) + d(z)\tau(w)d(x) + d(y)\sigma(z)\tau(w)d(x) + d(y)\sigma(z)\sigma[x, w] - \tau(\text{r}(z))\sigma[x, w] = 0 \]
\[ \tau(\text{r}(w)d(x) + d(z)\tau(w)d(x) + d(y)\sigma(z)\tau(w)d(x) + d(y)\sigma(z)\sigma[x, w] - \tau(z)\sigma[x, w] = 0, \text{ for all } x, y, z, w \in I. \]

We replacing \( z \) by \( y \) and \( w \) by \( y \) in the above equation, we get
\[ d(y)\sigma(x)y\tau(y)d(x) + d(y)\sigma(x)\sigma[x, y] - \tau(y)\sigma(y)d(x) = 0, \text{ for all } x, y \in I. \]

We replacing \( \tau(y) \) by \( \sigma(y) \) in the above equation, we get
\[ d(y)\sigma(y)d(x) + d(y)\sigma(y)d(x) - \sigma(y)d(x) = 0, \text{ for all } x, y \in I. \] (34)

The equation (34) is same similar equation (28) in theorem 5. Thus, by same argument of theorem 5, we can conclude the result \( I \subseteq Z \).

**Theorem 8:** Let \( R \) be a prime ring and \( I \) be a non-zero ideal on \( R \). Suppose that \( F \) is a generalized \((\sigma, \tau)\)-reverse derivation on \( R \) associated with \((\sigma, \tau)\)-reverse derivation \( \text{d} \) on \( R \) respectively, \( \tau(I) \neq 0 \) and \( \sigma(I) \neq 0 \). If \( F(xy) + d(y)F(x) + \sigma(xo) = 0, \text{ for all } x, y \in I \), then \( I \subseteq Z \).

**Proof:** We replacing \( F \) by \( F + \sigma \) in theorem 7, we get the required result.
Theorem 9: Let \( R \) be a prime ring and \( I \) be a non-zero ideal on \( R \). Suppose that \( F \) is a generalized \((\sigma,\tau)\)-reverse derivation on \( R \) associated with \((\sigma,\tau)\)-reverse derivation \( d \) on \( R \) respectively, \( \tau(I) \neq 0 \) and \( \sigma(I) \neq 0 \). If \( F(xy) + F(x)F(y) = 0 \), for all \( x, y \in I \). then \( I \subseteq Z \).

Proof: We have \( F(xy) + F(x)F(y) = 0 \), for all \( x, y \in I \). (35)

We replacing \( y \) by \( xy \) in equation (35), we obtain

\[ F(xxy) + F(x)F(xy) = 0, \text{ for all } x, y \in I \]

\[ F(xy)\sigma(x) + \tau(xy)d(x) + F(x)(F(y)\sigma(x) + \tau(y)d(x)) = 0 \]

\[ (F(xy) + F(x)F(y))\sigma(z) + \tau(xy)d(x) + F(x)\tau(y)d(x) = 0, \text{ for all } x, y \in I. \]

Using equation (35), it reduces to

\[ \tau(xy)d(x) + F(x)\tau(y)d(x) = 0, \text{ for all } x, y \in I. \] (36)

We replacing \( y \) by \( wy \) in equation (36), we get

\[ \tau(xwy)d(x) + F(x)\tau(wy)d(x) = 0, \text{ for all } x, y, w \in I. \] (37)

Left multiplying equation (36) by \( \tau(w) \), we get

\[ \tau(w)\tau(xy)d(x) + \tau(w)F(x)\tau(y)d(x) = 0, \text{ for all } x, y, z, w \in I. \] (38)

We subtracting equation (38) from equation (37), we get

\[ \tau(xwy)g(x) - \tau(wxy)g(x) + F(x)\tau(w)\tau(y)d(x) - \tau(w)F(x)\tau(y)d(x) = 0 \]

\[ (\tau(xwy) - \tau(wxy))g(x) + [F(x),\tau(w)]\tau(y)d(x) = 0, \text{ for all } x, y, w \in I. \] (39)

We replacing \( w \) by \( x \) and \( y \) by \( sy \), \( s \in R \) in equation (39), we get

\[ [F(x),\tau(x)]\tau(sy)d(x) = 0 \]

\[ [F(x),\tau(x)]\tau(ry)d(x) = 0, \text{ for all } x, y \in I, s \in R. \] (40)

Since \( R \) is prime, we get either \([F(x),\tau(x)] = 0 \), for all \( x \in I \) or \( \tau(y)d(x) = 0 \), for all \( x, y \in I \). Since \( \tau \) is an automorphism of \( R \) and \( \tau(I) \neq 0 \), we have either \([F(x),\tau(x)] = 0 \), for all \( x \in I \) or \( d(x) = 0 \), for all \( x \in I \).

