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Oscillation Condition for First Order Non-Linear Delay Differential Equations with Deviating Arguments and Oscillatory Coefficients

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***Abstract*:**

In this paper, we obtain a new oscillation condition for the first order non-linear delay differential equation of the form

$z'(t) + q(t)f(z(\tau(t))) = 0$, $t \geq t_0 > 0$

where q and τ are continuous functions on $[t_0, \infty)$, $t_0 > 0$, $q(t) \geq 0$ and $\lim_{t\to\infty} \tau(t) = \infty$ and

$f \in C(R, R)$, $\tau(t) < t$ and $zf(z) > 0$ for $z \neq 0$.Without imposing the nonnegative restriction on the coefficient functions q(t) a new sufficient oscillation criterion is obtained . An example illustrating the result is also given.

***Keywords*:** Delay differential equation, Non-monotone delays, Oscillatory solutions.

1. Classification

In the recent times, the study of the asymptotic and oscillatory behaviour of solutions of delay differential equations is concerned as major area of research. This is due to the development in science and technology and the challenges that the new classes of such equations provide in these application areas. Equations involving delay, and those involving advance and a combination of both arise in the models on lossless transmission lines in high speed computers which are used to interconnect switching circuits. The construction of these models using delays is complemented by the mathematical investigation of nonlinear equations. Moreover, the delay differential equations play an important role in many fields such as mathematical biology, economics, physics , biology, see [16,19,28,41].Oscillation phenomena appear in various models from real world applications; see[12,36,39] for models from mathematical biology where oscillation or delay actions may be formulated by means of cross – diffusion terms ,see [1 – 49].In particular ,the oscillation criteria of first – order differential equations with deviating arguments have many applications in the study of higher – order functional differential equations, see [13,37,38].

Recently, there has been a great interest in studying the oscillation of all the solutions of the first order delay differential equation of the form

$$z'(t) + q(t)f(z(\tau(t))) = 0, \quad t \geq t_0 > 0 \qquad (1.1)$$

where q and τ are continuous on $[t_0, \infty)$, $t_0 > 0$, $q(t) \geq 0$ and $\lim_{t\to\infty} \tau(t) = \infty$ and $f \in C(R, R)$,

$\tau(t) < t$ and $zf(z) > 0$ for $z \neq 0$. By a solution of (1.1) we mean a function which is continuous on $[\overline{t_*}, \infty]$ for some $t_* \geq t_0$, where $\overline{t_*} = \inf\{\tau(t): t \geq t_*\}$ and satisfies (1.1) for all $t \geq t_*$.

A solution of (1.1) is said to be oscillatory, if it has arbitrarily large zeroes. Otherwise, it is called non oscillatory.

In most of these works, the delay function is assumed to be nondecreasing ,see [14,16,29,31 -34 , 46,49] and the references therein. As shown in [8], the oscillation character of (1.1) with nonmonotone delay, is not an easy extension to the oscillation problem for the nondecreasing delay case. Many authors [1,3,5 – 11,15,20,25,27,40,47] have developed and generalized the methods used to study the oscillation of equation (1.1) with monotone delays and to study this property for the nonmonotone case. Only a few works, however, dealt with the oscillation of equation (1.1) with oscillatory coefficients. In [16, 48] the authors studied the oscillation of (1.1) where the delay function τ(t) is assumed to be nondecreasing and constant (i.e., $\tau(t) = t - \alpha$, $\alpha > 0$), respectively. Also, Kulenovic and Grammatikopoulos [29] studied the oscillation of a first-order nonlinear functional differential equation that contains (1.1). The authors obtained limit infimum and limit supremum including oscillation criteria for the case when the coefficient function does not need to be nonnegative.

