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Tree Domination Number in Total Graph *T(G)* of a Graph G.

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Abstract: Let G = (V, E) be a connected graph. A subset D of V is called a dominating set of G if N[D] = V. The minimum cardinality of a dominating set of G is called the domination number of G and is denoted by γ(G). A dominating set D of a graph G is called a tree dominating set (ntr - set) if the induced subgraph ⟨D⟩ is a tree. The tree domination number γ_{tr}(G) of G is the minimum cardinality of a tree dominating set. The total graph T(G) of a graph G is a graph such that the vertex set T(G) corresponds to the vertices and edges of G and two vertices are adjacent in T(G) if and only if their corresponding elements are either adjacent or incident in G. In this paper, we have studied some bounds for tree domination number of Total graph T(G) of a graph. Also, we have found the tree domination number of T(G) for some graphs.

In this paper, tree domination number of total graphs of some standard graphs are obtained. Also we have characterized graphs for which γ_{tr}(T(G)) = 1, 2 or n – 2.

Keywords: Domination number, connected domination number, tree domination number, total graph.

Mathematics Subject Classification: 05C69, 05C76

1 Introduction

Graphs are indispensable tools in every phase of human daily life and in many disciplines where mathematics permeates. Graph theory also has its own invariants. The concept of domination is one of the main parameters in graph theory. The domination is used in the solution of many problems and analysis of events in the historical process of human life. It is also used in solving network problems.

The graphs considered here are nontrivial, finite and undirected. The order and size of G are denoted by n and m respectively. If D ⊆ V, then $N(D) = \bigcup_{v \in D} N(v)$ and N[D] = N(D) ∪ D where N(v) is the set of vertices of G which are adjacent to v.

The concept of domination in graphs was introduced by Ore[5].

Definition 1.1: Let G = (V, E) be a connected graph. A subset D of V is called a dominating set of G if N[D] = V. The minimum cardinality of a dominating set of G is called the domination number of G and is denoted by γ(G).

Xuegang Chen, Liang Sun and Alice McRac [8] introduced the concept of tree domination in graphs.

Definition 1.2: A dominating set D of G is called a tree dominating set, if the induced subgraph ⟨D⟩ is a tree. The minimum cardinality of a tree dominating set of G is called the tree domination number of G and is denoted by γ_{tr}(G).

The graph G o K_1 is obtained from the graph G by attaching a pendent edge to all the vertices of G.

Definition 1.3: The total graph T(G) of a graph G is a graph such that the vertex set T(G) corresponds to the vertices and edges of G and two vertices are adjacent in T(G) if and only if their corresponding elements are either adjacent or incident in G.

Definition 1.4: A covering graph is a subgraph which contains either all the vertices or all the edges corresponding to some other graph. A subgraph which contains all the vertices is called a line (edge) covering. A subgraph which contains all the edges is called a vertex covering.

In this paper we study the concept of tree domination in total graphs.

2. Prior Results

Theorem 2.1: [3] For any graph G, $\kappa(G) \leq \delta(G)$.

Theorem 2.1: [4] For any cycle C_n, with $n \geq 4$, $\gamma(T(C_n)) = \left\lfloor \frac{n}{2} \right\rfloor$..

Theorem 2.1: [4] For any $K_{1,n}$, $\gamma(T(K_{1,n})) = 1$.

Theorem 2.2: [8] For any connected graph G with $n \geq 3$, $\gamma_{tr}(G) \leq n - 2$.

Theorem 2.3: [8] For any connected graph G with $\gamma_{tr}(G) = n - 2$ iff $G \cong P_n$ (or) C_n.

3. Main Results

In this section, tree domination number of total graphs is found.

TREE DOMINATION NUMBER OF TOTAL GRAPHS

Definition 3.1:

The total graph T(G) of a graph G is a graph such that the vertex set T(G) corresponds to the vertices and edges of G and two vertices are adjacent in T(G) if and only if their corresponding elements are either adjacent or incident in G.

Example 3.1:

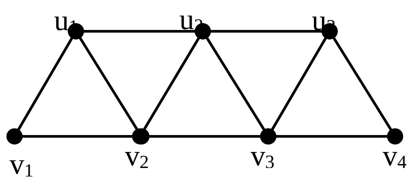

Figure 3.1

In the graph $T(P_4)$ given in Figure 3.1, minimum tree dominating set is $D = \{v_2, v_3\}$ and $\gamma_{tr}(T(P_4)) = 2$.

Example 3.2:

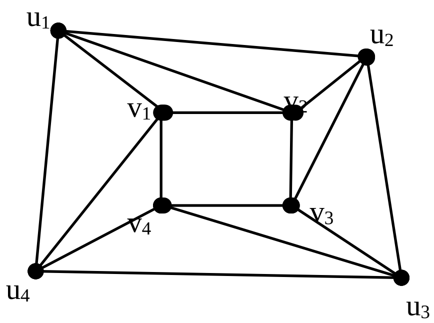

Figure 3.2

In the graph $T(C_4)$ given in Figure 3.2, minimum tree dominating set is $D = \{v_1, v_2, v_3\}$ and $\gamma_{tr}(T(C_4)) = 3$.

