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Some Covering Properties Using Fuzzy Maximal Covers

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Abstract
The aim of this article is to define fuzzy maximal open cover and discuss its few properties. we also defined and study fuzzy m-compact space and discussed its properties.Also we obtain few more results on fuzzy minimal c-regular and fuzzy minimal c-normalspaces. We have proved that a fuzzy Haussdorff m-compact space is fuzzy minimal c- normal.
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1 Introduction

Zadeh[8] established fuzzy set in 1965. Chang[1] introduced fuzzy topology in 1968. Consequent of fuzzy minimal (resp.maximal) open sets[2], Swaminathan developed fuzzy mean open sets in [3]. Swaminathan and Sivaraja studied various comparision resultsin fuzzy minimal, maximal and mean open sets in [4], [5] and [7]. The nature of fuzzy maximal open sets in fuzzy topology having significance in covering properties. Swaminathan and Sivaraja [**?**] introduced fuzzy s-refinement and extended maximal opencovers in fuzzy topology.

In section 2 of this article we study basic notions in fuzzy topology. In section 3 of this article fuzzy weakly m-compact, fuzzy weakly m-Lindelof, fuzzy m-Lindelof, fuzzycountably m-compact and fuzzy m-paracompact space and few properties discussed.

2 Preliminaries

Definition 2.1. ([2]) A proper fuzzy open set μ of X is said to be a fuzzy maximal open set if λ is an fuzzy open set such that $\mu < \lambda$, then $\lambda = \mu$ or $\lambda = 1_X$

Definition 2.2. ([2]) A proper fuzzy open set μ of X is said to be a fuzzy minimal open set if λ is an fuzzy open set such that $\lambda < \mu$, then $\lambda = \mu$ or $\lambda = 0_X$

Definition 2.3. [3] In a fts X, α is called a fuzzy mean open(resp. γ fuzzy mean closed)if $\exists\ \lambda, \mu(\neq \alpha)$ two distinct proper fuzzy open sets (resp. two distinct proper fuzzy closed sets $\beta, \delta(\neq \gamma)$) such that $\lambda < \alpha < \mu$ (resp. $\beta < \gamma < \delta$)

Definition 2.4. [6]Let C and D be two fuzzy covers of a fts X. C is an fuzzy s - refinement of D if $\forall\ \alpha \in C\ \exists\ \beta \in D$ such that $\alpha < \beta$. A fuzzy s -refinement C of D is said to be a fuzzy open s -refinement of D if all members of C and D are fuzzy open.

Definition 2.5. [6]If every FMAO cover of a fts X has a finite fuzzy open s-refinement then X is said to be fuzzy m-compact.

Definition 2.6. [6]A function $f : X \to Y$ for any two FTSs X and Y is said to be fuzzy m-continuous,if inverse image of each proper fuzzy open set in Y is FMAO in X.

Definition 2.7. [6]A fts X is called a fuzzy minimal c-regular if for each $p_x^\alpha \in X$ and each FMIC set γ with $p_x^\alpha \notin \gamma$, there exists disjoint fuzzy open sets λ, μ such that $p_x^\alpha \in \lambda$ and $\lambda < \mu$.

Definition 2.8. [6]A fts X is called a fuzzy minimal c-normal if for each pair of distinctFMIC sets η, γ there exists disjoint fuzzy open sets λ, μ such that $\eta < \lambda$ and $\gamma < \mu$.

Definition 2.9. [6]A fuzzy point p_x^α of a fts X is fuzzy m-complete accumulation point of any fuzzy subset M of X if $|U \wedge M| = |M|$ for each FMAO set U containing p_x^α.

Lemma 2.1. *[6] A fuzzy open cover containing a FMAO set is fuzzy maximal.*

Theorem 2.2. *[2] If α is fuzzy maximal open and β is fuzzy open in X, then either $\alpha \vee \beta = 1$ or $\beta \leq \alpha$. If β is also a fuzzy maximal open set distinct from α, then $\alpha \vee \beta$.*

Theorem 2.3. *[2] If λ is fuzzy minimal closed and μ is fuzzy closed in X, then either $\lambda \wedge \mu = 0$ or $\mu \leq \lambda$. If μ is also a fuzzy minimal closed set distinct from λ, then $\lambda \wedge \mu = 0$.*

Theorem 2.4. *[6] Every infinite T_1 fcts is fuzzy m-compact.*

3 Main Results

Definition 3.1. A fuzzy topological space X is said to be fuzzy weakly m-compact ifeach fuzzy maximal open cover of X has a fuzzy open finite refinement.