However, the delay and the coefficient functions are assumed to be nondecreasing and nonnegative on a sequence of intervals $\{(r_n, s_n)\}$, $n \geq 0$ such that $\lim_{n\to\infty} (s_n - r_n) = \infty$, respectively. Our aim in this paper is to obtain oscillation criteria for equation (1.1) where q(t) is a continuous function on $[t_0, \infty)$. We relax the nonnegative restriction on the coefficient functions q(t). To accomplish this goal, using the ideas of [27], we develop and enhance the work of Kwong [30]. This procedure leads to new sufficient oscillation criteria that improve and generalize those mentioned in [16, 29, 48].

2. Preliminary Results

In 2017, Ozkan Ocalan [43] proved the following theorem.

Theorem 2.1: (See [43], Theorem 2.1)

Suppose τ(t) is non – monotone or non – decreasing function. Set $h(t) = \sup_{s\leq t} \tau(s)$, $t \geq 0$.

Suppose that f in equation (1.1) satisfies the following condition

$$\lim_{z\to 0}\sup \frac{z}{f(z)} = \Omega \ , 0 \leq \Omega < \infty . \tag{2.1}$$

If $\lim_{t\to\infty}\inf \int_{\tau(t)}^{t} q(s)ds > \frac{\Omega}{e}$ holds, (2.2)

then all solutions of (1.1) oscillate.

If there exists a non-oscillatory solution z(t) of (1.1) , then -z(t) is also a solution of (1.1).The discussion is confined only to the case where the solution is eventually positive.

Thus from (1.1), we have $z'(t) + p(t)\, f(z(\tau(t))) \leq 0$ for all $t \geq t_1$. It means that z(t) is positive, nonincreasing and has a limit $\ell \geq 0$ as $t \to \infty$. Then, there exists a $t_1 > t_0$ such that $z(t) > 0$, $z(\tau(t)) > 0$ for all $t \geq t_1$.

Now we claim that $\ell = 0$. Condition (2.10) implies that

$$\int_a^\infty p(t)\, dt = \infty.$$

By [43, Theorem 3.1.5], we obtain $\lim_{t\to\infty} z(t) = 0$.

Then, by virtue of (2.1) we can choose $t_2 > t_1$ so large that

$$f(z(t)) \geq \frac{1}{3\Omega} z(t) \text{ for } t \geq t_2.$$

Since $\tau(t) \leq h(t)$ and z(t) is non increasing , by (1.1) we have

$$z'(t) + q(t)\left(\frac{1}{3\Omega} z(\tau(t))\right) \leq 0 , \quad t \geq t_3. \tag{2.3}$$

3. Main Result

Let M(t) and M_i (t) ,t ≥ 0,i ϵ N be defined as follows:

$$M(t) = \max \{u \geq t: \tau(u) \leq t\} \quad (3.1)$$

$M_1(t) = M(t)$,

$M_i(t) = M(t) \circ M_{i-1}(t)$, i = 2,3…

Also ,we define a function h(t) and a sequence $\{Q_n(v,u)\}_{n=0}^{\infty}$, $\tau(v) \leq u \leq v$, as follows.

Let $h(t) = \sup_{u \leq t} \tau(u)$, $t \geq t_0$. (3.2)

And

$Q_0(v,u) = 1$,

$Q_n(v,u) = e^{\int_u^v \frac{q(\zeta)}{3\Omega} Q_{n-1}(\zeta,\tau(\zeta))\, d\zeta}$, n ϵ N.

Lemma 3.1:

Let n ϵ N_0 , $N_0 = \{0,1,2\ldots\}$, $T \geq T^* > t_0$ and z(t) be a solution of (1.1) such that z(t) > 0 for all $t \geq T^*$.

If $q(t) \geq 0$ on $[T, T_1]$, $T_1 \geq M_{n+2}(T)$, then

$$\frac{z(u)}{z(v)} \geq Q_n(v,u) , \tau(v) \leq u \leq v, \text{ for } v \in [M_{n+2}(T), T_1] \quad (3.3)$$

Proof:

It follows from (1.1) that $z'(t) \leq 0$ on $[M_1(T),T_1]$.

