Some Results on Skewnormal Operator in Minkowski Space

Dr.K.Sindhuja [1], Dr.D.Krishnaswamy [2]
[1] *Research Scholar, Department of Mathematics, Annamalai University, Chidambaram, Tamil Nadu, India*
[2] *Professor, Department of Mathematics, Annamalai University, Chidambaram, Tamil Nadu, India*

Abstract:- In this paper, we have studied the relationship between m-symmetric and skew normal operators in Minkowski space M . Further, we have investigated some properties of the skew normal operator with example. In the study of operator theory, the classes of normal and skew normal operators [1,3] have received greater importance. A skew normal operator is nothing but the generalization of normal operator. Such kind of operators are examined and results are derived in this paper.

***Keywords**:* normal operator, polar decomposition, m-symmetric operator, skewnormal operator.

1. Introduction

Minkowski space is a four dimensional space, with fourth dimension as time. In 1918, Toeplitz introduced the concept of a normal matrix with entries from the complex field. And we briefly explained about skewnormal and skew n-normal operators in Minkowski space from Hilbert space with reference to Lalitha [2] and Meenambika [4]. Further, we have studied some properties of the skewnormal operator with example. Finally, relationship between m-symmetric and skewnormal operators in M are determind.

2. Skewnormal Operators in Minkowski Space

In this section, we analyses some of the algebraic results on skewnormal operators in Minkowski space M .

Definition 2.1. *An operator P is called skewnormal operator in M , if* $(PP^{\sim})P = P(P^{\sim}P)$

i.e., P commutes with normal operator and it is denoted by [sN].

Theorem 2.2. *If* $P \in M$ *then the following holds:*

(i) *If* λ *is any scalar which is real then* λP *is also skewnormal operator in* M .

(ii) *The restriction* P/M *of P to any closed subspace M of M that reduces P*

Proof. (i) Since

$(PP^{\sim})P = P(P^{\sim}P)$.

$= [(PQ)GQ^{*}GGP^{*}G](PQ)$

$(\lambda P)^{\sim} = G(\lambda P)^{*}G = \lambda GP^{*}G = \lambda P^{\sim}$ [since, $\lambda^{*} = \lambda$]

Consider, $[(\lambda P)(\lambda P)^{\sim}]\lambda P = \lambda^{3}(PP^{\sim})P$

$\lambda P[(\lambda P)^{\sim}(\lambda P)] = \lambda^{3}P(P^{\sim}P)$

$= \lambda^{3}(PP^{\sim})P$

Hence λP is skewnormal in M.

Consider, $[(P/M)(P/M)^{\sim}](P/M) = [(x + M)(x + M)^{\sim}](x + M)$

$= [(x + M)(Gx^{*}G + M)](x + M)$

$= [xGx^*G + xM + MGx^*G + M^2](x + M)$

$= (xx^{\sim} + xM + Mx^{\sim} + M^2)(x + M)$

$= [(x + M)x^{\sim} + M(x + M)](x + M)$

$= (x + M)(x^{\sim} + M)(x + M)$

$= (P/M)[(P/M)^{\sim}(P/M)]$

Hence P/M is skewnormal operator in Minkowski space M. □

Theorem 2.3. *Let P be skewnormal operator which is unitarily equivalent to an operator Q in M if* $PU = UP$, $PU^{\sim} = U^{\sim}P$, $P^{\sim}U = UP^{\sim}$. *Then Q is skewnormal in M .*

Proof. Since, P is unitarily eqivalent to Q, there is a unitary operator U such that

$Q = U^*PU$ which implies

$Q^{\sim} = (GQ^*G) = GU^*P^*UG$

$(PP^{\sim})P = P(P^{\sim}P)$

$(PGP^*G)P=P(GP^*GP)$

$(U^*UPGP^*U^*UG)(PU^*U)= (U^*UP)(GP^* UGU^*UP)$

$(U^*PUGU^*P^*UG)(U^*PU) =(U^*PU)(GU^*P^*UGU^*PU)$

$((U^*PU)(GU^*P^*UG))(U^*PU) =(U^*PU)((GU^*P^*UG)(U^*PU)$

$(QQ^{\sim})Q = Q(Q^{\sim}Q)$

Theorem 2.4. If P is normal in M then, it is skewnormal in M .

