Equitable Colouring of Cartesian Product of Semi-Total Point Graph with Certain Graphs

G. Murali[1], m.s. Paulraj[2]
[1] *Assistant Professor, PG and Research Department of Mathematics,*
Agurchand Manmull Jain College, University of Madras, Chennai, Tamilnadu, India.
[2] *Associate Professor, PG and Research Department of Mathematics,*
Agurchand Manmull Jain College, University of Madras, Chennai, Tamilnadu, India.

Abstract: An equitable proper colouring of graph G is the number of vertices with any two-colour classes that differ by one and is denoted by $\chi_{=}(G)$. The semi-total point graphs are graphs that has captivated the imagination of the graph theoretic researchers of the modern era. In this paper, the acceptance of equitable colouring to the Cartesian product of semi-total point graph with certain specific graphs has been proved.

Keywords: *Equitable colouring, Cartesian product, Semi-total point graph*

Mathematics Subject Classification: *05C15*

1. Introduction

The colouring problem in graphs are one of the most thought inspiring problems in graph theory that has extensive applications. An extension to proper colouring, was developed by Meyer in 1973[12]. Hanna Furmanczyk[6] proved the being of equitable colouring in product graphs. Following Furmanczyk, Wu-Hsiung Lin and Gerard J.Chang[19] proved that the Cartesian product of graphs is equitable. These things provided the impetus to apply the concept of equitable colouring and has come up with study Cartesian Product of Semi-Total Point Graph with Certain Graphs.

2. Basic Definitions

Definition 2.1 [12]

Let $G(V,E)$ be a graph and let the partition on the vertex set $\{V_i : 1 \leq i \leq k\}$ be a proper k-colouring of G. An equitable k-colouring is defined as $\big||V_i| - |V_j|\big| \leq 1,\ i,j = 1,2,\dots,k$. The Equitable colouring of G denoted by $\chi_{=}(G)$ is the smallest integer k for which G is equitably k-colourable.

Definition 2.2 [7]

For graphs G_1 and G_2, the Cartesian products of graphs G_1 and G_2 will be denoted by $G_1 \square G_2$ with vertex set $V(G_1 \square G_2) = \{(x,y): x \in V(G_1), y \in V(G_2)\}$ and edge set

$E(G_1 \square G_2) = \{(x,y)(u,v): x = u$ and $yv \in E(G_2)$ or $y = v$ and $xu \in E(G_1)\}$.

Definition 2.3 [16]

The semi-total point graph $T_2(G)$ of G is the graph whose vertex set is $V(G) \cup E(G)$. For $a,b \in V(T_2(G))$, a and b are adjacent if and only if the following conditions hold.

(i) $a,b \in V(G)$, a,b are adjacent vertices of G.

(ii) $a \in V(G)$ and $b \in E(G)$, b is incident with a in G.

3. Pre-Requisites

The following results are useful in proving the main results.

Theorem 3.1 [12] If G is a connected graph, different from C_{2n+1} and K_n $\forall\, n \geq 1$, then $\chi_{=}(G) \leq \Delta(G)$.

Theorem 3.2 [10] The Equitable Δ- Colouring Conjecture (EΔCC), a connected graph G is equitable $\Delta(G)$ – colourable if G is different from C_{2n+1}, K_n and $K_{2n+1,2n+1}$ $\forall\, n \geq 1$.

Theorem 3.3 [5] Let G be a connected graph with $\Delta(G) \geq \frac{|G|}{2}$. If G is different from K_m and $K_{2m+1,2m+1}$ for all $m \geq 1$ then G is equitably $\Delta(G)$-colourable.

Theorem 3.4 [7] $\chi(G_1 \square G_2)$ = max $\{\chi(G_1), \chi(G_2)$.

Theorem 3.5 [2] If G_1 and G_2 are equitably k colourable, then $G_1 \square G_2$ is equitably k - colourable.

Theorem 3.6 [19] $\chi_{=}(G_1 \square G_2) \leq \chi(G_1)\chi(G_2)$ for connected graph G_1 and G_2.

Theorem 3.7 [14] If n and m are non-negative integers then the equitable chromatic number of $T_2(P_m) \square P_n$ is 3.

Theorem 3.8 [13] If m and n are non-negative integers, $m \geq 3$, $n \geq 2$ then the equitable chromatic number of $T_2(C_m) \square P_n$ is 3

Theorem 3.9 [12] For any graph G, $\chi_{=}(G) \leq \Delta(G) + 1$.

4. Equitable Colouring Of Cartesian Product Of Graphs

In this paper, we first consider G as the path and H as the cycle, complete and bipartite graph. Secondly, we consider G as the cycle and H as the cycle, complete and bipartite graph. Next, we consider G as the complete graph and H as the path, cycle, complete and bipartite graph. Finally, we consider G as the complete bipartite graph and H as the path, cycle and complete graph.

The notion $i \equiv 02$ refers to $i \equiv 0(mod2)$.

Theorem 4.1 Let G and H be the two graphs, where G is a semi-total point graph of path, $T_2(P_m)$ on $m \geq 2$ vertices, the equitable colouring of the Cartesian product of G and H, for $n \geq 3$

(i) $\chi_{=}(G \square C_n) = \begin{cases} 3\,; n \neq 3k+1 \\ 4\,;\ n = 3k+1 \text{ and } n \neq 12k+1 \end{cases}$

(ii) $\chi_{=}(G \square K_n) = n\,;\, n = 3k$

(iii) $\chi_{=}(G \square K_{p,q}) = 3\,;\, m = 3k-1, p \geq 2, q \geq 1$ and $p-1 \leq q \leq p$

$\forall\, k \geq 1$.

Proof. Define the map $\sigma :\ V(G \square H) \ \to \{0,1,2,\dots,l\}\ \forall\, l \in W$

The proof of the theorem is divided into three cases,

The graph G is a semi-total point graph of path on $m \geq 2$.

Case 1: Let $H = C_n$ for $n \geq 3$.

Cleary, the number of vertices in $T_2(P_m) \square C_n$ is $n(2m-1)$ vertices and the number of edges is $n(5m-4)$ edges.

Let the vertex set and edge set of the Cartesian product $T_2(P_m) \square C_n$ be

$$V(\,G \,\square\, H) = \left(\bigcup_{i=0}^{m-1}\bigcup_{j=0}^{n-1}\{u_i v_j\}\right) \cup \left(\bigcup_{i=0}^{m-2}\bigcup_{j=0}^{n-1}\{e_i v_j\}\right)$$

$$E(\,G \,\square\, H) = \left(\bigcup_{j=0}^{n-1}\bigcup_{i=0}^{m-2}\{(u_i v_j)(u_{i+1} v_j);\,(u_i v_j)(e_i v_j);\,(e_i v_j)(u_{i+1} v_j)\}\right) \cup \left(\bigcup_{i=0}^{m-1}\bigcup_{j=0}^{n-1}\{(u_i v_j)(u_i v_{(j+1) mod\, n})\}\right) \cup \left(\bigcup_{i=0}^{m-2}\bigcup_{j=0}^{n-1}\{(e_i v_j)(e_i v_{(j+1) mod\, n})\}\right)$$

Claim (a): We will proof $\chi_{=}(G \,\square\, C_n\,) = 3$ for $n \geq 3$ and $n \neq 3k+1$, $k \geq 1$.

