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Eccentric Domination and Restrained Eccentric Domination in Circulant Graphs $C_p\langle 2, 3\rangle$

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***Abstract*:** A subset D of the vertex set V(G) of a graph G is said to be a dominating set if every vertex not in D is adjacent to at least one vertex in D. A dominating set D is said to be an eccentric dominating set if for every $v \in V-D$, there exists at least one eccentric vertex of v in D. The minimum cardinality of an eccentric dominating set is called the eccentric domination number and is denoted by $\gamma_{ed}(G)$. A subset D of V(G) is a restrained eccentric dominating set if D is a restrained dominating set of G and for every $v \in V - D$, there exists at least one eccentric vertex of v in D. The minimum of the cardinalities of the restrained eccentric dominating set of G is called the restrained eccentric domination number of G and is denoted by $\gamma_{red}(G)$. Let $p \geq 6$ be a positive integer. The circulant graph $C_p\langle 2, 3\rangle$ is the graph with vertex set $\{v_0, v_1, v_2, \ldots, v_{p-1}\}$ and edge set $\{\{v_i, v_{i+j}\}: i \in \{0, 1, 2, \ldots, p-1\}$ and $j \in \{2, 3\}\}$. In this paper, we initiate the study of domination number, restrained domination number, eccentric domination number and restrained eccentric domination number in the circulant graphs $C_p\langle 2, 3\rangle$.

***Keywords*:** Domination, Restrained Domination, Eccentric Domination, Restrained Eccentric Domination, Circulant Graphs.

Mathematics Subject Classification: 05C12, 05C69.

1. Introduction

Let G be a finite, simple, undirected (p, q) graph with vertex set V(G) and edge set E(G). For graph theoretic terminology refer to Harary [7], Buckley and Harary [5].

The concept of domination in graphs is originated from the chess games theory and that paved the way to the development of the study of various domination parameters and its relation to various other graph parameters. For details on domination theory, refer to Haynes, Hedetniemi and Slater [8]. Janakiraman, Bhanumathi and Muthammai [9] introduced Eccentric domination in Graphs. Bhanumathi, John Flavia and Kavitha [1] introduced and studied the concept of Restrained Eccentric domination in Graphs.

Definition 1.1: Let $p \geq 6$ be a positive integer. The **circulant graph $C_p\langle 2, 3\rangle$** is the graph with vertex set $\{v_0, v_1, v_2, \ldots, v_{p-1}\}$ and edge set $\{\{v_i, v_{i+j}\}: i \in \{0, 1, 2, \ldots, p-1\}$ and $j \in \{2, 3\}\}$.

Definition 1.2: Let G be a connected graph and v be a vertex of G. The **eccentricity** e(v) of v is the distance to a vertex farthest from v. Thus, $e(v) = \max\{d(u, v) : u \in V\}$. The radius r(G) is the minimum eccentricity of the vertices, whereas the diameter diam(G) = d(G) is the maximum eccentricity. For any connected graph G, $r(G) \leq diam(G) \leq 2r(G)$. The vertex v is a central vertex if e(v) = r(G). The **center** C(G) is the set of all central vertices.

For a vertex v, each vertex at a distance e(v) from v is an eccentric vertex of v. Eccentric set of a vertex v is defined as $E(v) = \{u \in V(G) / d(u, v) = e(v)\}$.

Definition 1.3: A graph G is called a **m-eccentric point graph** if each point of G has exactly $m \geq 1$ eccentric points.

Definition 1.4 [6, 8]: A set $D \subseteq V$ is said to be a **dominating set** in G, if every vertex in $V-D$ is adjacent to some vertex in D. The minimum cardinality of a dominating set is called the **domination number** and is denoted by $\gamma(G)$.

Definition 1.5 [7]: A set $D \subseteq V(G)$ is a **restrained dominating set** if every vertex not in D is adjacent to a vertex in D and to a vertex in $V-D$. The cardinality of minimum restrained dominating set is called the **restrained domination number** and is denoted by $\gamma_r(G)$.