Proof.

An operator P is normal in M if $P^{\sim}P = PP^{\sim}$,

when P commutes with normal operator it becomes,

$(PP^{\sim})P = P(P^{\sim}P)$.

Hence P is skewnormal in M.

Theorem 2.5. *let P, Q are skewnormal operators in M . If P and Q are doubly commuting then PQ is skewnormal in M .*

Proof. Consider,

$P \in [sN] \Rightarrow (PP^{\sim})P= P(P^{\sim}P)$,

$Q \in [sN] \Rightarrow (QQ^{\sim})Q= Q(Q^{\sim}Q)$

P and Q are doubly commuting

(a)To prove: $P^{\sim}Q^{\sim} = Q^{\sim}P^{\sim}$

We have, $P^*Q^* = Q^*P^*$

$\Rightarrow GGP^*GGQ^*GG = GGQ^*GGP^*GG$

$\Rightarrow GP^{\sim}Q^{\sim}G = GQ^{\sim}P^{\sim}G$

$\Rightarrow GGP^{\sim}Q^{\sim}G = GGQ^{\sim}P^{\sim}G$

$\Rightarrow P^{\sim}Q^{\sim} = Q^{\sim}P^{\sim}$ (1.1) (b)To prove $(QG)^{\sim}GP = PG(GQ)^{\sim}$

We have $Q^{*}P = PQ^{*}$

$\Rightarrow GGQ^{*}GGP = PGGQ^{*}GG$

$\Rightarrow GQ^{\sim}GP = PGQ^{\sim}G$

$\Rightarrow (QG)^{\sim}GP = PG(GQ)^{\sim}$ (1.2)

(c)To prove $(PG)^{\sim}GQ = QG(GP)^{\sim}$

We have $P^{*}Q = QP^{*}$

$\Rightarrow GGP^{*}GGQ = QGGP^{*}GG$

$\Rightarrow GP^{\sim}GQ = QGP^{\sim}G$

$\Rightarrow (PG)^{\sim}GQ = QG(GP)^{\sim}$ (1.3)

Now,

$[(PQ)(PQ)^{\sim}](PQ)=[(PQ)G(PQ)^{*}G](PQ)$

$= [(PQ)GQ^{*}P^{*}G](PQ)$

$= (PQ)GQ^{*}GGP^{*}(PQ)$

$= (PQ)Q^{\sim}P^{\sim}(PQ)$

$= PQGGQ^{\sim}GGGGP^{\sim}GGPQ$

$= PQG(QG)^{\sim}IGPGPQ$

$= PQG(PG)(GQ)^{\sim}GPQ$ [*by* 1.2]

$= PQGPGQ^{\sim}G^{\sim}GPQ$

$= PQP^{\sim}Q^{\sim}IPQ$

$= PQQ^{\sim}P^{\sim}PQ$ [*by* 1.1]

$= PQ[Q^{\sim}P^{\sim}(PQ)]$

$= PQ[(PQ)^{\sim}]PQ$

Hence PQ is skewnormal in M .

Theorem 2.6. *Let P and Q are skewnormal operators in M . If $P^{\sim}Q = QP^{\sim}$ and if $PQ^{\sim} = QP^{\sim} = 0$, $(PP^{\sim})Q = Q(P^{\sim}P)$, $(QQ^{\sim})P = P(Q^{\sim}Q)$. Then $P + Q$ is skewnormal in M .*

Proof. Since

$P \in [sN] \Rightarrow (PP^{\sim})P = P(P^{\sim}P)$

$Q \in [sN] \Rightarrow (QQ^{\sim})Q = Q(Q^{\sim}Q)$

$[(P + Q)(P + Q)^{\sim}](P + Q) = [(P + Q)(P^{\sim} + Q^{\sim})](P + Q)$

$= [PP^{\sim} + PQ^{\sim} + QP^{\sim} + QQ^{\sim}](P + Q)$

$= [PP^{\sim} + QQ^{\sim}](P + Q)$

$= (PP^{\sim})P + (QQ^{\sim})P + (PP^{\sim})Q + (QQ^{\sim})Q$ (1.4)