A fuzzy subset Y of X is said to be a fuzzy weakly m-compact subset of X if $(Y, \tau_Y)$is fuzzy weakly m-compact.

Theorem 3.1. *Let X be a fuzzy m-compact fuzzy topological space and K be fuzzyminimal closed in X. Then K is fuzzy weakly m-compact.*

Proof: Let $\mathcal{U}$ be a fuzzy maximal open cover of the fuzzy minimal closed set K. For each $U \in \mathcal{U}$, there is a fuzzy open set W in X such that $U = K \cap W$. Since by Lemma 1 and $1 - K$ is a fuzzy maximal open set in X, we write $\mathcal{W} = \{W : U \in \mathcal{U}\} \cup \{1 - K\}$ is a fuzzy maximal open cover of X. By fuzzy m-compactness of X, $\mathcal{W}$ has a fuzzy finite open s-refinement $\{V_1, V_2, ..., V_n\}$. Clearly $\{V_1 \cap K, V_2 \cap K, ..., V_n \cap K\}$ is a fuzzy finite open refinement of $\mathcal{U}$.

Definition 3.2. Let $x_\alpha \in X$ and $U \subseteq X$. A fuzzy point x_α is said to be a fuzzy m - accumulation point of U if for each fuzzy maximal open set containing x_α contains at least one point of U other than x_α.

Theorem 3.2. *Let X be a fuzzy m-accumulation fuzzy topological space. Then everyinfinite fuzzy subset of X has a fuzzy m-accumulation point.*

Proof: Assume that U be an infinite fuzzy subset of X. Let U have no fuzzy m - accumulation point. Then for each $x_\alpha \in X$, there is a fuzzy maximal open set V_{x_α} in X such that $x_\alpha \in V_{x_\alpha}$ and $V_{x_\alpha} \cap U = 0$ or $V_{x_\alpha} \cap U = \{x_\alpha\}$. Now $\mathcal{U} = \{V_{x_\alpha} : x_\alpha \in X\}$ is a fuzzy maximal cover of X (by Lemma 1). By the fuzzy m-compactness of X, there is a finite fuzzy s-refinement W of U. Let $W = \{W_{x_{\alpha_1}}, W_{x_{\alpha_2}}, \ldots, W_{x_{\alpha_n}}\}$. Then $U \subseteq X = \bigcup_{i=1} W_{x_\alpha}$. But for each $i \in \{1, 2, ..., n\}$, $U \bigcap W_{x_{\alpha_i}} = 0$ or $U \bigcap W_{x_{\alpha_i}} = \{x_{\alpha_i}\}$. It implies that cardinality of U is at most n. Which contradicts the fact that U is fuzzy infinite.

Definition 3.3. A fuzzy topological space X is said to be fuzzy m-Lindelof if everyfuzzy maximal open cover of X has a fuzzy open countable s-refinement.

Theorem 3.3. *Let X and Y be a fuzzy topological spaces, where X is fuzzy m-Lindelof and $f : X \to Y$ be a bijective fuzzy m-continuous function. Then Y is also fuzzy m - Lindelof.*

Proof: Let $S^{(Y)}$ be a fuzzy maximal open cover of Y. Since f is a fuzzy bijective m-continuous function, $S^{(X)} = \{ f^{-1}(U) : U \in S^{(Y)}\}$ is a fuzzy maximal open cover of X (by Definition 2.6 and Lemma 2.1). By fuzzy m-Lindelofness of X, $S^{(X)}$ has a fuzzy open countable s-refinement $S^{(X)} = \{W_\beta : \beta \in \Gamma\}$, say where the index set Γ is countable. Since f is bijective, it implies that $S^{(Y)} = \{f(W_\beta) : \beta \in \Gamma\}$ covers Y. Let $f(W_\beta)$ be a member of $S^{(Y)}$. Then $W_\beta \in S^{(X)}$. As $S^{(X)}$ is a fuzzy s-refinement of $S^{(X)}$, we have $W_\beta \subseteq f^{-1}(U)$, for some $U \in S^{(Y)}$. Further f is bijective gives that $f(W_\beta) \subseteq U$. Hence $S^{(Y)}$ is a fuzzy open countable s-refinement of $S^{(Y)}$.