Now let \( A = \{x \in I/\tau(x) = 0\} \) and \( B = \{x \in I/d(x) = 0\} \). Clearly, \( A \) and \( B \) are additive proper subgroups of \( I \) whose union is \( I \). Since a group cannot be the set theoretic union of two proper subgroups. Hence either \( A = I \) or \( B = I \).

If \( B = I \), then \( d(x) = 0 \), for all \( x \in I \). by lemma 2 implies that \( I \subseteq Z \).

On the other hand if \( A = I \), then \([F(x),\tau(x)] = 0 \), for all \( x \in I \).

If \([F(x),\tau(x)] = 0 \), for all \( x \in I \). (41)

We replacing \( x \) by \( x + y \) in equation (12), we get
\[ [F(x + y), \tau(x + y)] = 0 \]
\[ [F(x), \tau(x)] + [F(x), \tau(y)] + [F(y), \tau(x)] + [F(y), \tau(y)] = 0, \text{ for all } x, y \in I. \]

Using equation (41) in the above equation, we get
\[ [F(x), \tau(y)] + [F(y), \tau(x)] = 0, \text{ for all } x, y \in I. \quad (42) \]

We replacing \( y \) by \( xy \) in equation (42), we get
\[ [F(x), \tau(y)] + [F(yx), \tau(x)] = 0 \]
\[ [F(x), \tau(y)] \tau(x) + \tau(y)[F(x), \tau(x)] + [F(x)\sigma(y) + \tau(x)d(y), \tau(x)] = 0 \]
\[ [F(x), \tau(y)] \tau(x) + \tau(y)[F(x), \tau(x)] + [F(x)\sigma(y), \tau(x)] + [\tau(x)d(y), \tau(x)] = 0 \]
\[ [F(x), \tau(y)] \tau(x) + \tau(y)[F(x), \tau(x)] + [F(x), \tau(x)\sigma(y) + F(x)\sigma(y), \tau(x)] + \tau(x)[d(y), \tau(x)] + [\tau(x), \tau(x)]d(y) = 0 \]
\[ , \text{ for all } x, y \in I. \]

Using equation (41) in the above equation, we get
\[ [F(x), \tau(y)] \tau(x) + F(x)[\sigma(y), \tau(x)] + \tau(x)[d(y), \tau(x)] = 0, \text{ for all } x, y \in I. \]

We replacing \( \sigma(y) \) by \( \tau(x) \) in the above equation, we get
\[ [F(x), \tau(y)] \tau(x) + \tau(x)[d(y), \tau(x)] = 0, \text{ for all } x, y \in I. \]

We replacing \( y \) by \( x \) in the above equation, we get
\[ [F(x), \tau(x)] \tau(x) + \tau(x)[d(x), \tau(x)] = 0, \text{ for all } x, y \in I. \]

Using equation (41) in the above equation, we get
\[ \tau(x)[d(x), \tau(x)] = 0, \text{ for all } x, y \in I. \]

Since \( \tau \) is an automorphism of \( R \) and \( \tau(I) \neq 0 \), we get \([d(x), \tau(x)] = 0, \text{ for all } x, y \in I. \) (43)

The equation (43) is same as equation (12) in theorem 1. Thus, by same argument of theorem 1, we can conclude the result \( I \subseteq Z. \)

**Theorem 10:** Let \( R \) be a prime ring and \( I \) be a non-zero ideal on \( R \). Suppose that \( F \) is a generalized \((\sigma, \tau)\)-reverse derivation on \( R \) associated with \((\sigma, \tau)\)-reverse derivation \( d \) on \( R \) respectively, \( \tau(I) \neq 0 \) and \( \sigma(I) \neq 0 \). If \( F(xy) + F(y)F(x) = 0, \text{ for all } x, y \in I \), then \( I \subseteq Z. \)

**Proof:** We have \( F(xy) + F(y)F(x) = 0, \text{ for all } x, y \in I. \) (44)

We replacing \( x \) by \( xy \) in equation (44), we obtain
\[ F(xyw) + F(y)F(xw) = 0 \]
\[ F(wy)\sigma(x) + \tau(wy)d(x) + F(y)[F(w)\sigma(x) + \tau(w)d(x)] = 0 \]
\[ (F(wy) + F(y)F(w))\sigma(x) + \tau(wy)d(x) + F(y)\tau(w)d(x) = 0, \text{ for all } x, y, w \in I. \]

Using equation (44), it reduces to
\[ \tau(wy)d(x) + F(y)\tau(w)d(x) = 0, \text{ for all } x, y, w \in I. \]  
(45)