The colouring of vertices,

$\forall\ 0 \leq i \leq m-1$ and $0 \leq j \leq n-1$,

$$\sigma(u_i v_j) = (i+j)(mod\ 3)$$

$$\sigma(e_i v_j) = (i+j+2)(mod\ 3)$$

Now, the vertex set is partitioned into V_0, V_1 and V_2 as below

$$V_0 = \begin{cases} u_i v_j\,; & i+j \equiv 03 \\ e_i v_j\,; & i+j+2 \equiv 03 \end{cases} \qquad V_1 = \begin{cases} u_i v_j\,; & i+j \equiv 13 \\ e_i v_j\,; & i+j+2 \equiv 13 \end{cases} \qquad V_2 = \begin{cases} u_i v_j\,; & i+j \equiv 23 \\ e_i v_j\,; & i+j+2 \equiv 23 \end{cases}$$

The sets V_0, V_1 and V_2 are independent of $G \,\square\, H$, also

(i) If (a) $n \equiv 03$ and $\forall\, m$, (b) $n \equiv 23$ & $m \equiv 23$ then $|V_0| = |V_1| = |V_2| = \frac{n(2m-1)}{3}$.

(ii) If $n \equiv 23$ then

(a) For $m \equiv 03$ $|V_0| = \left\lceil \frac{n(2m-1)}{3} \right\rceil$ and $|V_1| = |V_2| = \left\lfloor \frac{n(2m-1)}{3} \right\rfloor$.

(b) For $m \equiv 13$ $|V_0| = |V_1| = \left\lceil \frac{n(2m-1)}{3} \right\rceil$ and $|V_2| = \left\lfloor \frac{n(2m-1)}{3} \right\rfloor$.

Obviously, from (i) and (ii) the inequality $\left||V_i| - |V_j|\right| \leq 1$ holds for every pair (i, j).

By theorem 3.4, we get, $\chi_{=}(G \,\square\, C_n\,) = 3$ for $n \geq 3$ and $n \neq 3k+1$, $k \geq 1$.

Claim (b): We will proof $\chi_{=}(G \,\square\, C_n\,) = 4$ for $n = 3k+1$ and $n \neq 12k+1$, $k \geq 1$.

The colouring of vertices,

$\forall\ 0 \leq i \leq m-1$ and $0 \leq j \leq n-1$,

$$\sigma(u_i v_j) = (i+j)(mod\ 4)$$

$$\sigma(e_i v_j) = (i+j+2)(mod\ 4)$$

Now, the vertex set is partitioned into V_0, V_1 and V_2 as below

$$V_0 = \begin{cases} u_i v_j\,; & i+j \equiv 04 \\ e_i v_j\,; & i+j+2 \equiv 04 \end{cases} \qquad V_1 = \begin{cases} u_i v_j\,; & i+j \equiv 14 \\ e_i v_j\,; & i+j+2 \equiv 14 \end{cases}$$

$$V_2 = \begin{cases} u_i v_j\,; & i+j \equiv 24 \\ e_i v_j\,; & i+j+2 \equiv 24 \end{cases} \qquad V_3 = \begin{cases} u_i v_j\,; & i+j \equiv 34 \\ e_i v_j\,; & i+j+2 \equiv 34 \end{cases}$$

The sets V_0, V_1, V_2 and V_3 are independent of $G \,\square\, H$, also

(i) If $n \equiv 04$, $\forall\, m$ then $|V_0| = |V_1| = |V_2| = |V_3| = \frac{n(2m-1)}{4}$.

(ii) If $n \equiv 24$ then

(a) For $m \equiv 04$, $|V_0| = |V_3| = \left\lceil \frac{n(2m-1)}{4} \right\rceil$ and $|V_1| = |V_2| = \left\lfloor \frac{n(2m-1)}{4} \right\rfloor$.
(b) For $m \equiv 14$, $|V_0| = |V_1| = \left\lceil \frac{n(2m-1)}{4} \right\rceil$ and $|V_2| = |V_3| = \left\lfloor \frac{n(2m-1)}{4} \right\rfloor$.
(c) For $m \equiv 24$, $|V_1| = |V_2| = \left\lceil \frac{n(2m-1)}{4} \right\rceil$ and $|V_0| = |V_3| = \left\lfloor \frac{n(2m-1)}{4} \right\rfloor$.
(d) For $m \equiv 34$, $|V_2| = |V_3| = \left\lceil \frac{n(2m-1)}{4} \right\rceil$ and $|V_0| = |V_1| = \left\lfloor \frac{n(2m-1)}{4} \right\rfloor$.

(iii) If $n \equiv 34$ then

(a) For $m \equiv 04$, $|V_0| = \left\lceil \frac{n(2m-1)}{4} \right\rceil$ and $|V_1| = |V_2| = |V_3| = \left\lfloor \frac{n(2m-1)}{4} \right\rfloor$.
(b) For $m \equiv 14$, $|V_0| = |V_1| = |V_2| = \left\lceil \frac{n(2m-1)}{4} \right\rceil$ and $|V_3| = \left\lfloor \frac{n(2m-1)}{4} \right\rfloor$.
(c) For $m \equiv 24$, $|V_2| = \left\lceil \frac{n(2m-1)}{4} \right\rceil$ and $|V_0| = |V_1| = |V_3| = \left\lfloor \frac{n(2m-1)}{4} \right\rfloor$.
(d) For $m \equiv 34$, $|V_0| = |V_2| = |V_3| = \left\lceil \frac{n(2m-1)}{4} \right\rceil$ and $|V_1| = \left\lfloor \frac{n(2m-1)}{4} \right\rfloor$.

Obviously, from (i), (ii) and (iii) the inequality $\left| |V_i| - |V_j| \right| \le 1$ holds for every pair (i, j).

By theorem 3.4, we get, $\chi_{=}(G \,\square\, C_n) = 4$ for $n = 3k + 1$ and $n \ne 12k + 1$, $k \ge 1$.

Case 2: Let $H = K_n$ for $n = 3k$, $k \ge 1$

Cleary, the number of vertices in $T_2(P_m) \,\square\, K_n$ is $n(2m - 1)$ vertices and the number of edges is $\frac{n}{2}(2mn + 4m - n - 5)$ edges.