Theorem 3.4. *Let X be a fuzzy m-Lindelof topological space and M be a fuzzy subsetof X with $|M| \geq p$. Then M has a fuzzy complete m-accumulation point.*

Proof: Consider for each $x_\alpha \in X$, there is a fuzzy maximal open set U_α containing x_α and satisfying $|U_{x_\alpha} \cap M| < |M|$. Then $|U_{x_\alpha} \cap M| \leq \omega_0$, for each $x_\alpha \in X$. As $\{U_{x_\alpha} : x_\alpha \in X\}$ is a fuzzy open cover of X consists of fuzzy maximal open sets by Lemma2.1, $\{U_{x_\alpha} : x_\alpha \in X\}$ is a fuzzy maximal open cover of X. Then there is a fuzzy open countable s-refinement $\{U_{x_{\alpha_i}} : x_{\alpha_i}, i \in \Omega\}$, where the index set Ω is a countable subset of $\{U_{x_\alpha} : x_\alpha \in X\}$ Now $|M| = \bigcup_{i \in \Omega} (U_{x_{\alpha_i}} \cap M) \leq \omega_0$. This gives that $|M| \leq \omega_0 \leq p \leq |M|$, which is a contradiction.

Theorem 3.5. *Let X be a fuzzy m-Lindelof topological space and M be a uncountablefuzzy subset of X. Then M has a fuzzy m-accumulation point.*

Proof: If possible, let M have no fuzzy m-accumulation point. Then for each $x_\alpha \in X$,there is a fuzzy maximal open set $V_{x_\alpha} \in X$ such that $x_\alpha \in V_{x_\alpha}$ and $(V_{x_\alpha} \cap M) = 0$

or $(V_{x_\alpha} \cap M) = \{x_\alpha\}$. Now $U = \{V_{x_\alpha} : x_\alpha \in X\}$ is a fuzzy maximal open cover of X (by the Lemma 1). By the fuzzy m-Lindelofness of X, there is a fuzzy open countable s-refinement W of U. Let us write $W = \{W_{x_{\alpha_1}}, W_{x_{\alpha_2}}, \ldots, W_{x_{\alpha_n}}\}$. Then $M = M \cap X = \bigcup (M \cap W_{x_{\alpha_i}})$. But for each $i \in \{1, 2, \ldots, n\}$, $M \cap W_{x_{\alpha_i}} = 0$ or $M \cap W_{x_{\alpha_i}} \subseteq \{x_{\alpha_i}\}$. It gives that cardinality of M is at most ω_0. Which contradicts the fact that M is fuzzy uncountable.

Definition 3.4. A fuzzy topological space X is said to be fuzzy weakly m-Lindelof ifeach fuzzy maximal open cover of X has a fuzzy countable open refinement.

Let $Y \subseteq X$. Then Y is said to be a fuzzy weakly m-compact subset of X if (Y, τ_Y) is fuzzy weakly m-compact.

Theorem 3.6. *Let X be a fuzzy m-Lindelof topological space and K be fuzzy minimalclosed in X. Then K is fuzzy weakly m-Lindelof.*

Proof: Proof is similar to the proof of Theorem 3.1.

Definition 3.5. A fuzzy topological space X is said to be fuzzy countably m-compact ifevery countable fuzzy maximal open cover has a finite fuzzy open s-refinement.

Obviously, fuzzy m-compact topological space is fuzzy countably m-compact.

Theorem 3.7. *Let X be a fuzzy Lindelof topological space containing a fuzzy minimalclosed set K. Then following are equivalent:*

(i) *X is fuzzy m-compact.*

(ii) *X is fuzzy countably m-compact.*

Proof: (i) ⇒ (ii) It is obvious.

(ii) ⇒ (i) Let U be a fuzzy maximal open cover of X. By fuzzy Lindelofness of X, U has a countable fuzzy subcollection W, say, that covers X. Then by the Lemma 1, $W \cup \{1 - K\}$ is a countable fuzzy maximal open cover of X. By the countably fuzzy m-compactness of X, $W \cup \{1 - K\}$ has a finite fuzzy open s-refinement of X, i.e., X is fuzzy m-compact.

Theorem 3.8. *An infinite fuzzy T_1-connected topological space is countably fuzzy m-compact.*

Proof: Proof follows from Theorem 2.4.

Definition 3.6. A fuzzy topological space X is said to be a fuzzy m-paracompact topological space if each fuzzy maximal open cover of X has a fuzzy open locally finite fuzzy s-refinement.

Theorem 3.9. *If X is a fuzzy m-paracompact topological space, then each fuzzy maximalopen cover of X has a fuzzy open locally finite fuzzy s-refinement.*

Proof: Proof is trivial.