Theorem 3.1: For any path P_n on n vertices, $\gamma_{tr}(T(P_n)) = n - 2, n \geq 3$.

Proof:

Let $V(P_n) = \{v_1, v_2, v_3, \ldots, v_n\}$ and let $D = \bigcup_{i=1}^{n-2}\{v_{i+1}\}$. Then D is a dominating set of $T(P_n)$ and the induced subgraph $\langle D\rangle \cong P_{n-2}$. Therefore, D is a tree dominating set of $T(P_n)$ and $\gamma_{tr}(T(P_n)) \leq |D| = n - 2$. Also, since $\gamma_{tr}(P_n) = n - 2$ and $\gamma_{tr}(T(P_n)) \geq \gamma_{tr}(P_n) = n - 2$, $\gamma_{tr}(T(P_n)) = n - 2$.

Theorem 3.2: For any cycle C_n on n vertices, $\gamma_{tr}(T(C_n)) = n - 1, n \geq 3$.

Proof:

Let $V(C_n) = \{v_1, v_2, v_3, \ldots, v_n\}$ and let $D = \bigcup_{i=1}^{n-1}\{v_i\}$.. Then D is a dominating set of $T(C_n)$ and the induced subgraph $\langle D\rangle$ is isomorphic to P_{n-1}. Therefore, D is a tree dominating set of $T(C_n)$ and $\gamma_{tr}(T(C_n)) \leq |D| = n - 1$. Let D′ be a minimum tree dominating set of $T(C_n)$. Since $\gamma(T(C_n)) = n - 2$, atleast one vertex of $T(C_n)$ is to be added with a dominating set of $T(C_n)$. Therefore, D′ contains atleast $n - 1$ vertices and $\gamma_{tr}(T(C_n)) \geq n - 1$ and hence $\gamma_{tr}(T(C_n)) = n - 1$.

Theorem 3.3: $\gamma_{tr}(T(K_{r,s})) = \min\{r, s\} + 1, r, s \geq 2$.

Proof:

Let $[V_1, V_2]$ be the bipartition of $K_{r,s}$ and let $w \in V_2$. Assume $r \leq s$. The set $D = V_1 \cup \{w\}$ is a minimum tree dominating set of $K_{r,s}$. Since $\langle V_1 \cup \{w\}\rangle \cong K_{1,r}$, D is also a tree dominating set of $T(K_{r,s})$.

Therefore, $\gamma_{tr}(T(K_{r,s})) = |D| = |V_1 \cup \{w\}| = r + 1 = \min\{r, s\} + 1$.

Remark 3.1:

$\gamma_{tr}(T(K_{1,n})) = 1$, since the central vertex of $K_{1,n}$ is the central vertex of $T(K_{1,n})$.

Theorem 3.4: $\gamma_{tr}(T(P_n \circ K_1)) = n, n \geq 2$, where $P_n \circ K_1$ is the corona of P_n with K_1.

Proof:

In $P_n \circ K_1$, each vertex of degree greater than or equal to 2 is a support. Since for any connected graph of order atleast two, every support is a member of every tree dominating set, $\gamma_{tr}(P_n \circ K_1) = n$. Let D be a minimum tree dominating set of $P_n \circ K_1$.

Then $D = \{$all supports of $P_n \circ K_1\}$. Since each edge in $P_n \circ K_1$ is incident with atleast one vertex in D, D is also a minimum tree dominating set of $T(P_n \circ K_1)$.

Therefore, $\gamma_{tr}(T(P_n \circ K_1)) = n$.

Theorem 3.5: $\gamma_{tr}(T(\overline{P_n})) = n - 2, 6 \leq n \leq 10$, where $\overline{P_n}$ is the complement of P_n.

Proof:

Let $V(\overline{P_n}) = \{v_1, v_2, v_3, \ldots, v_n\}$ and let $e_{i,j} = (v_i, v_{i+j})$, $i = 1, 2, 3, \ldots, n-2$ and $j = 2, 3, \ldots, n-i$ and $e_{1,n} = (v_1, v_n)$ be the edges of $\overline{P_n}$. Then $v_1, v_2, \ldots, v_n, e_{i,j} \in V(T(\overline{P_n}))$.

Let D = $(\cup_{i=1}^{4}\{v_i\}) \cup (\cup_{i=1}^{n-6}\{e_{i,i+4}\})$. Then D is a dominating set of T($\overline{P_n}$). The induced subgraph ⟨D⟩ is a tree obtained from $P_4 = \{v_1, v_2, v_3, v_4\}$ by attaching a pendant edge at atmost four vertices of P_4. Therefore, D is a tree dominating set of T($\overline{P_n}$) and is minimum.