Thus $z(u) \geq z(v)$ as $\tau(v) \leq u \leq v$ for v ϵ $[M_2(T),T_1]$.

Therefore, $\frac{z(u)}{z(v)} \geq 1 = Q_0(v,u)$, $\tau(v) \leq u \leq v$, for v ϵ $[M_2(T),T_1]$.

Dividing (2.3) by z(t) and integrating from u to v, $\tau(v) \leq u \leq v$ and using (2.1), we get

$$\int_u^v \frac{z'(\zeta)}{z(\zeta)} d\zeta + \int_u^v \frac{q(\zeta)}{3\Omega} \frac{z(\tau(\zeta))}{z(\zeta)} d\zeta \leq 0$$

or

$$\ln \frac{z(u)}{z(v)} \geq \int_u^v \frac{q(\zeta)}{3\Omega} \frac{z(\tau(\zeta))}{z(\zeta)} d\zeta$$

or

$$\frac{z(u)}{z(v)} \geq e^{\int_u^v \frac{q(\zeta)}{3\Omega} \frac{z(\tau(\zeta))}{z(\zeta)} d\zeta} . \quad (3.4)$$

As $\tau(\zeta) \leq \zeta$ and $z'(t) \leq 0$ on $[M_1(T),T_1]$, we have

$$\frac{z(u)}{z(v)} \geq e^{\int_u^v \frac{q(\zeta)}{3\Omega} d\zeta} = e^{\int_u^v \frac{q(\zeta)}{3\Omega} Q_0(\zeta,\tau(\zeta))\, d\zeta}$$

$= Q_1(v,u)$, $\tau(v) \leq u \leq v$, for v ϵ $[M_3(T),T_1]$ and consequently, for $u \leq \zeta \leq v$, we have

$$\frac{z(\tau(\zeta))}{z(\zeta)} \geq Q_1(\zeta,\tau(\zeta)), \tau(v) \leq u \leq v, \text{ for } v \in [M_4(T),T_1].$$

Substituting in (3.4), we get

$$\frac{z(u)}{z(v)} \geq e^{\int_u^v \frac{q(\zeta)}{3\Omega} \cdot Q_1(\zeta,\tau(\zeta))\, d\zeta}$$

$= Q_2(v,u)$ for v ϵ $[M_4(T),T_1]$.

Repeating this argument n times, we obtain

$$\frac{z(u)}{z(v)} \geq e^{\int_u^v \frac{q(\zeta)}{3\Omega} \cdot Q_{n-1}(\zeta,\tau(\zeta))\, d\zeta}$$

$= Q_n(v,u)$ for $t \in [M_{n+2}(T), T_1]$.

The proof of the lemma is complete. □

Let $\{T_k\}_{k\geq 0}$ be a sequence of real numbers such that $\lim_{k\to\infty} T_k = \infty$ and $q(t) \geq 0$ for

$t \in [T_k, M_{n+4}(T)]$ for all $k \in N$, for some $n \in N_0$. (3.5)

Also, we define the sequence $\{\beta_n\}_{n\geq 1}$, $\beta_n > 1$, for all $n \in N$ as follows:

$Q_n(t,h(t)) > \beta_n$, $t \in [h(M_{n+3}(T_k)), (M_{n+3}(T_k)]$ for all $k \in N_0$ for some $n \in N$. (3.6)

Theorem 3.1:

Suppose (3.5) and (3.6) are satisfied. If

$$\int_{h(M_{n+4}(T_k))}^{M_{n+4}(T_k)} q(\zeta)\; Q_{n+1}(h(\zeta),\tau(\zeta))\, d\zeta \geq 3\Omega \left(\frac{\ln(\beta_{n+1}) + 1}{\beta_{n+1}}\right) \text{ for all } n \in N \text{ and } k \in N_0, \quad (3.7)$$

then every solution of (1.1) is oscillatory.