Definition 1.6 [9]: A set $D \subseteq V(G)$ is an **eccentric dominating set** if D is a dominating set of G and for every $v \in V-D$, there exists at least one eccentric vertex of v in D. The minimum cardinality of an eccentric dominating set is called the **eccentric domination number** and is denoted by $\gamma_{ed}(G)$.

Definition 1.7 [1]: A subset D of V(G) is a **restrained eccentric dominating set** if D is a restrained dominating set of G and for every $v \in V-D$, there exists at least one eccentric vertex of v in D. The minimum of the cardinalities of the restrained eccentric dominating set of G is called the **restrained eccentric domination number** of G and is denoted by $\gamma_{red}(G)$.

Theorem 1.1 [8]: For any graph G, $\lceil p/(1+\Delta(G)) \rceil \leq \gamma(G) \leq p-\Delta(G)$.

Theorem 1.2 [3]: Let G be a connected graph. Let $u \in V(G)$ be eccentric to atmost m vertices, then $\lceil p/(1+m) \rceil \leq \gamma_{ed}(G)$.

2. Domination, Restrained domination, Eccentric domination and Restrained eccentric domination in Circulant Graph $C_p\langle 2, 3\rangle$

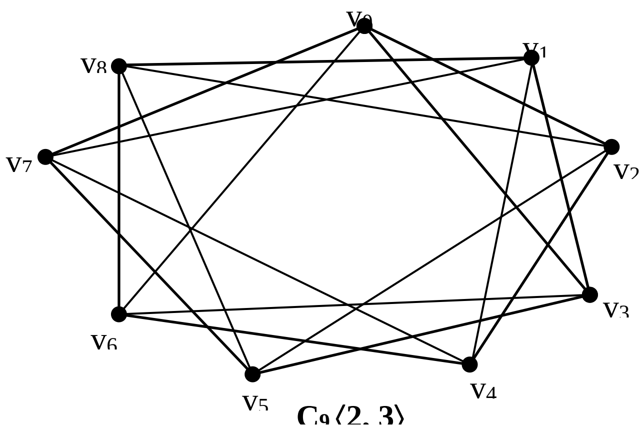
Let $p \geq 6$ be a positive integer. The circulant graph $C_p\langle 2, 3\rangle$ is the graph with vertex set $\{v_0, v_1, v_2, \ldots, v_{p-1}\}$ and edge set $\{\{v_i, v_{i+j}\}: i \in \{0, 1, 2, \ldots, p-1\}$ and $j \in \{2, 3\}\}$.

In this section, the domination number, the restrained domination number, the eccentric domination number and the restrained eccentric domination number of circulant graph $C_p\langle 2, 3\rangle$, for any integer $p \geq 6$ are determined. Cleary, $C_p\langle 2, 3\rangle$ is a 4-regular graph on p vertices.

Clearly, $\gamma(C_p\langle 2, 3\rangle) \leq \gamma_{ed}(C_p\langle 2, 3\rangle) \leq \gamma_{red}(C_p\langle 2, 3\rangle)$ and

$$\gamma(C_p\langle 2, 3\rangle) \leq \gamma_{ed}(C_p\langle 2, 3\rangle) \leq \gamma_{red}(C_p\langle 2, 3\rangle)$$

Example 2.1:



$C_9\langle 2, 3\rangle$

Figure 2.1

In Figure 2.1, $S_1 = \{v_0, v_4, v_8\}$ is a minimum dominating set. It is also a minimum restrained dominating set. Therefore, $\gamma(C_9\langle 2, 3\rangle) = \gamma_r(C_9\langle 2, 3\rangle) = 3$.

$S_2 = \{v_0, v_3, v_6\}$ is a minimum eccentric dominating set and is also a minimum restrained eccentric dominating set. Therefore, $\gamma_{ed}(C_9\langle 2, 3\rangle) = \gamma_{red}(C_9\langle 2, 3\rangle) = 3$.

Theorem 2.1: For any integer $p > 6$, $\gamma(C_p\langle 2, 3\rangle) = \gamma_r(C_p\langle 2, 3\rangle) = \lceil p/4 \rceil$.