Therefore, $\gamma_{tr}(T(\overline{P_n})) = 4 + n - 6 = n - 2$.

Remark 3.2:

(i) For n = 4, $\gamma_{tr}(T(\overline{P_n})) = 2$.

(ii) If n = 5, then $\gamma_{tr}(T(\overline{P_n})) = 3$, since the set $\{v_1, v_2, v_5\}$ is a γ_{tr} - set of T($\overline{P_n}$).

(iii) For $n \geq 11$, $\gamma_{tr}(T(\overline{P_n})) = 0$.

Theorem 3.6: $\gamma_{tr}(T(\overline{C_n})) = n - 2$, $6 \leq n \leq 10$, where $\overline{C_n}$ is the complement of C_n.

Proof:

Let V($\overline{C_n}$) = { $v_1, v_2, v_3, \dots ,v_n$} and let $e_{i,j} = (v_i, v_{i+j})$, i = 1, 2, 3, … , n – 2 and j = 2, 3, … , n – i be the edges of $\overline{C_n}$. Then $v_1, v_2, \dots ,v_n, e_{i,j} \in V(T(\overline{C_n}))$.

Let D = $(\cup_{i=1}^{4}\{v_i\}) \cup (\cup_{i=1}^{n-6}\{e_{i,i+4}\})$. Then D is a dominating set of T($\overline{C_n}$) and the induced subgraph ⟨D⟩ is a tree obtained from $P_4 = \{v_1, v_2, v_3, v_4\}$ by attaching a pendant edge at atmost four vertices of P_4. Therefore, D is a tree dominating set of T($\overline{C_n}$) and is minimum. Therefore, $\gamma_{tr}(T(\overline{C_n})) = 4 + n - 6 = n - 2$.

Remark 3.3:

(i) If n = 5, then $\gamma_{tr}(T(\overline{C_n})) = 3$, since $\{v_1, v_2, v_5\}$ is a γ_{tr} - set of T($\overline{C_n}$).

(ii) For $n \geq 11$, $\gamma_{tr}(T(\overline{C_n})) = 0$.

Theorem 3.7: If G is a tree obtained from a star $K_{1,n-1}$ by subdividing k – 1 edges, then $\gamma_{tr}(T(G)) = k$, $4 \leq k \leq n$.

Proof:

Let $V(K_{1,n-1}) = \{v, v_1, v_2, \dots ,v_{n-1}\}$, where v is the central vertex. Let G be a tree obtained from the star $K_{1,n-1}$ by subdividing k – 1 edges $vv_1, vv_2, vv_3, \dots ,vv_{k-1}$, $4 \leq k \leq n$. Then Δ(T) = k – 1. Let w_i be the subdivided vertex adjacent to v_i, i = 1, 2, 3, …, k – 1.

Then the set D = $\{v, w_1, w_2, \dots ,w_{k-1}\}$ is a minimum tree dominating set of T(G). Thus, $\gamma_{tr}(T(G)) = |D| = 1 + k - 1 = k$.

Remark 3.4:

(i) If T_1 is a tree obtained from a star $K_{1,n-1}$ by subdividing atmost n – 1 edges, $1 \leq k - 1 \leq n - 2$ ($n \geq 5$), then $\gamma_{tr}(T(T_1)) = k - 1$.

(ii) If T_2 is a tree obtained from a star $K_{1,n-1}$ by subdividing exactly one edge twice, then $\gamma_{tr}(T(T_2)) = 2$.

In the following, the graphs for which $\gamma_{tr}(T(G)) = 1$, 2 or n – 2 are characterized.

Theorem 3.8: Let G be a connected graph with n vertices and m edges. Then $\gamma_{tr}(T(G)) = 1$ if and only if G is a star.

Proof:

Maximum degree of T(G) = $\Delta(T(G)) = 2\Delta(G)$. But for every connected graph G with n vertices, $\gamma_{tr}(G) = 1$ if and only if $\Delta(G) = n - 1$. Therefore $\gamma_{tr}(T(G)) = 1$ if and only if $\Delta(T(G)) = n + m - 1$. But $\Delta(T(G)) = n + m - 1$ if and only if there exists a vertex v of G with maximum degree such that all the edges are adjacent to v. That is, G is a star.

Conversely, if G is a star, $\gamma_{tr}(T(G)) = 1$.

Theorem 3.9: Let G be a connected graph with atleast four vertices and is not a star. Then $\gamma_{tr}(T(G)) = 2$ if and only if there exists two adjacent vertices u, v in G such that each edge in G is incident with atleast one of u and v.