$[(P + Q)(P + Q)^{\sim}](P + Q) = (P + Q)[(P^{\sim} + Q^{\sim})(P + Q)]$

$= (P + Q)[(P^{\sim}P + P^{\sim}Q + Q^{\sim}P + Q^{\sim}Q]$

$= (P + Q)[P^{\sim}P + Q^{\sim}Q]$

$= P(P^{\sim}P) + P(Q^{\sim}Q) + Q(P^{\sim}P) + Q(Q^{\sim}Q)$ (1.5)

From (1.4) and (1.5), and since, P and Q are skewnormal in M and

$(PP^{\sim})Q = Q(P^{\sim}P), (QQ^{\sim})P = P(Q^{\sim}Q)$.

$[(P + Q)(P + Q)^{\sim}](P + Q) = [(P + Q)(P + Q)^{\sim}](P + Q)$.

The following example shows that the converse need not be true.

Example 2.1. Let $P = \begin{pmatrix} 1 & i \\ -i & 0 \end{pmatrix}$ is skewnormal but not normal in M.

Theorem 2.7. *Let, $P = U|P|$ be the polar decomposition of an operator P in M . Then, $P = U|P|$ is skewnormal in M if $U|P| = |P|U$.*

Proof. $P = U|P| \Rightarrow P^{*} = (U|P|)^{*} = |P|*U^{*}$,$P^{\sim} = GP^{*}G = G|P|*U^{*}G$

Assume, $U|P| = |P|U$, then

$(PP^{\sim})P = (U|P|G|P|*U^{*}G)U|P|$

$= (U|P|G|P|*GGU^{*}G)U|P|$

$= (U|P||P|\sim U^{\sim})U|P|$

$= U|P||P|\sim U^{\sim}U|P|$

$= U|P||P|\sim|P|$ (1.6)

$P(P^{\sim}P) = U|P|(G|P|*U^{*}GU|P|)$

$= U|P|(G|P|*GGU^{*}GU|P|)$

$= U|P|(|P|\sim U^{\sim}U|P|)$

$= U|P||P|\sim U^{\sim}U|P|$

$= U|P||P|\sim|P|$ (1.7)

From equation (1.6) and (1.7),

P is skewnormal in M.

Theorem 2.8. *Let P be decomposed as $P = U + iV$ in M , then P is skewnormal in M if $UV^2 = V^2U$ and $U^2V = VU^2$*

Proof.

To prove P is skewnormal in M $\Rightarrow (PP^{\sim})P = P(P^{\sim}P)$

$(PP^{\sim})P = [(U + iV)G(U - iV)G](U + iV)$

$= [(U + iV)(GUG - iGVG)](U + iV)$

$= [GU^2G - iGUVG + iGVUG + GV^2G](U + iV)$

$= [GU^3G - iGU^2VG + iGVU^2G + GV^2UG]$

$+ [iGVU^2G + GV^2UG - iGV^2UG + iGV^3G]$

$= [GU^3G + iGVU^2G + GV^2UG + iGV^3G]$ (1.8)

$P(P^{\sim}P) = (U + iV)[G(U - iV)G(U + iV)]$

$= (U + iV)[(GUG - iGVG)(U + iV)]$

$= (U + iV)[GU^2G - iGUVG + iGVUG + GV^2G]$

$= [GU^3G - iGU^2VG + iGU^2VG + GUV^2G]$

$+ [iGU^2VG + GUV^2G - GUV^2G + iGV^3G]$

$= GU^3G + GUV^2G + iGU^2VG + iGV^3G$ (1.9)

From equation (1.8) and (1.9)

$(PP^{\sim})P = P(P^{\sim}P)$ in M , if $(i) UV^2 = V^2U$ and $U^2V = VU^2$.