Proof:

On the contrary, suppose that z(t) is an eventually positive solution of (1.1).Then there exists a sufficiently large $T^* > t_0$, such that $z(t) > 0$ for all $t > T^*$. Suppose that $T_{k_1} \in \{T_k\}_{k\geq 0}$

such that $T_{k_1} > T^*$.From (3.3), (3.5) and (3.6),

$$\frac{z(h(M_{n+4}(T_{k_1})))}{z(M_{n+4}(T_{k_1}))} \geq Q_{n+1}(M_{n+4}(T_{k_1}),$$

$h(M_{n+4}(T_{k_1})) > \beta_{n+1} > 1$. Then there exists $t_* \in (h(M_{n+4}(T_{k_1})), M_{n+4}(T_{k_1}))$ such that

$$\frac{z(h(M_{n+4}(T_{k_1})))}{z(t_*)} = \beta_{n+1} \quad (3.8)$$

Integrating (2.3) from t_* to t gives

$$\int_{t_*}^t z'(\zeta)\, d\zeta + \int_{t_*}^t \frac{q(\zeta)}{3\Omega} z(\tau(\zeta))\, d\zeta \leq 0,$$

or

$$z(M_{n+4}(T_{k_1})) - z(t_*) + \int_{t_*}^{M_{n+4}(T_{k_1})} \frac{q(\zeta)}{3\Omega} z(\tau(\zeta))\, d\zeta \leq 0, \quad (3.9)$$

where we have used the monotonicity of z .

Now dividing (2.3) by z(t) and integrating between $\tau(\zeta)$ and $h(\zeta)$ gives

$$\int_{\tau(\zeta)}^{h(\zeta)} \frac{z'(\zeta_1)}{z(\zeta_1)}\, d\zeta_1 + \int_{\tau(\zeta)}^{h(\zeta)} \frac{q(\zeta_1)}{3\Omega} \frac{z(\tau(\zeta_1))}{z(\zeta_1)}\, d\zeta_1 \leq 0.$$ That is,

$$\ln \frac{z(h(\zeta))}{z(\tau(\zeta))} + \int_{\tau(\zeta)}^{h(\zeta)} \frac{q(\zeta_1)}{3\Omega} \frac{z(\tau(\zeta_1))}{z(\zeta_1)}\, d\zeta_1 \leq 0$$

That is,

$$\frac{z(\tau(\zeta))}{z(h(\zeta))} \geq e^{\int_{\tau(\zeta)}^{h(\zeta)} \frac{q(\zeta_1)}{3\Omega} \frac{z(\tau(\zeta_1))}{z(\zeta_1)} d\zeta_1}.$$

Or

$$z(\tau(\zeta) \geq z(h(\zeta))\, e^{\int_{\tau(\zeta)}^{h(\zeta)} \frac{q(\zeta_1)}{3\Omega}\frac{z(\tau(\zeta_1))}{z(\zeta_1)}d\zeta_1}. \qquad (3.10)$$

Using (3.10) in (3.9) ,we get

$$z(M_{n+4}(T_{k_1})) - z(t_*) + \int_{t_*}^{M_{n+4}(T_{k_1})} \frac{q(\zeta)}{3\Omega}\, e^{\int_{\tau(\zeta)}^{h(\zeta)} \frac{q(\zeta_1)}{3\Omega}\frac{z(\tau(\zeta_1))}{z(\zeta_1)}d\zeta_1}\, z(h(\zeta))\, d\zeta \leq 0.$$

Since $z'(t) \leq 0$ on $[M_1(T_{k_1}), M_{n+4}(T_{k_1})]$, it follows that

$$z(M_{n+4}(T_{k_1})) - z(t_*) + z(h(M_{n+4}(T_{k_1}))) \int_{t_*}^{M_{n+4}(T_{k_1})} \frac{q(\zeta)}{3\Omega}\, e^{\int_{\tau(\zeta)}^{h(\zeta)} \frac{q(\zeta_1)}{3\Omega}\frac{z(\tau(\zeta_1))}{z(\zeta_1)}d\zeta_1}\, d\zeta \leq 0. \quad (3.11)$$