Lemma 3.10. *Let U be a fuzzy s-refinement (resp. fuzzy refinement) of W and W bea refinement(resp., fuzzy s-refinement) of V. Then U is a fuzzy s-refinement of V*

Proof: Obvious.

Theorem 3.11. *If X is a fuzzy m-paracompact topological space, then each fuzzymaximal open cover of X has a locally finite fuzzy s-refinement (not necessarily open).*

Proof: Proof follows from the Lemma 3.10..

Theorem 3.12. *A fuzzy Hausdorff m-paracompact topological space is fuzzy minimalm-regular.*

Proof: Let X be a fuzzy Hausdorff m-compact fuzzy topological space. Suppose $K \in X$ be a fuzzy minimal closed set and $x_\alpha \in X$ such that $x_\alpha \notin K$. Then for each $z_\beta \in K$ there exists disjoint fuzzy open sets $U_{z\beta}, V_{z\beta}$ such that $x_\alpha \in U_{z\beta}$ and $z_\beta \in V_{z\beta}$. Clearly$x_\alpha \notin cl(V_{z\beta})$. Then $V = \{V_{z\beta} : z_\beta \in K\} \cup \{1 - K\}$ is a fuzzy maximal open cover of X, by the Lemma 1. Since X is fuzzy m-paracompact, there is a fuzzy open locally finite$s$-refinement W say $V = \{V_{z\beta} : z_\beta \in K\} \cup \{1 - K\}$.

Let $V = \bigcup\{W \in W \mid W \cap K \neq 0\}$. Then V is a fuzzy open set which contains K. Since $\{W \in W \mid W \cap K \neq 0\}$ is a subcollection of a fuzzy locally finite family, it is fuzzy locally finite and therefore $cl(V) = \bigcup\{cl(W) : W \in W \mid W \cap K \neq 0\}$. Now for each $W \in W$, there is a $V_{z\beta} \in V$ such that $W \subseteq V_{z\beta}$ such that $W \subseteq V_{z\beta}$, that is $cl(W) \subseteq cl(V_{z\beta})$. Thus $x_\alpha \notin cl(V)$ i.e., $x_\alpha \in 1 - cl(V)$. Thus X is fuzzy minimal c-regular.

Corollary 3.13. *A fuzzy Hausdorff m-paracompact topological space is fuzzy minimalfuzzy c-normal.*

Proof: Let G and K distinct fuzzy minimal closed sets in fuzzy Hausdorff m - paracompact topological space. For each $z_\beta \in G$, by Theorem 3.12, there exist disjoint fuzzy open sets $U_{z\beta}$ and $V_{z\beta}$ such that $z_\beta \in V_{z\beta}$ and $W \subseteq V_{z\beta}$. By Lemma 2.1,$U = \{U_{z\beta} : z_\beta \in G\} \cup \{1 - G\}$ is a fuzzy maximal cover of X. Now if we proceed in a similar way as the proof of the Theorem, we can get disjoint fuzzy open sets U, Vsuch that $G \subseteq U$ and $K \subseteq V$.

Definition 3.7. A fuzzy topological space X is said to be a fuzzy weakly m-paracompact if each fuzzy maximal open cover of X has a fuzzy open locally finite s-refinement.

Let $Y \subseteq X, Y$ is said to be a fuzzy weakly m-paracompact subset of X if (Y, τ_Y) is fuzzy weakly m-paracompact.

Theorem 3.14. *Let X be a fuzzy m-paracompact topological space and K be fuzzy minimal closed in X. Then K is fuzzy weakly m-paracompact*

Proof: Proof is trivial.

Theorem 3.15. *Let X be a fuzzy topological space in which all proper open sets are fuzzymean open. Then X is fuzzy m-compact.*

Proof: If possible, let X be not fuzzy m-compact. Then there exists a fuzzy maximal open cover A of X which has no fuzzy open finite s-refinement. Now by the Definition 2.3,for each $A \in A$ we have can a proper fuzzy open set W such that $A \subsetneq W$. Then A is a fuzzy s-refinement of $\{W : A \in A\}$, which contradicts the fuzzy maximality of A. Thus X is fuzzy m-compact.

Corollary 3.16. *Let X be a fuzzy topological space in which all proper fuzzy open sets are fuzzy mean open. Then X is fuzzy m-Lindelof, fuzzy countably m-compact, fuzzy m-paracompact,fuzzy weakly m-compact, fuzzy weakly m-Lindelof and fuzzy weakly m- paracompact.*

Proof: Obvious.

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