Proof: Let $p > 6$ and let C_p represent the cycle in $C_p\langle 2, 3\rangle$ with vertices $v_0, v_1, v_2, \ldots, v_{p-1}$ and edges v_0v_2, v_0v_3, …, $v_{p-3}v_0$, $v_{p-2}v_0$. $C_p\langle 2, 3\rangle$ is a four regular graph.

Hence, $\gamma(C_p\langle 2, 3\rangle) \geq \lceil p/1 + 4 \rceil = \lceil p/5 \rceil$.

Let S be a minimum dominating set of $C_p\langle 2, 3\rangle$. Suppose $v_0 \in S$. Vertex v_0 dominate v_0, v_2, v_3, v_{p-3} and v_{p-2}. So, to dominate v_1 any one of v_1, v_3 or v_4 must lie in S. As a worst case, we can assume that $v_4 \in S$.

So, for every consecutive four vertices of C_p, there must be a vertex in S.

Therefore, $\gamma(C_p\langle 2, 3\rangle) \geq \lceil p/4 \rceil$. (1)

Case (i): p = 4k, k ≥ 2.

In this case, $S = \{v_0, v_4, v_8, \ldots, v_{p-8}, v_{p-4}\}$ is a dominating set of $C_p\langle 2, 3\rangle$ and $|S| = \lceil p/4 \rceil$. Thus, $\gamma(C_p\langle 2, 3\rangle) \leq \lceil p/4 \rceil$.

Case (ii): p = 4k + 1, k ≥ 2.

In this case, $S = \{v_0, v_4, v_8, \ldots, v_{p-5}, v_{p-1}\}$ is a dominating set of $C_p\langle 2, 3\rangle$ and $|S| = \lceil p/4 \rceil$. Thus, $\gamma(C_p\langle 2, 3\rangle) \leq \lceil p/4 \rceil$.

Case (iii): p = 4k + 2, k ≥ 3.

In this case, $S = \{v_0, v_4, v_8, \ldots, v_{p-6}, v_{p-3}\}$ is a dominating set of $C_p\langle 2, 3\rangle$ and $|S| = \lceil p/4 \rceil$. Thus, $\gamma(C_p\langle 2, 3\rangle) \leq \lceil p/4 \rceil$.

Case (iv): p = 4k + 3, k ≥ 1.

In this case, $S = \{v_0, v_4, v_8, \ldots, v_{p-7}, v_{p-3}\}$ is a dominating set of $C_p\langle 2, 3\rangle$ and $|S| = \lceil p/4 \rceil$. Thus, $\gamma(C_p\langle 2, 3\rangle) \leq \lceil p/4 \rceil$.

So, in all cases, $\gamma(C_p\langle 2, 3\rangle) \leq \lceil p/4 \rceil$. (2)

From (1) and (2), we get $\gamma(C_p\langle 2, 3\rangle) = \lceil p/4 \rceil$.

In all the above cases, S is also a restrained dominating set of $C_p\langle 2, 3\rangle$. Therefore, $\gamma_r(C_p\langle 2, 3\rangle) = \gamma(C_p\langle 2, 3\rangle)$.

Remark 2.1:

(i) $S_1 = \{v_0, v_3\}$ is a minimum dominating set of $C_6\langle 2, 3\rangle$ and is also a minimum restrained dominating set of $C_6\langle 2, 3\rangle$. Hence, $\gamma(C_6\langle 2, 3\rangle) = \gamma_r(C_6\langle 2, 3\rangle) = 2$.

(ii) $S_2 = \{v_0, v_4, v_7\}$ is a minimum dominating set of $C_{10}\langle 2, 3\rangle$ and is also a minimum restrained dominating set of $C_{10}\langle 2, 3\rangle$. Hence, $\gamma(C_{10}\langle 2, 3\rangle) = \gamma_r(C_{10}\langle 2, 3\rangle) = 3$.