Let the vertex set and edge set of the Cartesian product $T_2(P_m) \,\square\, K_n$ be

$$V(G \,\square\, H) = \left(\bigcup_{i=0}^{m-1} \bigcup_{j=0}^{n-1} \{u_i v_j\} \right) \cup \left(\bigcup_{i=0}^{m-2} \bigcup_{j=0}^{n-1} \{e_i v_j\} \right)$$

$$E(G \,\square\, H) = \left(\bigcup_{j=0}^{n-1} \bigcup_{i=0}^{m-2} \{(u_i v_j)(u_{i+1} v_j); (u_i v_j)(e_i v_j); (e_i v_j)(u_{i+1} v_j)\} \right) \cup \left(\bigcup_{i=0}^{m-1} \bigcup_{j=0}^{n-2} \bigcup_{k>j}^{n-1} \{(u_i v_j)(u_i v_{\mathrm{k}})\} \right)$$
$$\cup \left(\bigcup_{i=0}^{m-2} \bigcup_{j=0}^{n-2} \bigcup_{k>j}^{n-1} \{(e_i v_j)(e_i v_k)\} \right)$$

The colouring of vertices,

$\forall\ 0 \le i \le m - 1$ and $0 \le j \le n - 1$,

$$\sigma(u_i v_j) = (i + j)(mod\ 3) + 3\left\lfloor \frac{j}{3} \right\rfloor$$

$$\sigma(e_i v_j) = (i + j + 2)(mod\ 3) + 3\left\lfloor \frac{j}{3} \right\rfloor$$

Each colour $(0,1,2, \dots, n - 1)$ appears exactly $2m - 1$ times. Thus, the difference does not exceed one.

By theorem 3.4, we get, $\chi_{=}(G \,\square\, K_n) = n$ for $n = 3k, k \ge 1$.

Case 3: Let $H = K_{p,q}$ for $m = 3k - 1, k \ge 1, p \ge 2, q \ge 1$ and $p - 1 \le q \le p$.

Cleary, the number of vertices in $T_2(P_m) \,\square\, K_{p,q}$ is $2p(2m - 1)$ vertices and the number of edges is $3(m - 1)(p + q) + pq(2m - 1)$ edges.

Let the vertex set and edge set of the Cartesian product $T_2(P_m) \square K_{p,q}$ be

$$V(G \square H) = \left(\bigcup_{i=0}^{m-1}\bigcup_{j=0}^{p-1}\{u_i v_j\}\right) \cup \left(\bigcup_{i=0}^{m-2}\bigcup_{j=0}^{p-1}\{e_i v_j\}\right) \cup \left(\bigcup_{i=0}^{m-1}\bigcup_{j'=0}^{q-1}\{u_i v_{j'}\}\right) \cup \left(\bigcup_{i=0}^{m-2}\bigcup_{j'=0}^{q-1}\{e_i v_{j'}\}\right)$$

$$E(G \square H) = \left(\bigcup_{j=0}^{p-1}\bigcup_{i=0}^{m-2}\{(u_i v_j)(u_{i+1} v_j); (u_i v_j)(e_i v_j); (e_i v_j)(u_{i+1} v_j)\}\right)$$
$$\cup \left(\bigcup_{j'=0}^{q-1}\bigcup_{i=0}^{m-2}\{(u_i v_{j'})(u_{i+1} v_{j'}); (u_i v_{j'})(e_i v_{j'}); (e_i v_{j'})(u_{i+1} v_{j'})\}\right)$$
$$\cup \left(\bigcup_{i=0}^{m-1}\bigcup_{j=0}^{p-1}\bigcup_{j'=0}^{q-1}\{(u_i v_j)(u_i v_{j'})\}\right) \cup \left(\bigcup_{i=0}^{m-2}\bigcup_{j=0}^{p-1}\bigcup_{j'=0}^{q-1}\{(e_i v_j)(e_i v_{j'})\}\right)$$

The colouring of vertices,

$\forall\ 0 \le i \le m-1$ and $0 \le j \le p-1$,

$$\sigma(u_i v_j) = i (mod\ 3)$$

$$\sigma(e_i v_j) = (i+2)(mod\ 3)$$

$\forall\ 0 \le j' \le q-1$,

$$\sigma(u_i v_{j'}) = (i+2)(mod\ 3)$$

$$\sigma(e_i v_{j'}) = (i+1)(mod\ 3)$$

Now, the vertex set is partitioned into V_0, V_1, V_2 and V_3 as below

$$V_0 = \{u_{i\equiv 03} v_j; e_{i\equiv 13} v_j; u_{i\equiv 13} v_{j'}; e_{i\equiv 23} v_{j'}\}$$

$$V_1 = \{u_{i\equiv 13} v_j; e_{i\equiv 23} v_j; u_{i\equiv 23} v_{j'}; e_{i\equiv 03} v_{j'}\}$$

$$V_2 = \{u_{i\equiv 23} v_j; e_{i\equiv 03} v_j; u_{i\equiv 03} v_{j'}; e_{i\equiv 13} v_{j'}\}$$

The sets V_0, V_1 and V_2 are independent of $G \square H$, also $|V_0| = |V_1| = |V_2| = \frac{(2m-1)(p+q)}{3}$.

The inequality $\big||V_i| - |V_j|\big| \le 1$ holds for every pair (i,j). By theorem 3.4, we get, $\chi_=(G \square K_{p,q}) = 3$ for $m = 3k-1, k \ge 1, p \ge 2, q \ge 1$ and $p-1 \le q \le p$.

Corollary 4.1.1 For $n = 3k+2, k \ge 1$, $\chi_=(T_2(P_2) \square K_n) = n$.

Proof. Define the map $\sigma : V(G \square H) \to \{0,1,2,\dots,l\}\ \forall\ l \in W$

For $n = 3k+2, k \ge 1$, The colouring of vertices as follows from theorem 4.1 case 2.

Each colour $(0,1,2,\dots,n-4)$ appears exactly 3 times and each colour $(n-3, n-2, n-1)$ appears exactly 2 times. Thus, the difference does not exceed one.

By theorem 3.4, we get, $\chi_=(T_2(P_2) \square K_n) = n$ for $n = 3k+2$, $k \ge 1$.

Theorem 4.2 If m and n are non-negative integers, $m \ge 3$, $n \ge 3$ then the equitable colouring of

$$\chi_=(T_2(C_m) \square C_n) = \begin{cases} 3\ ; m = 3k \text{ and } n \ne 3k+1 \\ 4\ ; m = 2(3k-1), 2(3k-2), n \ne 4k+1 \text{ and } m = 6k, n = 3k+1, n \ne 12k+1 \\ 5\ ; m = 2k-1, m \ne 3k, 5k-1, 5k+1 \text{ and } n \ne 5k+1 \end{cases}$$

$\forall\ k \geq 1$.

Proof. Define the map $\sigma : V(G \ \square\ H) \ \rightarrow \{0,1,2,\dots,l\}\ \forall\ l \in W$

For $m \geq 3$ and $n \geq 3$,

Cleary, the number of vertices in $T_2(C_m) \ \square\ C_n$ is $2mn$ vertices and the number of edges is $5mn$ edges.