3. M-Symmetric Operator and Skewnormal Operator in Minkowski Space

In this section, we have obtained the properties of skewnormal operator with respect to m-symmetric operator in Minkowski space *M.*

Theorem 3.1. *If P is skewnormal operator in M which is a m-symmetric operator, then $P^{\sim}$ is also skewnormal operator in M .*

Proof. Since, P is skewnormal operator, we have $(PP^{\sim})P = P(P^{\sim}P)$. Since, P is m-symmetric operator we have $P = P^{\sim}$

Replace $P^{\sim}$ by P ,

$(PP^{\sim})P = (P^{\sim}(P^{\sim})^{\sim})P^{\sim} = (P^{\sim}P)P^{\sim}$

$P(P^{\sim}P) = P^{\sim}((P^{\sim})^{\sim}P^{\sim}) = P^{\sim}(PP^{\sim})$ [since, P is skewnormal in M]

Hence, $P^{\sim}$ is skewnormal in M .

Theorem 3.2. *If P is m-symmetric operator in M , then P is skewnormal operator in M .*

Proof.

Since, P is m-symmetric operator in M we have $P = P^{\sim}$

Now, $(PP^{\sim})P = (PP)P = P^3$

and $P(P^{\sim}P) = P(PP) = P^3$

Hence, P is skewnormal in M .

Theorem 3.3. *Let P be any operator on a Minkowski space M ,Then*

(i) $(P + P^{\sim})$ *is skewnormal in M .*

(ii) $PP^{\sim}$ *is skewnormal in M .*

(iii) $P^{\sim}P$ *is skewnormal in M .*

(iv) $I + P^{\sim}P, I + PP^{\sim}$ *are skewnormal in M .*

Proof. (i) Let, $N = P + P^{\sim}$

$N^{\sim} = (P + P^{\sim})^{\sim} = P^{\sim} + P = N$

Hence, N is m-symmetric.

By theorem 3.2, every m-symmetric operator is skewnormal in M

Hence $(P + P^{\sim})$ is skewnormal in M .

(ii) $(PP^{\sim})^{\sim} = (P^{\sim})^{\sim}P^{\sim} = PP^{\sim}$.

Hence $PP^{\sim}$ is m-symmetric, and hence skewnormal in M .

(iii) $(P^{\sim}P)^{\sim} = P^{\sim}(P^{\sim})^{\sim} = P^{\sim}P$.

Hence $P^{\sim}P$ is m-symmetric, and hence skewnormal in M .

(v) $(I + P^{\sim}P)^{\sim} = (I^{\sim} + P^{\sim}(P^{\sim})^{\sim}) = (I + P^{\sim}P)$

$(I + PP^{\sim})^{\sim} = (I^{\sim} + (P^{\sim})^{\sim}P^{\sim}) = (I + PP^{\sim})$

Hence, $I + P^{\sim}P$, $I + PP^{\sim}$ are m-symmetric in M ,

and hence $I + P^{\sim}P$, $I + PP^{\sim}$ are Skewnormal in M .

Theorem 3.4. *If P is a m-symmetric operator in M , then $Q^{\sim}PQ$ is skewnormal in M.*

Proof. Since, P is m-symmetric in M , we have $P = P^{\sim}$

Consider, $(Q^{\sim}PQ)^{\sim} = Q^{\sim}P^{\sim}(Q^{\sim})^{\sim} = Q^{\sim}PQ$

$\Rightarrow Q^{\sim}PQ$ is m-symmetric in M .

Hence $Q^{\sim}PQ$ is skewnormal in M .

Another way of proving the above result is: Assume that $Q^{\sim}PQ$ is m-symmetric operator in M ,

$[(Q^{\sim}PQ)(Q^{\sim}PQ)^{\sim}](Q^{\sim}PQ) = [(Q^{\sim}PQ)(Q^{\sim}PQ)](Q^{\sim}PQ) = (Q^{\sim}PQ)^3$

$(Q^{\sim}PQ)[(Q^{\sim}PQ)^{\sim}(Q^{\sim}PQ)] = (Q^{\sim}PQ)[(Q^{\sim}PQ)Q^{\sim}PQ] = (Q^{\sim}PQ)^3$

4. Skew N-Normal Operator in Minkowski Space

In this section, we generalized some results on skewnormal operator in M . □

Definition 4.1. *The operator p is called skew n-normal*

Operator in M if

$(P^nP^{\sim})P = P(P^{\sim}P^n)$ *, where $P^{\sim}$ is the adjoint of the operator P in M*