By (3.1) and $\zeta_1 \in [M_{n+2}(T_{k_1}), (M_{n+3}(T_{k_1})]$, for $\tau(\zeta) < \zeta_1 < h(\zeta)$,

$h(M_{n+4}(T_{k_1}) < \zeta < M_{n+4}(T_{k_1})$, we get

$$z(h(M_{n+4}(T_{k_1}))) \int_{t_*}^{M_{n+4}(T_{k_1})} \frac{q(\zeta)}{3\Omega}\, e^{\int_{\tau(\zeta)}^{h(\zeta)} \frac{q(\zeta_1)}{3\Omega}\frac{z(\tau(\zeta_1))}{z(\zeta_1)}d\zeta_1}\, d\zeta \leq z(t_*) - z(M_{n+4}(T_{k_1})).$$

So,

$$\int_{t_*}^{M_{n+4}(T_{k_1})} \frac{q(\zeta)}{3\Omega}\, e^{\int_{\tau(\zeta)}^{h(\zeta)} \frac{q(\zeta_1)}{3\Omega}\frac{z(\tau(\zeta_1))}{z(\zeta_1)}d\zeta_1}\, d\zeta \leq \frac{z(t_*)}{z(h(M_{n+4}(T_{k_1})))} - \frac{z(M_{n+4}(T_{k_1}))}{z(h(M_{n+4}(T_{k_1})))}.$$

Since $\frac{z(M_{n+4}(T_{k_1})}{z(h(M_{n+4}(T_{k_1})))} > 0$, we have

$$\int_{t_*}^{M_{n+4}(T_{k_1})} \frac{q(\zeta)}{3\Omega}\, e^{\int_{\tau(\zeta)}^{h(\zeta)} \frac{q(\zeta_1)}{3\Omega}\frac{z(\tau(\zeta_1))}{z(\zeta_1))}d\zeta_1}\, d\zeta < \frac{z(M_{n+4}(T_{k_1})}{z(h(M_{n+4}(T_{k_1})))} = \frac{1}{\beta_{n+1}} \qquad (3.12)$$

Again, dividing (2.3) by z(t) and integrating between $h(M_{n+4}(T_{k_1}))$ and t_*, we obtain

$$\int_{h(M_{n+4}(T_{k_1}))}^{t_*} \frac{z'(\zeta)}{z(\zeta)} d\zeta + \int_{h(M_{n+4}(T_{k_1}))}^{t_*} \frac{q(\zeta)}{3\Omega}\frac{z(\tau(\zeta))}{z(\zeta)} d\zeta \leq 0,$$

or

$$\ln \frac{z(h(M_{n+4}(T_{k_1})))}{z(t_*)} \geq \int_{h(M_{n+4}(T_{k_1}))}^{t_*} \frac{q(\zeta)}{3\Omega}\frac{z(\tau(\zeta))}{z(\zeta)} d\zeta$$

$$\geq \int_{h(M_{n+4}(T_{k_1}))}^{t_*} \frac{q(\zeta)}{3\Omega} \frac{z(h(\zeta))\, e^{\int_{\tau(\zeta)}^{h(\zeta)} \frac{q(\zeta_1)}{3\Omega}\frac{z(\tau(\zeta_1))}{z(\zeta_1))}d\zeta_1}}{z(\zeta)} d\zeta, \qquad (3.13)$$

where we have used (3.10).