Let the vertex set and edge set of the Cartesian product $T_2(C_m) \ \square\ C_n$ be

$$V(\ G \ \square\ H) = \bigcup_{i=0}^{m-1}\bigcup_{j=0}^{n-1}\{u_iv_j; e_iv_j\}$$

$$E(\ G \ \square\ H) = \left(\bigcup_{j=0}^{n-1}\bigcup_{i=0}^{m-1}\{(u_iv_j)(u_{(i+1)(mod\ m)}v_j); (u_iv_j)(e_iv_j); (e_iv_j)(u_{(i+1)(mod\ m)}v_j)\}\right)$$
$$\cup \left(\bigcup_{i=0}^{m-1}\bigcup_{j=0}^{n-1}\{(u_iv_j)(u_iv_{(j+1)(mod\ n)}); (e_iv_j)(e_iv_{(j+1)(mod\ n)})\}\right)$$

Case 1: We will proof $\chi_{=}(T_2(C_m) \ \square\ C_n\ ) = 3$ for $m = 3k$ and $n \neq 3k+1$, $k \geq 1$.

The colouring of vertices as follows from theorem 4.1 case 1 claim (a),

The sets V_0, V_1 and V_2 are independent of $G \ \square\ H$, also $|V_0| = |V_1| = |V_2| = \frac{2mn}{3}$.

The inequality $\left||V_i| - |V_j|\right| \leq 1$ holds for every pair (i,j).

By theorem 3.4, we get, $\chi_{=}(T_2(C_m) \ \square\ C_n\ ) = 3$ for $m = 3k$ and $n \neq 3k+1$, $k \geq 1$.

Case 2: We will proof $\chi_{=}(T_2(C_m) \ \square\ C_n\ ) = 4$ for $m = 2(3k-1), 2(3k-2), n \neq 4k+1$ and $m = 6k, n = 3k+1, n \neq 12k+1$, $k \geq 1$.

The colouring of vertices as follows from theorem 4.1 case 1 claim (b),

The sets V_0, V_1 and V_2 are independent of $G \ \square\ H$, also $|V_0| = |V_1| = |V_2| = \frac{mn}{2}$.

The inequality $\left||V_i| - |V_j|\right| \leq 1$ holds for every pair (i,j).

By theorem 3.4, we get, $\chi_{=}(T_2(C_m) \ \square\ C_n\ ) = 4$ for $m = 2(3k-1), 2(3k-2), n \neq 4k+1$ and $m = 6k, n = 3k+1, n \neq 12k+1$, $k \geq 1$.

Case 3: We will proof $\chi_{=}(T_2(C_m) \ \square\ C_n\ ) = 5$ for $m = 2k-1, m \neq 3k, 5k-1, 5k+1$ and $n \neq 5k+1$.

The colouring of vertices,

$\forall\ \ 0 \leq i \leq m-1$ and $0 \leq j \leq n-1$,

$$\sigma(u_iv_j) = (i+j)(mod\ 5)$$

$$\sigma(e_iv_j) = (i+j+2)(mod\ 5)$$

Now, the vertex set is partitioned into V_0, V_1, V_2 and V_3 as below

$$V_0 = \begin{cases} u_iv_j\ ; & i+j \equiv 0_5 \\ e_iv_j\ ; i+j+2 \equiv 0_5 \end{cases} \quad V_1 = \begin{cases} u_iv_j\ ; & i+j \equiv 1_5 \\ e_iv_j\ ; i+j+2 \equiv 1_5 \end{cases} \quad V_2 = \begin{cases} u_iv_j\ ; & i+j \equiv 2_5 \\ e_iv_j\ ; i+j+2 \equiv 2_5 \end{cases}$$

$$V_3 = \begin{cases} u_iv_j\ ; & i+j \equiv 3_5 \\ e_iv_j\ ; i+j+2 \equiv 3_5 \end{cases} \quad V_4 = \begin{cases} u_iv_j\ ; & i+j \equiv 4_5 \\ e_iv_j\ ; i+j+2 \equiv 4_5 \end{cases}$$

The sets V_0, V_1, V_2, V_3 and V_4 are independent of $G \square H$, also

(i) If $m \equiv 15$ and $n \equiv 15$ then $|V_0| = |V_2| = \left\lceil \frac{2nm}{5} \right\rceil$ and $|V_1| = |V_3| = |V_4| = \left\lfloor \frac{2nm}{5} \right\rfloor$.

(ii) If $m \equiv 25$ and $n \equiv 25$ then $|V_0| = |V_4| = \left\lfloor \frac{2nm}{5} \right\rfloor$ and $|V_1| = |V_2| = |V_3| = \left\lceil \frac{2nm}{5} \right\rceil$.

(iii) If $m \equiv 35$ and $n \equiv 35$ then $|V_0| = |V_1| = \left\lfloor \frac{2nm}{5} \right\rfloor$ and $|V_2| = |V_3| = |V_4| = \left\lceil \frac{2nm}{5} \right\rceil$.

(iv) If $m \equiv 45$ and $n \equiv 45$ then $|V_0| = |V_3| = \left\lceil \frac{2nm}{5} \right\rceil$ and $|V_1| = |V_2| = |V_4| = \left\lfloor \frac{2nm}{5} \right\rfloor$.

(v) If (a) $m \equiv 05 \;\forall\, n$, (b) $n \equiv 05$ and $m \equiv r5,\ 1 \le r \le 4$ then $|V_0| = |V_1| = |V_2| = |V_3| = |V_4| = \frac{2nm}{5}$.

(vi) If (a) $m \equiv 15$ and $n \equiv 25$, (b) $m \equiv 25$ and $n \equiv 15$ then $|V_0| = |V_1| = |V_2| = |V_3| = \left\lceil \frac{2nm}{5} \right\rceil$ and $|V_4| = \left\lfloor \frac{2nm}{5} \right\rfloor$.

(vii) If (a) $m \equiv 15$ and $n \equiv 35$, (b) $m \equiv 35$ and $n \equiv 15$ then $|V_0| = |V_1| = |V_3| = |V_4| = \left\lfloor \frac{2nm}{5} \right\rfloor$ and $|V_2| = \left\lceil \frac{2nm}{5} \right\rceil$

(viii) If (a) $m \equiv 15$ and $n \equiv 45$, (b) $m \equiv 45$ and $n \equiv 15$ then $|V_0| = |V_2| = |V_3| = \left\lceil \frac{2nm}{5} \right\rceil$ and $|V_1| = |V_4| = \left\lfloor \frac{2nm}{5} \right\rfloor$.

(ix) If (a) $m \equiv 25$ and $n \equiv 35$, (b) $m \equiv 35$ and $n \equiv 25$ then $|V_0| = |V_1| = |V_4| = \left\lfloor \frac{2nm}{5} \right\rfloor$ and $|V_2| = |V_3| = \left\lceil \frac{2nm}{5} \right\rceil$.