Theorem 2.2: For any integer $p > 13$,

$$\gamma_{ed}(C_p\langle 2,3\rangle)=\begin{cases}\left\lceil \frac{p}{4}\right\rceil & if \quad p=12k, \quad k>1.\\ \left\lceil \frac{p}{4}\right\rceil & if \quad p=12k+1, \quad k>1.\\ \left\lceil \frac{p}{2}\right\rceil & if \quad p=12k+2, \quad k>1.\\ \left\lceil \frac{p}{3}\right\rceil & if \quad p=12k+3, \quad k>1.\\ \left\lceil \frac{p}{4}\right\rceil & if \quad p=12k+5, \quad k>1.\\ \left\lceil \frac{p}{4}\right\rceil & if \quad p=12k+6, \quad k>1.\\ \left\lceil \frac{p}{4}\right\rceil & if \quad p=12k+7, \quad k>1.\\ \left\lceil \frac{p}{2}\right\rceil & if \quad p=12k+8, \quad k>1.\\ \left\lceil \frac{p}{3}\right\rceil & if \quad p=12k+9, \quad k>1.\\ \left\lceil \frac{p}{4}\right\rceil & if \quad p=12k+11, \quad k>1.\end{cases}$$

Also, $\lceil p/4\rceil \leq \gamma_{ed}(C_p\langle 2, 3\rangle) \leq \lceil p/3\rceil - 1$ if p = 12k + 4, k is a multiple of odd number and p = 12k + 10, $\gamma_{ed}(C_p\langle 2, 3\rangle) = \lceil p/4\rceil$ if p = 12k + 4, k is a multiple of two.

Proof: Let $v_0, v_1, v_2, \ldots, v_{p-1}$ be the vertices of $C_p\langle 2, 3\rangle$.

By Theorem 2.1, $\gamma(C_p\langle 2, 3\rangle) = \lceil p/4\rceil$. (3)

Case (i): p = 12k, k > 1.

In this case, $C_p\langle 2, 3\rangle$ is a p/6 self-centered graph. The vertices $v_{\frac{p-4}{2}+i}, v_{\frac{p-2}{2}+i}, v_{\frac{p}{2}+i}, v_{\frac{p+2}{2}+i}, v_{\frac{p+4}{2}+i}$ are the eccentric vertices of v_i (i = 0, 1, 2, ..., p – 1). Therefore, $C_p\langle 2, 3\rangle$ is a 5-eccentric point graph.

$S = \{v_0, v_4, v_8, \ldots, v_{p-8}, v_{p-4}\}$ is an eccentric dominating set of $C_p\langle 2, 3\rangle$ and $|S| = \lceil p/4\rceil$.

Thus, $\gamma_{ed}(C_p\langle 2, 3\rangle) \leq \lceil p/4\rceil$. (4)

From (3) and (4), $\gamma_{ed}(C_p\langle 2, 3\rangle) = \lceil p/4\rceil$.

Case (ii): p = 12k + 1, k > 1.

In this case, $C_p\langle 2, 3\rangle$ is a (p – 1)/6 self-centered graph. The vertices $v_{\frac{p-5}{2}+i}, v_{\frac{p-3}{2}+i}, v_{\frac{p-1}{2}+i}, v_{\frac{p+1}{2}+i}, v_{\frac{p+3}{2}+i}, v_{\frac{p+5}{2}+i}$ are the eccentric vertices of v_i (i = 0, 1, 2, ..., p – 1). Therefore, $C_p\langle 2, 3\rangle$ is a 6-eccentric point graph.

$S = \{v_0, v_4, v_8, \ldots, v_{p-5}, v_{p-1}\}$ is an eccentric dominating set of $C_p\langle 2, 3\rangle$ and $|S| = \lceil p/4\rceil$.

Thus, $\gamma_{ed}(C_p\langle 2, 3\rangle) \leq \lceil p/4\rceil$. (5)

From (3) and (5), $\gamma_{ed}(C_p\langle 2, 3\rangle) = \lceil p/4\rceil$.

Case (iii): p = 12k + 2, k ≥ 1.