Using (3.3) in the last inequality ,we get

$$\ln \frac{z(h(M_{n+4}(T_{k_1})))}{z(t_*)} \geq \int_{h(M_{n+4}(T_{k_1}))}^{t_*} \frac{q(\zeta)}{3\Omega} Q_{n+1}(\zeta, h(\zeta))\, e^{\int_{\tau(\zeta)}^{h(\zeta)} \frac{q(\zeta_1)}{3\Omega} Q_n(\zeta_1, \tau(\zeta_1))d\zeta_1}\, d\zeta$$

or

$$\ln \frac{z(h(M_{n+4}(T_{k_1})))}{z(t_*)} \geq \beta_{n+1} \int_{h(M_{n+4}(T_{k_1}))}^{t_*} \frac{q(\zeta)}{3\Omega}\, e^{\int_{\tau(\zeta)}^{h(\zeta)} \frac{q(\zeta_1)}{3\Omega} Q_n(\zeta_1, \tau(\zeta_1))d\zeta_1}\, d\zeta$$

$$\beta_{n+1} \int_{h(M_{n+4}(T_{k_1}))}^{t_*} \frac{q(\zeta)}{3\Omega}\, e^{\int_{\tau(\zeta)}^{h(\zeta)} \frac{q(\zeta_1)}{3\Omega} Q_n(\zeta_1, \tau(\zeta_1))d\zeta_1}\, d\zeta \leq \ln(\beta_{n+1}) \text{ by (3.8)}$$

That is,

$$\int_{h(M_{n+4}(T_{k_1}))}^{t_*} \frac{q(\zeta)}{3\Omega}\, e^{\int_{\tau(\zeta)}^{h(\zeta)} \frac{q(\zeta_1)}{3\Omega} Q_n(\zeta_1, \tau(\zeta_1))d\zeta_1}\, d\zeta \leq \frac{\ln(\beta_{n+1})}{\beta_{n+1}} \qquad (3.14)$$

Combining (3.12) and (3.14),

$$\int_{h(M_{n+4}(T_{k_1}))}^{M_{n+4}(T_{k_1})} \frac{q(\zeta)}{3\Omega} e^{\int_{\tau(\zeta)}^{h(\zeta)} \frac{q(\zeta_1)}{3\Omega} Q_n(\zeta_1,\tau(\zeta_1))d\zeta_1} d\zeta < \frac{\ln(\beta_{n+1})}{\beta_{n+1}} + \frac{1}{\beta_{n+1}}$$

$$\int_{h(M_{n+4}(T_{k_1}))}^{M_{n+4}(T_{k_1})} q(\zeta)\, e^{\int_{\tau(\zeta)}^{h(\zeta)} \frac{q(\zeta_1)}{3\Omega} Q_n(\zeta_1,\tau(\zeta_1))d\zeta_1} d\zeta < 3\Omega \left[\frac{\ln(\beta_{n+1})+1}{\beta_{n+1}} \right]$$

$$\int_{h(M_{n+4}(T_{k_1}))}^{M_{n+4}(T_{k_1})} q(\zeta)\; Q_{n+1}(h(\zeta),\tau(\zeta))\; d\zeta < 3\Omega \left[\frac{\ln(\beta_{n+1})+1}{\beta_{n+1}} \right]$$ which is a contradiction to (3.7).

Hence the proof of the theorem is complete.

Theorem 3.2:

Suppose (3.5) holds.

If $\int_{h(M_{n+4}(T_k))}^{M_{n+4}(T_k)} \frac{q(\zeta)}{3\Omega} Q_{n+1}(h(M_{n+4}(T_k)), T(\zeta))\, d\zeta \geq 1$ for all $k \in N_0$. (3.15)

Then , every solution of (1.1) is oscillatory.

Proof:

Let z(t) be an eventually positive solution of (1.1).Then there exists a $T^* > t_0$ such that $z(t) > 0$ for all $t \geq T^*$.