(x) If (a) $m \equiv 25$ and $n \equiv 45$, (b) $m \equiv 45$ and $n \equiv 25$ then $|V_3| = \left\lceil \frac{2nm}{5} \right\rceil$ and $|V_0| = |V_1| = |V_2| = |V_4| = \left\lfloor \frac{2nm}{5} \right\rfloor$.

(xi) If (a) $m \equiv 35$ and $n \equiv 45$, (b) $m \equiv 45$ and $n \equiv 35$ then $|V_0| = |V_2| = |V_3| = |V_4| = \left\lceil \frac{2nm}{5} \right\rceil$ and $|V_1| = \left\lfloor \frac{2nm}{5} \right\rfloor$.

The inequality $\left| |V_i| - |V_j| \right| \le 1$ holds for every pair (i, j).

By theorem 3.4, we get, $\chi_{=}(T_2(C_m) \square C_n) = 5$ for $m = 2k-1, m \ne 3k, 5k-1, 5k+1$ and $n \ne 5k+1$.

Theorem 4.3 Let G and H be the two graphs, where G is a semi-total point graph of cycle, $T_2(C_m)$ on $m \ge 3$ vertices, the equitable colouring of the Cartesian product of G and H, for $n \ge 3$

(i) $\chi_{=}(G \square K_n) = n\ ;\ m, n = 3k$

(ii) $\chi_{=}(G \square K_{p,q}) = 3\ ;\ m = 3k, p \ge 2, q \ge 1$ and $p - 1 \le q \le p$

$\forall\, k \ge 1$.

Proof. Define the map $\sigma : V(G \square H) \to \{0,1,2,\dots,l\}\ \forall\, l \in W$

The proof of the theorem is divided into two cases,

The graph G is a semi-total point graph of cycle on $m \ge 3$.

Case 1: Let $H = K_n$ for $m, n = 3k,\ k \ge 1$.

Cleary, the number of vertices in $T_2(C_m) \square K_n$ is $2mn$ vertices and the number of edges is $mn(n+2)$ edges.

Let the vertex set and edge set of the Cartesian product $T_2(C_m) \square K_n$ be

$$V(G \square H) = \bigcup_{i=0}^{m-1} \bigcup_{j=0}^{n-1} \{u_i v_j; e_i v_j\}$$

$$E(\, G \,\square\, H) = \left(\bigcup_{j=0}^{n-1}\bigcup_{i=0}^{m-1}\{(u_iv_j)(u_{(i+1)(mod\ m)}v_j);\,(u_iv_j)(e_iv_j);\,(e_iv_j)(u_{(i+1)(mod\ m)}v_j)\}\right) \cup \left(\bigcup_{i=0}^{m-1}\bigcup_{j=0}^{n-2}\bigcup_{k>j}^{n-1}\{(u_iv_j)(u_iv_k);\,(e_iv_j)(e_iv_k)\}\right)$$

The colouring of vertices as follows from theorem 4.1 case 2.

Each colour $(0,1,2,\dots,n-1)$ appears exactly $2m$ times. Thus, the difference does not exceed one.

By theorem 3.4, we get, $\chi_{=}(G \,\square\, K_n\,) = n$ for $m = 3k$ and $n = 3k$, $k \geq 1$.

Case 2: Let $H = K_{p,q}$ for $m = 3k, k \geq 1$ and $p \geq 2, q \geq 1$ and $p - 1 \leq q \leq p$

Cleary, the number of vertices in $T_2(C_m) \,\square\, K_{p,q}$ is $2m(p+q)$ vertices and the number of edges is $m(3p + 3q + pq)$ edges.

Let the vertex set and edge set of the Cartesian product $T_2(C_m) \,\square\, K_{p,q}$ be

$$V(\, G \,\square\, H) = \left(\bigcup_{i=0}^{m-1}\bigcup_{j=0}^{p-1}\{u_iv_j;\, e_iv_j\}\right) \cup \left(\bigcup_{i=0}^{m-1}\bigcup_{j'=0}^{q-1}\{u_iv_{j'};\, e_iv_{j'}\}\right)$$

$$E(\, G \,\square\, H) = \left(\bigcup_{j=0}^{p-1}\bigcup_{i=0}^{m-1}\{(u_iv_j)(u_{(i+1)(mod\ m)}v_j);\,(u_iv_j)(e_iv_j);\,(e_iv_j)(u_{(i+1)(mod\ m)}v_j)\}\right) \cup \left(\bigcup_{j'=0}^{q-1}\bigcup_{i=0}^{m-1}\{(u_iv_{j'})(u_{(i+1)(mod\ m)}v_{j'});\,(u_iv_{j'})(e_iv_{j'});\,(e_iv_{j'})(u_{(i+1)(mod\ m)}v_{j'})\}\right) \cup \left(\bigcup_{i=0}^{m-1}\bigcup_{j=0}^{p-1}\bigcup_{j'=0}^{q-1}\{(u_iv_j)(u_iv_{j'});\,(e_iv_j)(e_iv_{j'})\}\right)$$

The colouring of vertices as follows from theorem 4.1 case 3.

The sets V_0, V_1 and V_2 are independent of $G \,\square\, H$, also $|V_0| = |V_1| = |V_2| = \frac{2m(p+q)}{3}$.

The inequality $\big||V_i| - |V_j|\big| \leq 1$ holds for every pair (i,j). By theorem 3.4, we get, $\chi_{=}(G \,\square\, K_{p,q}) = 3$ for $m = 3k, k \geq 1$ and $p \geq 2, q \geq 1$ and $p - 1 \leq q \leq p$.

Theorem 4.4 Let G and H be the two graphs, where G is a semi-total point graph of complete, $T_2(K_m)$ on $m \geq 4$ vertices, the equitable colouring of the Cartesian product of G and H,

(i) $\chi_{=}(G \,\square\, P_n) = m\,;\, n \geq 2$
(ii) $\chi_{=}(G \,\square\, C_n\,) = m\,;\; n \neq mk + 1$
(iii) $\chi_{=}(G \,\square\, K_n) = m\,;\, m \geq n$
(iv) $\chi_{=}(G \,\square\, K_{p,q}) = m\,; m = 2k, p \geq 2, q \geq 1\ \&\ p - 1 \leq q \leq p + 1$ and also $m = 2k + 3\ \&\ p = 2$

$\forall\, k \geq 1$.

Proof. Define the map $\sigma :\ V(G \,\square\, H)\ \rightarrow \{0,1,2,\dots,l\}\ \forall\, l \in W$

The proof of the theorem is divided into four cases,

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The graph G is a semi-total point graph of complete on $m \geq 4$.

Case 1: Let $H = P_n$ for $n \geq 2$.

Cleary, the number of vertices in $T_2(K_m) \square P_n$ is $\frac{nm(m+1)}{2}$ vertices and the number of edges is $\frac{m}{2}(4mn - 2n - m - 1)$ edges.