Proposition 4.2. *If P is skew n-normal operator in M . Then*

(i) *αP is skew n-normal operator in M for every scalar α .*

(ii) *$P^{\sim}$ is skew n-normal operator in M .*

(iii) *If Q is unitarily equiuvalent to P , then Q is skew n-normal operator in M .*

(iv) *If M is closed subspace of M , then (P/M) is skew n-normal operator in M .*

Proposition 4.3. *Let Q be a normal operator and P is skew n-normal operator in M . If Q and P are doubly commuting . Then QP is skew n-normal operator in M .*

Theorem 4.4. *If P is skew n-normal operator, then P is skew $n + k(n-1)$ -normal operator in M , for every positive integer k .*

Proof. Since, P is skew n-normal operator in M , then $(P^nP^{\sim})P = P(P^{\sim}P^n)$

We prove by induction that P is skew $n + k(n-1)$ -normal operator for positive integer

k in M .

when $k = 1$, $(P^{n+(n-1)}P^{\sim})P = P^{(n-1)}(P^nP^{\sim})P$

$(P^2P^{\sim})\,P = P(-P^{\sim})P^{\sim}P$

$=P(P^{\sim})(-P^{\sim})P$

$= P(P^{\sim}P^2)$

$= P^{(n-1)}P(P^{\sim}P^{n})$

$= (Pn\ P\sim)PP(n-1)$

$= P(P\sim Pn)P(n-1)$

$= P(P\sim Pn+(n-1))$

$\Rightarrow (Pn+(n-1)\ P\sim)P = P(P\sim Pn+(n-1))$

(Inductive step): Suppose the result is true for $n = k$

$\Rightarrow (Pn+K(n-1)\ P\sim)P = P(P\sim Pn+K(n-1))$

Now,

$(Pn+(k+1)(n-1)\ P\sim)P = P(n-1)[(Pn+k(n\text{-}1)\ P\sim)P]$

$= P(n-1)[P(P\sim Pn+k(n-1))]$

$= [(Pn\ P\sim)P]Pn+k(n-1)-1$

$= [P(P\sim Pn)]P(n-1)+k(n-1)$

$= [P(P\sim Pn)]P(k+1)(n-1)$

$= P(P\sim Pn+(k+1)(n-1))$

Therefore P is skew $n + (k + 1)(n - 1)$ -normal operator in M .

Corollary 4.5. *If P is skew 2-normal operator in M , then P is skew n-normal operator in M $\forall n \geq 2$.*

Proof. The proof is an immediate consequence of the above theorem. □

Proposition 4.6. *If $P = -P^{\sim}$, then P is skew n-normal operator for every n in M .*

Proof. We show P is skew 2-normal operator in M , By corollary 4.5, P is skew n- normal operator for every n.

Theorem 4.7. *Let $P_1, P_2, \cdots, P_m$ be skew n-normal operators in M . Then*

$(P_1 \oplus P_2 \oplus \ldots \oplus P_m)$ and $(P_1 \otimes P_2 \otimes \cdots P_m)$ are the skew n-normal operators in M .

Proposition 4.8. *Every quasinormal operator is skew n-normal operator in M .*

Proof. Let P be a quasinormal operator $\Rightarrow P(P^{\sim}P) = (P^{\sim}P)P$

Then $P^{(n-1)}$ commutes with quasinormal operartor in M . So that, $(P^{n}P^{\sim})P = PP^{(n-1)}(P^{\sim}P)$

$= P(P^{\sim}P)P^{(n-1)}$

$= P(P^{\sim}P^{n})$

$\Rightarrow P$ is skew n-normal operator in M .

Proposition 4.9. *If P is n-normal operator and quasi n-normal operator in M . Then P is skew n-normal operator in M .*

Proof. Since, P is n-normal operator in $M \Rightarrow P^{n}P^{\sim} = P^{\sim}P^{n}$

P is quasi n-normal operator in $M \Rightarrow P(P^{\sim}P^{n}) = (P^{\sim}P^{n})P$

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