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In this case, $C_p\langle 2, 3\rangle$ is a (p + 4)/6 self-centered graph. The vertex $v_{\frac{p}{2}+i}$ is the eccentric vertex of v_i (i = 0, 1, 2, ..., p – 1). Therefore, $C_p\langle 2, 3\rangle$ is a self-centered unique eccentric point graph.

Hence, by Theorem 1.2, $\gamma_{ed}(C_p\langle 2, 3\rangle) \geq \lceil p/2 \rceil$. (6)

S = {v_0, v_2, v_4, …, v_{p-4}, v_{p-2}} is an eccentric dominating set of $C_p\langle 2, 3\rangle$ and $|S| = \lceil p/2 \rceil$.

Thus, $\gamma_{ed}(C_p\langle 2, 3\rangle) \leq \lceil p/2 \rceil$. (7)

From (6) and (7), $\gamma_{ed}(C_p\langle 2, 3\rangle) = \lceil p/2 \rceil$.

Case (iv): p = 12k + 3, k ≥ 1.

In this case, $C_p\langle 2, 3\rangle$ is a (p + 3)/6 self-centered graph. The vertices $v_{\frac{p-1}{2}+i}, v_{\frac{p+1}{2}+i}$ are the eccentric vertices of v_i (i = 0, 1, 2, ..., p – 1). Therefore, $C_p\langle 2, 3\rangle$ is a 2-eccentric point graph.

Hence, by Theorem 1.2, $\lceil p/3 \rceil \leq \gamma_{ed}(C_p\langle 2, 3\rangle)$. (8)

S = {v_0, v_3, v_6, …, v_{p-6}, v_{p-3}} is an eccentric dominating set of $C_p\langle 2, 3\rangle$ and $|S| = \lceil p/3 \rceil$.

Thus, $\gamma_{ed}(C_p\langle 2, 3\rangle) \leq \lceil p/3 \rceil$. (9)

From (8) and (9), $\gamma_{ed}(C_p\langle 2, 3\rangle = \lceil p/3 \rceil$.

Case (v): p = 12k + 5, k ≥ 1.

In this case, $C_p\langle 2, 3\rangle$ is a (p + 1)/6 self-centered graph. The vertices $v_{\frac{p-3}{2}+i}, v_{\frac{p-1}{2}+i}, v_{\frac{p+1}{2}+i}, v_{\frac{p+3}{2}+i}$ are the eccentric vertices of v_i (i = 0, 1, 2, ..., p – 1). Therefore, $C_p\langle 2, 3\rangle$ is a 4-eccentric point graph.

S = {v_0, v_4, v_8, …, v_{p-5}, v_{p-1}} is an eccentric dominating set of $C_p\langle 2, 3\rangle$ and $|S| = \lceil p/4 \rceil$.

Thus, $\gamma_{ed}(C_p\langle 2, 3\rangle) \leq \lceil p/4 \rceil$. (10)

From (3) and (10), $\gamma_{ed}(C_p\langle 2, 3\rangle) = \lceil p/4 \rceil$.

Case (vi): p = 12k + 6, k ≥ 1.

In this case, $C_p\langle 2, 3\rangle$ is a p/6 self-centered graph. The vertices $v_{\frac{p-4}{2}+i}, v_{\frac{p-2}{2}+i}, v_{\frac{p}{2}+i}, v_{\frac{p+2}{2}+i}, v_{\frac{p+4}{2}+i}$ are the eccentric vertices of v_i (i = 0, 1, 2, ..., p – 1). Therefore, $C_p\langle 2, 3\rangle$ is a 5-eccentric point graph.

S = {v_0, v_4, v_8, …, v_{p-6}, v_{p-3}} is an eccentric dominating set of $C_p\langle 2, 3\rangle$ and $|S| = \lceil p/4 \rceil$.

Thus, $\gamma_{ed}(C_p\langle 2, 3\rangle) \leq \lceil p/4 \rceil$. (11)

From (3) and (11), $\gamma_{ed}(C_p\langle 2, 3\rangle) = \lceil p/4 \rceil$.