Let the vertex set and edge set of the Cartesian product $T_2(K_m) \square P_n$ be

$$V(G \square H) = \left(\bigcup_{i=0}^{m-1}\bigcup_{j=0}^{n-1}\{u_i v_j\}\right) \cup \left(\bigcup_{i=0}^{m-2}\bigcup_{k>i}^{m-1}\bigcup_{j=0}^{n-1}\{e_{ik} v_j\}\right)$$

$$E(G \square H) = \left(\bigcup_{j=0}^{n-1}\bigcup_{i=0}^{m-2}\bigcup_{k>i}^{m-1}\{(u_i v_j)(u_k v_j); (u_i v_j)(e_{ik} v_j); (e_{ik} v_j)(u_k v_j)\}\right) \cup \left(\bigcup_{i=0}^{m-1}\bigcup_{j=0}^{n-2}\{(u_i v_j)(u_i v_{j+1})\}\right) \cup \left(\bigcup_{i=0}^{m-2}\bigcup_{k>i}^{m-1}\bigcup_{j=0}^{n-2}\{(e_{ik} v_j)(e_{ik} v_{j+1})\}\right)$$

The colouring of vertices,

$\forall\ 0 \leq i \leq m-1$ and $0 \leq j \leq n-1$,

$$\sigma(e_{ik} v_j) = \begin{cases} (k + j + 1)(mod\ m)\ ; & i = 0 \text{ and } 1 \leq k \leq (m-3) \\ (j + 2)(mod\ m)\ ; & i = 0 \text{ and } k = (m-2) \\ (j + 1)(mod\ m)\ ; & i = 0 \text{ and } k = (m-1) \\ (i + j + 2)(mod\ m)\ ; & 1 \leq i < k \leq (m-1) \text{ and } k = i + 1 \\ (i + k + j)(mod\ m)\ ; & 1 \leq i < k \leq (m-1) \text{ and } k \neq i + 1 \end{cases}$$

$\forall\ k = 1,2,3, \dots, (m-1)$ & $\forall\ m$

For $m = 4$

$$\sigma(u_i v_j) = \begin{cases} (i + j)(mod\ 4)\ ; & i = 0 \text{ and } i = 3 \\ (j + 2)(mod\ 4)\ ; & i = 1 \\ (j + 1)(mod\ 4)\ ; & i = 2 \end{cases}$$

For $m > 4$

$$\sigma(u_i v_j) = (i + j)(mod\ m)$$

When m is even and n is odd then each odd colour appears exactly $\left\lceil \frac{n(m+1)}{2} \right\rceil$ times and even colour appears exactly $\left\lfloor \frac{n(m+1)}{2} \right\rfloor$. Otherwise, each colour appears exactly $\frac{n(m+1)}{2}$. Thus, the difference does not exceed one.

The inequality $\left||V_i| - |V_j|\right| \leq 1$ holds for every pair (i, j).

By theorem 3.4, we get, $\chi_{=}(G \square P_n) = m\ ; n \geq 2$.

Case 2: Let $H = C_n$ for $n \geq 2$.

Cleary, the number of vertices in $T_2(K_m) \square C_n$ is $\frac{nm(m+1)}{2}$ vertices and the number of edges is $\frac{mn}{2}(4m - 3)$ edges.

Let the vertex set and edge set of the Cartesian product $T_2(K_m) \square C_n$ be

$$V(G \,\square\, H) = \left(\bigcup_{i=0}^{m-1}\bigcup_{j=0}^{n-1}\{u_i v_j\}\right) \cup \left(\bigcup_{i=0}^{m-2}\bigcup_{k>i}^{m-1}\bigcup_{j=0}^{n-1}\{e_{ik} v_j\}\right)$$

$$E(G \,\square\, H) = \left(\bigcup_{j=0}^{n-1}\bigcup_{i=0}^{m-2}\bigcup_{k>i}^{m-1}\{(u_i v_j)(u_k v_j); (u_i v_j)(e_{ik} v_j); (e_{ik} v_j)(u_k v_j)\}\right)$$
$$\cup \left(\bigcup_{i=0}^{m-1}\bigcup_{j=0}^{n-1}\{(u_i v_j)(u_i v_{(j+1)(mod\ n)})\}\right) \cup \left(\bigcup_{i=0}^{m-2}\bigcup_{k>i}^{m-1}\bigcup_{j=0}^{n-1}\{(e_{ik} v_j)(e_{ik} v_{(j+1)(mod\ n)})\}\right)$$

The colouring of vertices and results as follows from theorem 4.4 case 1.

By theorem 3.4, we get, $\chi_{=}(G \,\square\, C_n) = m\,;\ n \neq mk+1, k \geq 1$.

Case 3: Let $H = K_n$ for $m \geq n$.

Cleary, the number of vertices in $T_2(K_m) \,\square\, K_n$ is $\frac{nm(m+1)}{2}$ vertices and the number of edges is $\frac{mn}{4}(mn + n + 5m - 7)$ edges.

Let the vertex set and edge set of the Cartesian product $T_2(K_m) \,\square\, K_n$ be

$$V(G \,\square\, H) = \left(\bigcup_{i=0}^{m-1}\bigcup_{j=0}^{n-1}\{u_i v_j\}\right) \cup \left(\bigcup_{i=0}^{m-2}\bigcup_{k>i}^{m-1}\bigcup_{j=0}^{n-1}\{e_{ik} v_j\}\right)$$

$$E(G \,\square\, H) = \left(\bigcup_{j=0}^{n-1}\bigcup_{i=0}^{m-2}\bigcup_{k>i}^{m-1}\{(u_i v_j)(u_k v_j); (u_i v_j)(e_{ik} v_j); (e_{ik} v_j)(u_k v_j)\}\right) \cup \left(\bigcup_{i=0}^{m-1}\bigcup_{j=0}^{n-2}\bigcup_{k'>j}^{n-1}\{(u_i v_j)(u_i v_{k'})\}\right)$$
$$\cup \left(\bigcup_{i=0}^{m-2}\bigcup_{k>i}^{m-1}\bigcup_{j=0}^{n-2}\bigcup_{k'>j}^{n-1}\{(e_{ik} v_j)(e_{ik} v_{k'})\}\right)$$

The colouring of vertices and results as follows from theorem 4.4 case 1.

By theorem 3.4, we get, $\chi_{=}(G \,\square\, K_n) = m\,;\ m \geq n$.

Case 4: Let $H = K_{p,q}$ for $p \geq 2, q \geq 1$.

Cleary, the number of vertices in $T_2(K_m) \,\square\, K_{p,q}$ is $\frac{m(m+1)(p+q)}{2}$ vertices and the number of edges is $\frac{m}{2}\{3(m-1)(p+q) + pq(m+1)\}$ edges.