Proof:

Let G be a connected graph with atleast four vertices and is not a star. Let u and v be two adjacent vertices in G and each edge in G is incident with atleast one of u and v. Then u, v $\in$ V(T(G)).

Let D = {u, v}. Then D $\subseteq$ V(T(G)). Since each edge in G is incident with atleast one of u and v, the vertices of T(G) corresponding to these edges of G are adjacent to atleast one of u and v. Also, all the vertices of G in T(G) are adjacent to atleast one of u and v. Also $\langle D\rangle$ is isomorphic to K_2. Therefore, D is a tree dominating set of T(G). Thus $\gamma_{tr}(T(G)) \leq |D| = 2$. Since G is not a star, $\gamma_{tr}(T(G)) \geq 2$. Hence $\gamma_{tr}(T(G)) = 2$.

` Conversely assume $\gamma_{tr}(T(G)) = 2$. Then there exists a tree dominating set D of T(G) containing two vertices of T(G) and $\langle D\rangle$ is isomorphic to K_2 in T(G).

Let u, v$\in$D.

Case 1. u, v$\in$V(G)

Then vertices of G and L(G) in T(G) are adjacent to atleast one of u and v. That is, each edge in G is incident with atleast one of u and v.

Case 2. u$\in$V(G), v$\in$V(L(G))

Let v = e. Then e is an edge in G incident with v, since $\langle D\rangle$ is a tree. Since D is a dominating set of T(G), all the vertices of G in T(G) are adjacent to u and all the edges of G are adjacent to e. Therefore, G is a star, which is a contradiction since G is not a star.

Case 3. u, v$\in$V(L(G))

Let u = e_1, v = e_2. Then e_1 and e_2 are adjacent edges in G. Since D is a dominating set, all the edges of G are adjacent to e_1, e_2 or both. Let e_3 be an edge of G adjacent to e_1 and $e_3 = (u_3, v_3)$, where $u_3, v_3 \in V(G)$. Then $u_3, v_3 \in V(T(G))$ are adjacent to none of e_1 and e_2 in T(G). Therefore, G is isomorphic to P_3. But $\gamma_{tr}(T(P_3)) = 1$. Therefore, by Case 1 - Case 3, if $\gamma_{tr}(T(G)) = 2$, then there exist two adjacent vertices u, v in G such that each edge in G is incident with atleast one of u and v. Hence the Theorem follows.

Theorem 3.10: Let G be a connected graph with n vertices and m edges. Then $\gamma_{tr}(T(G)) = n + m - 2$ if and only if G is isomorphic to K_2.

Proof:

By Theorem 2.3, " For every connected graph G with n vertices, $\gamma_{tr}(G) = n - 2$ if and only if G is isomorphic to P_n or C_n", $\gamma_{tr}(T(G)) = n + m - 2$ if and only if T(G) is isomorphic to P_{n+m} or C_{n+m}. Since each edge in G gives rise to a triangle in T(G), T(G) is not isomorphic to P_{n+m}. Therefore T(G) is isomorphic to C_{n+m}.

When $n + m \geq 4$, T(G) is not isomorphic to C_{n+m}. Hence $T(G) \cong C_3$ and $G \cong K_2$.

Theorem 3.11: Any tree dominating set D of G is also a tree dominating set of T(G) if and only if each edge in G is incident with atleast one vertex in D.

Proof:

Let D be a tree dominating set of both G and T(G). Let there exist an edge e in G not incident with any of the vertices in D. Then $e \in V(T(G))$ and e is not adjacent to any of the vertices in $D \subseteq V(T(G))$, which is a contradiction. Therefore, each edge in G is incident with atleast one vertex in D.

Conversely, let D be a tree dominating set of G such that each edge in G is incident with atleast one vertex in D. Then D dominates all the vertices of G in T(G) and the vertices in T(G) corresponding to the edges of G. Also $\langle D \rangle$ is a tree in T(G). Therefore, D is a tree dominating set of T(G).

Theorem 3.12: Any tree dominating set D′ of L(G) is a tree dominating set of T(G) if and only if the set of edges of G corresponding to the vertices of D′ is an edge cover of G.

Proof:

Let D′ be a tree dominating set of L(G) and F be the set of edges of G corresponding to the vertices of D′. If F is not an edge cover of G, there exists a vertex v of G such that $v \in V(T(G))$ is not adjacent to any of the vertices in D′.

Conversely, if the set of edges of G corresponding to the vertices of D′ is an edge cover of G, D′ dominates all the vertices of G in T(G) and since D′ is a tree dominating set of L(G), D′ dominates vertices of L(G) in T(G) and $\langle D' \rangle$ is a tree. Therefore, D′ is a tree dominating set of T(G).

4. Conclusion

In this paper, tree domination number of total graphs of some standard graphs are obtained. Also we have characterized graphs for which $\gamma_{tr}(T(G)) = 1, 2$ or $n - 2$.

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