Case (vii): p = 12k + 7, k ≥ 1.

In this case, $C_p\langle 2, 3\rangle$ is a (p – 1)/6 self-centered graph. The vertices $v_{\frac{p-5}{2}+i}, v_{\frac{p-3}{2}+i}, v_{\frac{p-1}{2}+i}, v_{\frac{p+1}{2}+i}, v_{\frac{p+3}{2}+i}, v_{\frac{p+5}{2}+i}$ are the eccentric vertices of v_i (i = 0, 1, 2, ..., p – 1). Therefore, $C_p\langle 2, 3\rangle$ is a 6-eccentric point graph.

S = {v_0, v_4, v_8, …, v_{p-7}, v_{p-3}} is an eccentric dominating set of $C_p\langle 2, 3\rangle$ and $|S| = \lceil p/4 \rceil$.

Thus, $\gamma_{ed}(C_p\langle 2, 3\rangle) \leq \lceil p/4 \rceil$. (12)

From (3) and (12), $\gamma_{ed}(C_p\langle 2, 3\rangle) = \lceil p/4 \rceil$.

Case (viii): p = 12k + 8, k ≥ 1.

In this case, $C_p\langle 2, 3\rangle$ is a (p + 4)/6 self-centered graph. The vertex $v_{\frac{p}{2}+i}$ is the eccentric vertex of v_i (i = 0, 1, 2, ..., p – 1). Therefore, $C_p\langle 2, 3\rangle$ is a self-centered unique eccentric point graph.

Hence, by Theorem 1.2, $\gamma_{ed}(C_p\langle 2, 3\rangle) \geq \lceil p/2 \rceil$. (13)

S = $\{v_0, v_2, v_4, \ldots, v_{(p-4)/2}, v_{(p+2)/2}, v_{(p+6)/2}, v_{(p+10)/2}, \ldots, v_{p-3}, v_{p-1}\}$ is an eccentric dominating set of $C_p\langle 2, 3\rangle$ and $|S| = \lceil p/2 \rceil$.

Thus, $\gamma_{ed}(C_p\langle 2, 3\rangle) \leq \lceil p/2 \rceil$. (14)

From (13) and (14), $\gamma_{ed}(C_p\langle 2, 3\rangle) = \lceil p/2 \rceil$.

Case (ix): p = 12k + 9, k ≥ 1.

In this case, $C_p\langle 2, 3\rangle$ is a (p + 3)/6 self-centered graph. The vertices $v_{\frac{p-1}{2}+i}, v_{\frac{p+1}{2}+i}$ are the eccentric vertices of v_i (i = 0, 1, 2, ..., p – 1). Therefore, $C_p\langle 2, 3\rangle$ is a 2-eccentric point graph.

Hence, by Theorem 2.1, $\lceil p/3 \rceil \leq \gamma_{ed}(C_p\langle 2, 3\rangle)$. (15)

S = $\{v_0, v_3, v_6, \ldots, v_{p-6}, v_{p-3}\}$ is an eccentric dominating set of $C_p\langle 2, 3\rangle$ and $|S| = \lceil p/3 \rceil$.

Thus, $\gamma_{ed}(C_p\langle 2, 3\rangle) \leq \lceil p/3 \rceil$. (16)

From (15) and (16), $\gamma_{ed}(C_p\langle 2, 3\rangle) = \lceil p/3 \rceil$.

Case (x): p = 12k + 11, k ≥ 1.

In this case, $C_p\langle 2, 3\rangle$ is a (p + 1)/6 self-centered graph. The vertices $v_{\frac{p-3}{2}+i}, v_{\frac{p-1}{2}+i}, v_{\frac{p+1}{2}+i}, v_{\frac{p+3}{2}+i}$ are the eccentric vertices of v_i (i = 0, 1, 2, ..., p – 1). Therefore, $C_p\langle 2, 3\rangle$ is a 4-eccentric point graph.

S = $\{v_0, v_4, v_8, \ldots, v_{p-7}, v_{p-3}\}$ is an eccentric dominating set of $C_p\langle 2, 3\rangle$ and $|S| = \lceil p/4 \rceil$.