Let the vertex set and edge set of the Cartesian product $T_2(K_m) \,\square\, K_{p,q}$ be

$$V(G \,\square\, H) = \left(\bigcup_{i=0}^{m-1}\bigcup_{j=0}^{p-1}\{u_i v_j\}\right) \cup \left(\bigcup_{i=0}^{m-2}\bigcup_{k>i}^{m-1}\bigcup_{j=0}^{p-1}\{e_{ik} v_j\}\right) \cup \left(\bigcup_{i=0}^{m-1}\bigcup_{j'=0}^{q-1}\{u_i v_{j'}\}\right) \cup \left(\bigcup_{i=0}^{m-2}\bigcup_{k>i}^{m-1}\bigcup_{j'=0}^{q-1}\{e_{ik} v_{j'}\}\right)$$

$$E(G \,\square\, H) = \left(\bigcup_{j=0}^{p-1}\bigcup_{i=0}^{m-2}\bigcup_{k>i}^{m-1}\{(u_i v_j)(u_k v_j);\,(u_i v_j)(e_{ik} v_j);\,(e_{ik} v_j)(u_k v_j)\}\right)$$
$$\cup\left(\bigcup_{j'=0}^{q-1}\bigcup_{i=0}^{m-2}\bigcup_{k>i}^{m-1}\{(u_i v_j)(u_k v_j);\,(u_i v_j)(e_{ik} v_j);\,(e_{ik} v_j)(u_k v_j)\}\right)$$
$$\cup\left(\bigcup_{i=0}^{m-1}\bigcup_{j=0}^{p-1}\bigcup_{j'=0}^{q-1}\{(u_i v_j)(u_i v_{j'})\}\right)\cup\left(\bigcup_{i=0}^{m-2}\bigcup_{k>i}^{m-1}\bigcup_{j=0}^{p-1}\bigcup_{j'=0}^{q-1}\{(e_{ik} v_j)(e_{ik} v_{j'})\}\right)$$

The colouring of vertices,

$\forall\ 0 \le i \le m-1,\ 0 \le j \le p-1$ and $0 \le j' \le q-1$

$$\sigma(e_{ik} v_j) = \begin{cases} (k+1)(mod\ m); & i = 0 \text{ and } 1 \le k \le (m-3) \\ 2\,; & i = 0 \text{ and } k = (m-2) \\ 1\,; & i = 0 \text{ and } k = (m-1) \\ (i+2)(mod\ m)\,; & 1 \le i < k \le (m-1) \text{ and } k = i+1 \\ (i+k)(mod\ m)\,; & 1 \le i < k \le (m-1) \text{ and } k \ne i+1 \end{cases}$$

$$\sigma(e_{ik} v_{j'}) = \begin{cases} (k+2)(mod\ m) & ;\ i = 0 \text{ and } 1 \le k \le (m-3) \\ 3 & ;\ i = 0 \text{ and } k = (m-2) \\ 2 & ;\ i = 0 \text{ and } k = (m-1) \\ (i+3)(mod\ m) & ;\ 1 \le i < k \le (m-1) \text{ and } k = i+1 \\ (i+k+1)(mod\ m) & ;\ 1 \le i < k \le (m-1) \text{ and } k \ne i+1 \end{cases}$$

$\forall\ k = 1,2,3,\dots,(m-1)$

For $m = 4$

$$\sigma(u_i v_j) = \begin{cases} i\,; & i = 0 \text{ and } i = 3 \\ 2\,; & i = 1 \\ 1\,; & i = 2 \end{cases}$$
$$\sigma(u_i v_{j'}) = \begin{cases} (i+1)(mod\ 4)\,; & i = 0 \text{ and } i = 3 \\ 3\,; & i = 1 \\ 2\,; & i = 2 \end{cases}$$

For $m > 4$

$$\sigma(u_i v_j) = i$$
$$\sigma(u_i v_{j'}) = (i+1)(mod\ m)$$

Claim (i): When $m = 2k, k \ge 1, p \ge 2, q \ge 1\ \&\ p-1 \le q \le p+1$.

(a) If $q = p$, then each colour $(0,1,2,\dots,m-1)$ appears exactly $\frac{(m+1)(p+q)}{2}$.

(b) If $q = p-1$, then each odd colour appears exactly $\left\lceil\frac{(m+1)(p+q)}{2}\right\rceil$ and each even colour appears exactly $\left\lfloor\frac{(m+1)(p+q)}{2}\right\rfloor$.

(c) If $q = p+1$, then each even colour appears exactly $\left\lceil\frac{(m+1)(p+q)}{2}\right\rceil$ and each odd colour appears exactly $\left\lfloor\frac{(m+1)(p+q)}{2}\right\rfloor$.

Claim (ii): When $m = 2k+3, k \ge 1\ \&\ p = 2, q \ge 1$, each colour $(0,1,2,\dots,m-1)$ appears exactly $\frac{(m+1)(p+q)}{2}$ times.

Thus, the difference does not exceed one. By theorem 3.4, we get, $\chi_{=}(G \,\square\, C_n) = m$ for $m = 2k, k \ge 1$, $p \ge 2, q \ge 1\ \&\ p-1 \le q \le p+1$ and also $m = 2k+3\ \&\ p = 2$.

Theorem 4.5 Let G and H be the two graphs, where G is a semi-total point graph of complete bipartite, $T_2(K_{p,p})$ on $p \geq 1$ vertices, the equitable colouring of the Cartesian product of G and H, for $n = 3k, k \geq 1$

(i) $\chi_=(G \,\square\, P_n) = \chi_=(G \,\square\, C_n) = 3$

(ii) $\chi_=(G \,\square\, K_n) = n$

Proof. Define the map $\sigma : V(G \,\square\, H) \rightarrow \{0,1,2, \dots, l\}\ \forall\, l \in W$

The proof of the theorem is divided into three cases,

Let the graph G is a semi-total point graph of bipartite on $p, q \geq 1$.

Case 1a: Let $H = P_n$ for $n = 3k,\ k \geq 1$

Cleary, the number of vertices in $T_2(K_{p,q}) \,\square\, P_n$ is $np(q+1)$ vertices and the number of edges is $4npq - pq + np - 2nq - p$ edges.