We replacing \( y \) by \( zy \) in equation (45), we get
\[ \tau(wzy)d(x) + F(zy)\tau(w)d(x) = 0 \]
\[ \tau(wzy)d(x) + F(y)\sigma(z)\tau(w)d(x) + \tau(y)d(z)\tau(w)d(x) = 0, \text{ for all } x, y, z, w \in I. \]  
(46)

We replacing \( y \) by \( z \) in equation (45), we get
\[ \tau(wz)d(x) + F(z)\tau(w)d(x) = 0, \text{ for all } x, z, w \in I. \]  
(47)

Left multiplying equation (47) by \( \tau(y) \), we get
\[ \tau(y)\tau(wz)d(x) + \tau(y)F(z)\tau(w)d(x) = 0, \text{ for all } x, y, z, w \in I. \]  
(48)

We subtracting equation (48) from equation (46), we get
\[ (\tau(wzy) - \tau(ywz))d(x) + F(y)\sigma(z)\tau(w)d(x) + \tau(y)d(z)\tau(w)d(x) - \tau(y)F(z)\tau(w)d(x) = 0 \]
\[ \tau[wz,y]d(x) + (F(y)\sigma(z) + \tau(y)d(z))\tau(w)d(x) - \tau(y)F(z)\tau(w)d(x) = 0 \]
\[ \tau(\{w, y\}z + w[z, y])d(x) + (F(zy) - \tau(y)F(z))\tau(w)d(x) = 0, \text{ for all } x, y, z, w \in I. \]

We replacing \( z \) by \( y \) and \( w \) by \( y \) in the above equation, we get
\[ (F(yy) - \tau(y)F(y))\tau(y)d(x) = 0, \text{ for all } x, y \in I. \]  
(49)

We replacing \( x \) by \( sx, s \in R \) in equation (49), we get
\[ (F(yy) - \tau(y)F(y))\tau(x)d(s) = 0 \]
\[ (F(yy) - \tau(y)F(y))\tau(x)\sigma(s) + (F(yy) - \tau(y)F(y))\tau(y)\tau(x)d(s) = 0, \text{ for all } x, y, s \in I. \]

Using equation (49) in the above equation, we get
\[ (F(yy) - \tau(y)F(y))\tau(x)d(s) = 0, \text{ for all } x, y, s \in I. \]

We replacing \( x \) by \( rx, r \in R \) in the above equation, we get
\[ (F(yy) - \tau(y)F(y))\tau(x)\tau(s) = 0, \text{ for all } x, y, s \in I. \]
\[ (F(yy) - \tau(y)F(y))\tau(x)d(s) = 0, \text{ for all } x, y, s \in I. \]

Since \( R \) is prime, we get either \( (F(yy) - \tau(y)F(y))\tau(y) \) or \( \tau(x)d(s) = 0 \) for all \( x, s \in I \). Since \( \tau \) is an automorphism of \( R \) and \( \tau(I) \neq 0 \), we have either \( (F(yy) - \tau(y)F(y)) = 0 \) for all \( y \in I \) or \( \tau(x)d(s) = 0 \) for all \( x \in I \).
Now let $A = \{ x \in I / [F(x^2) - \tau(x)F(x)] = 0 \}$ and $B = \{ x \in I / d(x) = 0 \}$. Clearly, $A$ and $B$ are additive proper subgroups of $I$ whose union is $I$. Since a group cannot be the set theoretic union of two proper subgroups. Hence either $A = I$ or $B = I$.

If $B = I$, then $d(x) = 0$, for all $x \in I$, by lemma 2 implies that $I \subseteq Z$.

On the other hand if $A = I$, then $F(x^2) - \tau(x)F(x) = 0$, for all $x \in I$. (50)

We replacing $y$ by $xx$ in equation (44), we get

$$G(xxx) = -F(xx)F(x), \text{ for all } x \in I.$$ (51)

We replacing $x$ by $xx$ and $y$ by $x$ in equation (44), we get

$$G(xxx) = -F(x)F(xx), \text{ for all } x \in I.$$ (52)

From equation (51) and equation (52), we get

$$F(x)F(x^2) = F(x^2)F(x), \text{ for all } x \in I.$$ (50)

Using equation (50), it reduces to

$$F(x)\tau(x)F(x) = \tau(x)F(x)F(x)$$

$$\left(F(x)\tau(x) - \tau(x)F(x)\right)F(x) = 0$$

We conclude that $[F(x), \tau(x)] = 0$, for all $x \in I$. (53)

The equation (53) is same as equation (41) in theorem 9. Thus, by same argument of theorem 9, we can conclude the result $I \subseteq Z$.

References

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