Thus, $\gamma_{ed}(C_p\langle 2, 3\rangle) \leq \lceil p/4 \rceil$. (17)

From (3) and (17), $\gamma_{ed}(C_p\langle 2, 3\rangle) = \lceil p/4 \rceil$.

Case (xi): p = 12k + 4, k ≥ 1.

Subcase (i): k is a multiple of odd number.

In this case, $C_p\langle 2, 3\rangle$ is a (p + 2)/6 self-centered graph. The vertices $v_{\frac{p-2}{2}+i}, v_{\frac{p}{2}+i}, v_{\frac{p+2}{2}+i}$ are the eccentric vertices of v_i (i = 0, 1, 2, ..., p – 1). Therefore, $C_p\langle 2, 3\rangle$ is a 3-eccentric point graph.

S = $\{v_0, v_3, v_6, \ldots, v_{p-7}, v_{p-4}\}$ is an eccentric dominating set of $C_p\langle 2, 3\rangle$ and $|S| = \lceil p/3 \rceil - 1$.

Thus, $\gamma_{ed}(C_p\langle 2, 3\rangle) \leq \lceil p/3 \rceil - 1$. (18)

From (3) and (18), $\lceil p/4 \rceil \leq \gamma_{ed}(C_p\langle 2, 3\rangle) \leq \lceil p/3 \rceil - 1$.

Subcase (ii): k is a multiple of two.

In this case, $C_p\langle 2, 3\rangle$ is a (p + 2)/6 self-centered graph. The vertices $v_{\frac{p-2}{2}+i}, v_{\frac{p}{2}+i}, v_{\frac{p+2}{2}+i}$ are the eccentric vertices of v_i (i = 0, 1, 2, ..., p – 1). Therefore, $C_p\langle 2, 3\rangle$ is a 3-eccentric point graph.

$S = \{v_0, v_4, v_8, \ldots, v_{p-8}, v_{p-4}\}$ is an eccentric dominating set of $C_p\langle 2, 3\rangle$ and $|S| = \lceil p/4 \rceil$.

Thus, $\gamma_{ed}(C_p\langle 2, 3\rangle) \leq \lceil p/4 \rceil$. (19)

From (3) and (19), $\gamma_{ed}(C_p\langle 2, 3\rangle) = \lceil p/4 \rceil$.

Case (xii): p = 12k + 10, k ≥ 1.

In this case, $C_p\langle 2, 3\rangle$ is a (p + 2)/6 self-centered graph. The vertices $v_{\frac{p-2}{2}+i}, v_{\frac{p}{2}+i}, v_{\frac{p+2}{2}+i}$ are the eccentric vertices of v_i (i = 0, 1, 2, ..., p – 1). Therefore, $C_p\langle 2, 3\rangle$ is a 3-eccentric point graph.

$S = \{v_0, v_3, v_6, \ldots, v_{p-7}, v_{p-4}\}$ is an eccentric dominating set of $C_p\langle 2, 3\rangle$ and $|S| = \lceil p/3 \rceil - 1$.

Thus, $\gamma_{ed}(C_p\langle 2, 3\rangle) \leq \lceil p/3 \rceil - 1$. (20)

From (3) and (20), $\lceil p/4 \rceil \leq \gamma_{ed}(C_p\langle 2, 3\rangle) \leq \lceil p/3 \rceil - 1$.

Remark 2.2:

- $S_1 = \{v_0, v_3\}$ is a minimum eccentric dominating set of $C_6\langle 2, 3\rangle$.
Hence, $\gamma_{ed}(C_6\langle 2, 3\rangle) = 2$.
- $S_2 = \{v_0, v_3, v_6\}$ is a minimum eccentric dominating set of $C_p\langle 2, 3\rangle$.
Hence, $\gamma_{ed}(C_p\langle 2, 3\rangle) = 3$, p = 7, 8, 9.
- $S_3 = \{v_0, v_4, v_7\}$ is a minimum eccentric dominating set of $C_{10}\langle 2, 3\rangle$.