Let the vertex set and edge set of the Cartesian $T_2(K_{p,q}) \,\square\, P_n$ be

$$V(\,G \,\square\, H) = \left(\bigcup_{j=0}^{n-1}\bigcup_{i=0}^{p-1}\{u_i v_j\}\right) \cup \left(\bigcup_{j=0}^{n-1}\bigcup_{i'=0}^{q-1}\{u_{i'} v_j\}\right) \cup \left(\bigcup_{j=0}^{n-1}\bigcup_{i=0}^{p-2}\bigcup_{i'=0}^{q-1}\{e_{ii'} v_j\}\right)$$

$$E(\,G \,\square\, H) = \left(\bigcup_{j=0}^{n-1}\bigcup_{i=0}^{p-1}\bigcup_{i'=0}^{q-1}\{(u_i v_j)(u_{i'} v_j)\}\right) \cup \left(\bigcup_{j=0}^{n-1}\bigcup_{i=0}^{p-2}\bigcup_{i'=0}^{q-1}\{(u_i v_j)(e_{ii'} v_j); (e_{ii'} v_j)(u_{i'} v_j)\}\right)$$
$$\cup \left(\bigcup_{i=0}^{p-1}\bigcup_{j=0}^{n-2}\{(u_i v_j)(u_i v_{j+1})\}\right) \cup \left(\bigcup_{i=0}^{q-1}\bigcup_{j=0}^{n-2}\{(u_{i'} v_j)(u_{i'} v_{j+1})\}\right)$$
$$\cup \left(\bigcup_{i=0}^{p-2}\bigcup_{i=0}^{q-1}\bigcup_{j=0}^{n-2}\{e_{ii'} v_j)(e_{ii'} v_{j+1})\}\right)$$

The colouring of vertices

$\forall\ 0 \leq i \leq p-1,\ 0 \leq i' \leq q-1$ and $0 \leq j \leq n-1$,

$$\sigma(u_i v_j) = j (mod\ 3)$$

$$\sigma(u_{i'} v_j) = (j+1)(mod\ 3)$$

$$\sigma(e_{ii'} v_j) = (j+2)(mod\ 3)$$

If $p = q$, then G is a semi-total point graph of complete bipartite, $T_2(K_{p,p})$.

Each colour $(0,1,2)$ appears exactly $\frac{np(p+2)}{3}$ times. Thus, the difference does not exceed one.

By theorem 3.4, we get, $\chi_=(G \,\square\, P_n\,) = 3$ for $n = 3k,\ k \geq 1$.

Case 1b: Let $H = C_n$ for $n = 3k,\ k \geq 1$

Cleary, the number of vertices in $T_2(K_{p,q}) \,\square\, C_n$ is $np(q+1)$ vertices and the number of edges is $n(4pq - 2q + p)$ edges.

Let the vertex set and edge set of the Cartesian $T_2(K_{p,q}) \,\square\, C_n$ be

$$V(\,G \,\square\, H) = \left(\bigcup_{j=0}^{n-1}\bigcup_{i=0}^{p-1}\{u_i v_j\}\right) \cup \left(\bigcup_{j=0}^{n-1}\bigcup_{i'=0}^{q-1}\{u_{i'} v_j\}\right) \cup \left(\bigcup_{j=0}^{n-1}\bigcup_{i=0}^{p-2}\bigcup_{i'=0}^{q-1}\{e_{ii'} v_j\}\right)$$

$$E(G \square H) = \left(\bigcup_{j=0}^{n-1}\bigcup_{i=0}^{p-1}\bigcup_{i'=0}^{q-1}\{(u_i v_j)(u_{i'} v_j)\}\right) \cup \left(\bigcup_{j=0}^{n-1}\bigcup_{i=0}^{p-2}\bigcup_{i'=0}^{q-1}\{(u_i v_j)(e_{ii'} v_j); (e_{ii'} v_j)(u_{i'} v_j)\}\right)$$
$$\cup \left(\bigcup_{i=0}^{p-1}\bigcup_{j=0}^{n-1}\{(u_i v_j)(u_i v_{(j+1)(mod\ n)})\}\right) \cup \left(\bigcup_{i=0}^{q-1}\bigcup_{j=0}^{n-1}\{(u_{i'} v_j)(u_{i'} v_{(j+1)(mod\ n)})\}\right)$$
$$\cup \left(\bigcup_{i=0}^{p-2}\bigcup_{i=0}^{q-1}\bigcup_{j=0}^{n-1}\{e_{ii'} v_j)(e_{ii'} v_{(j+1)(mod\ n)})\}\right)$$

The colouring of vertices and the result as follows from theorem 4.5 case 1a.

By theorem 3.4, we get, $\chi_{=}(G \square C_n) = 3$ for $n = 3k,\ k \geq 1$.

Case 2: Let $H = K_n$ for $n = 3k,\ k \geq 1$

Cleary, the number of vertices in $T_2(K_{p,q}) \square K_n$ is $np(q+1)$ vertices and the number of edges is $\frac{n}{2}(npq - 5pq + np - 4q - p)$ edges.

Let the vertex set and edge set of the Cartesian $T_2(K_{p,q}) \square K_n$ be

$$V(G \square H) = \left(\bigcup_{j=0}^{n-1}\bigcup_{i=0}^{p-1}\{u_i v_j\}\right) \cup \left(\bigcup_{j=0}^{n-1}\bigcup_{i'=0}^{q-1}\{u_{i'} v_j\}\right) \cup \left(\bigcup_{j=0}^{n-1}\bigcup_{i=0}^{p-2}\bigcup_{i'=0}^{q-1}\{e_{ii'} v_j\}\right)$$

$$E(G \square H) = \left(\bigcup_{j=0}^{n-1}\bigcup_{i=0}^{p-1}\bigcup_{i'=0}^{q-1}\{(u_i v_j)(u_{i'} v_j)\}\right) \cup \left(\bigcup_{j=0}^{n-1}\bigcup_{i=0}^{p-2}\bigcup_{i'=0}^{q-1}\{(u_i v_j)(e_{ii'} v_j); (e_{ii'} v_j)(u_{i'} v_j)\}\right)$$
$$\cup \left(\bigcup_{i=0}^{p-1}\bigcup_{j=0}^{n-2}\bigcup_{k>j}^{n-1}\{(u_i v_j)(u_i v_k)\}\right) \cup \left(\bigcup_{i=0}^{q-1}\bigcup_{j=0}^{n-2}\bigcup_{k>j}^{n-1}\{(u_{i'} v_j)(u_{i'} v_k)\}\right)$$
$$\cup \left(\bigcup_{i=0}^{p-2}\bigcup_{i=0}^{q-1}\bigcup_{j=0}^{n-2}\bigcup_{k>j}^{n-1}\{e_{ii'} v_j)(e_{ii'} v_{(j+1)(mod\ n)})\}\right)$$

The colouring of vertices as follows from theorem 4.5 case 1a.

If $p = q$, then G is a semi-total point graph of complete bipartite, $T_2(K_{p,p})$.

Each colour $(0,1,2,\ldots,n-1)$ appears exactly $p(p+2)$ times. Thus, the difference does not exceed one. By theorem 3.4, we get, $\chi_{=}(G \square K_n) = n$ for $n = 3k,\ k \geq 1$.

5. Conclusion

In this research article, the Cartesian product is considered for a variety of graphs of semi-total point graph such as path, cycle, complete and complete bipartite with path, cycle, complete, bipartite and complete bipartite. The admittance of equitable colouring to these graphs has been established by defining suitable colouring function. The same concept can be applied to different types of product graphs and thereof.

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