Integrating (2.3) from $h(M_{n+4}(T_{k_1}))$ to $M_{n+4}(T_{k_1})$ and using (3.10), we get

$z(M_{n+4}(T_{k_1})) - z(h(M_{n+4}(T_{k_1})))+$

$$z(h(M_{n+4}(T_{k_1}))) \int_{h(M_{n+4}(T_{k_1}))}^{M_{n+4}(T_{k_1})} \frac{q(\zeta)}{3\Omega}\, e^{\int_{\tau(\zeta)}^{h(M_{n+4}(T_{k_1}))} \frac{q(\zeta_1)}{3\Omega}\frac{z(\tau(\zeta_1))}{z(\zeta_1)} d\zeta_1} d\zeta \leq 0.$$

$$z(M_{n+4}(T_{k_1})) + z(h(M_{n+4}(T_{k_1}))) \left[\int_{h(M_{n+4}(T_{k_1}))}^{M_{n+4}(T_{k_1})} \frac{q(\zeta)}{3\Omega}\, e^{\int_{\tau(\zeta)}^{h(M_{n+4}(T_{k_1}))} \frac{q(\zeta_1)}{3\Omega}\frac{z(\tau(\zeta_1))}{z(\zeta_1)} d\zeta_1} d\zeta - 1 \right] \leq 0 .$$

Since $z(M_{n+4}(T_{k_1}))$ is positive , we have

$$\int_{h(M_{n+4}(T_{k_1}))}^{M_{n+4}(T_{k_1})} \frac{q(\zeta)}{3\Omega}\, e^{\int_{\tau(\zeta)}^{h(M_{n+4}(T_{k_1}))} \frac{q(\zeta_1)}{3\Omega}\frac{z(\tau(\zeta_1))}{z(\zeta_1)} d\zeta_1} d\zeta < 1.$$

Then by using (3.3)

$\int_{h(M_{n+4}(T_{k_1}))}^{M_{n+4}(T_{k_1})} \frac{q(\zeta)}{3\Omega}\, e^{\int_{\tau(\zeta)}^{h(M_{n+4}(T_{k_1}))} \frac{q(\zeta_1)}{3\Omega} Q_n(\zeta_1,\tau(\zeta_1))\, d\zeta_1} d\zeta < 1$.Now using the definition of $Q_n(v,u)$, the last inequality becomes

$\int_{h(M_{n+4}(T_k))}^{M_{n+4}(T_k)} \frac{q(\zeta)}{3\Omega} Q_{n+1}(h(M_{n+4}(T_k)), T(\zeta))\, d\zeta < 1$,which contradicts (3.15).

The proof of the theorem is complete. □

Example:

Consider the delay differential equation

$z'(t) + q(t)z(\tau(t)) \ln(9 + |z(\tau(t))|) = 0, \; t \geq 1,$ (3.16)

where $q \in C([1,\infty),R)$ such that $q(t) = \eta > 0$ for $t \in [\, 3r_k , 3r_k + \frac{161}{30}]$ for all $k \in N_0$,

$\{r_k\}_{k\geq 0}$ is a sequence of positive integers such that $r_{k+1} > r_k + \frac{53}{30}$ and $\lim_{k\to\infty} r_k = \infty$, and with

$$\tau(t) = \begin{cases} t-1, & t \in [3\ell, 3\ell+2] \\ -t+6\ell+3, & t \in [3\ell+2, 3\ell+2.1],\ \ell \in N_0 \\ \frac{11}{9}t - \frac{2}{3}\ell - \frac{5}{3}, & t \in [3\ell+2.1, 3\ell+3] \end{cases}$$

Here $f(z) = z(\tau(t)) \ln(9 + |z(\tau(t))|$

In view of (3.1) and (3.2) ,we have

$$h(t) = \begin{cases} t-1, & t \in [3\ell, 3\ell+2] \\ 3\ell+1, & t \in [3\ell+2, 3\ell+\frac{24}{11}],\ \ell \in N_0 \\ \frac{11}{9}t - \frac{2}{3}\ell - \frac{5}{3}, & t \in [3\ell+\frac{24}{11}, 3\ell+3] \end{cases}$$

and

$$M(t) = \begin{cases} t+1, & t \in [3\ell, 3\ell+0.9] \\ \frac{9}{11}t + \frac{6}{11}\ell + \frac{15}{11}, & t \in [3\ell+0.9, 3\ell+2],\ \ell \in N_0 \\ t+1, & t \in [3\ell+2, 3\ell+3] \end{cases}$$

respectively.