Hence, $\gamma_{ed}(C_{10}\langle 2, 3\rangle) = 3$.

- $S_4 = \{v_0, v_4, v_8\}$ is a minimum eccentric dominating set of $C_p\langle 2, 3\rangle$.

Hence, $\gamma_{ed}(C_p\langle 2, 3\rangle) = 3$, p = 11, 12.

- $S_5 = \{v_0, v_4, v_8, v_{12}\}$ is a minimum eccentric dominating set of

$C_{13}\langle 2, 3\rangle$. Hence, $\gamma_{ed}(C_{13}\langle 2, 3\rangle) = 4$.

Corollary 2.1: For any integer p > 13,

$$\gamma_{red}(C_p\langle 2,3\rangle) = \begin{cases} \left\lceil \frac{p}{4} \right\rceil & if \quad p = 12k, \quad k > 1. \\ \left\lceil \frac{p}{4} \right\rceil & if \quad p = 12k+1, \quad k > 1. \\ \left\lceil \frac{p}{2} \right\rceil & if \quad p = 12k+2, \quad k > 1. \\ \left\lceil \frac{p}{3} \right\rceil & if \quad p = 12k+3, \quad k > 1. \\ \left\lceil \frac{p}{4} \right\rceil & if \quad p = 12k+5, \quad k > 1. \\ \left\lceil \frac{p}{4} \right\rceil & if \quad p = 12k+6, \quad k > 1. \\ \left\lceil \frac{p}{4} \right\rceil & if \quad p = 12k+7, \quad k > 1. \\ \left\lceil \frac{p}{2} \right\rceil & if \quad p = 12k+8, \quad k > 1. \\ \left\lceil \frac{p}{3} \right\rceil & if \quad p = 12k+9, \quad k > 1. \\ \left\lceil \frac{p}{4} \right\rceil & if \quad p = 12k+11, \quad k > 1. \end{cases}$$

Also, $\lceil p/4 \rceil \leq \gamma_{red}(C_p\langle 2, 3\rangle) \leq \lceil p/3 \rceil - 1$ if p = 12k + 4, k is a multiple of odd number and p = 12k + 10, $\gamma_{red}(C_p\langle 2, 3\rangle) = \lceil p/4 \rceil$ if p = 12k + 4, k is a multiple of two.

Proof: The γ_{ed}-sets found in Theorem 2.2 are also restrained eccentric dominating sets. Hence, the theorem follows.

Remark 2.3:

- $S_1 = \{v_0, v_3\}$ is a minimum eccentric dominating set of $C_6\langle 2, 3\rangle$.

Hence, $\gamma_{red}(C_6\langle 2, 3\rangle) = 2$.

- $S_2 = \{v_0, v_3, v_6\}$ is a minimum eccentric dominating set of $C_p\langle 2, 3\rangle$.

Hence, $\gamma_{red}(C_p\langle 2, 3\rangle) = 3$, p = 7, 8, 9.

- $S_3 = \{v_0, v_4, v_7\}$ is a minimum eccentric dominating set of $C_{10}\langle 2, 3\rangle$.

Hence, $\gamma_{red}(C_{10}\langle 2, 3\rangle) = 3$.

- $S_4 = \{v_0, v_4, v_8\}$ is a minimum eccentric dominating set of $C_p\langle 2, 3\rangle$.

Hence, $\gamma_{red}(C_p\langle 2, 3\rangle) = 3$, p = 11, 12.

- $S_5 = \{v_0, v_4, v_8, v_{12}\}$ is a minimum eccentric dominating set of

$C_{13}\langle 2, 3\rangle$. Hence, $\gamma_{red}(C_{13}\langle 2, 3\rangle) = 4$.

Conclusion:

In this paper, we have found out the exact values of domination number and restrained domination number of $C_p\langle 2, 3\rangle$. Also, we have evaluated the exact values of eccentric domination number and restrained eccentric domination number of circulant graphs $C_p\langle 2, 3\rangle$.

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