Letting $T_k = 3r_k$, $k \in N_0$, so $M_5(T_k) = 3r_k + \frac{161}{30}$,

and we have

$$q(t) = \eta \quad \text{for } t \in [T_k, M_5(T_k)] \text{ for all } k \in N_0 \tag{3.17}$$

we have

$h((M_5(T_k)) = 3r_k + \frac{131}{30}$

and

$t - 1.2 \le \tau(t) \le h(t) \le t - 1$ for all $t \ge 1$.

Therefore,

$$Q_2(t, h(t)) = e^{(\int_{h(t)}^{t} q(\zeta) e^{(\int_{\tau(\zeta)}^{\zeta} q(\zeta_1)\, d\zeta_1)} d\zeta}$$

$$\ge e^{(\int_{t-1}^{t} q(\zeta)\, e^{(\int_{\zeta-1}^{\zeta} q(\zeta_1) d\zeta_1)}\, d\zeta)}$$

$$\ge e^{(\eta e^{(\eta)})} \quad \text{for } t \in [3r_k + \tfrac{131}{30}, 3r_k + \tfrac{161}{30}].$$

Denote $\beta_2 = e^{(\eta e^{(\eta)})} > 1$.

Then , $Q_2(t,h(t)) > \beta_2$ for $t \in [h((M_5(T_k)), M_5(T_k)]$ for all $k \in N_0$. (3.18)

Also ,

$$\int_{h((M5(Tk))}^{M5(Tk)} q(\zeta) Q_2(h(\zeta), \tau(\zeta))\, d\zeta = \int_{3rk+\frac{131}{30}}^{3rk+\frac{161}{30}} q(\zeta) e^{(\int_{\tau(\zeta)}^{h(\zeta)} Q_1(\zeta_1, \tau(\zeta_1)) q(\zeta_1)}) d\zeta$$

$$= \int_{3rk+\frac{131}{30}}^{3rk+\frac{161}{30}} \eta e^{(\eta e^{(\eta)})} d\zeta$$

$$= e^{(\eta e^{(\eta)})}\, \eta[\,(3rk + \tfrac{161}{30}) - (3rk + \tfrac{131}{30})\,]$$

$$> e^{(\eta e^{(\eta)})}\, \eta\,(1)$$

$$> 0.62\, e^{(0.62 e^{(0.62)})} \quad \text{for } \eta > 0.61 \text{ and } k \in N_0$$

$$> 1.95$$

Also,

$(\frac{1 + ln(\beta 2)}{\beta 2}) < 0.679$ for all $\eta \geq 0.61$ and $k \in N_0$.

Also from (2.1) ,

$\lim_{z \to 0} \sup \frac{z}{f(z)} = \Omega$, $0 \leq \Omega < \infty$

$\Omega = 0.11$

So $3\,\Omega = 0.33$.

It is obvious that

$\int_{h((M5(Tk))}^{M5(Tk)} q(\zeta) Q_2(h(\zeta), \tau(\zeta)\, d\zeta > 1.95 > (3\,\Omega)\,(\frac{1 + ln(\beta 2)}{\beta 2}) = (0.33)\, 0.679) = 0.224$.

In view of this,(3.17) and (3.18) ,all conditions of Theorem 3.1 with n=1 are satisfied for all $\eta \geq 0.61$.

Therefore all solutions of (3.16) are oscillatory for $\eta \geq 0.61$.

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