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Generalized Forms of Fuzzy Closure and Fuzzy Interior

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Abstract: In this paper we defined and characterized the concept of generalized fuzzy topological space($g_f -$ topological space) and obtained some significant results. Further, we introduced and studied the concept of generalized fuzzy closure ($g_f -$ closure) and generalized fuzzy interior ($g_f -$ interior) and obtained some significant results in this context with help of various supporting examples

Keywords: Fuzzy topological space, $g_f -$ topological space, $g_f -$closure, $g_f -$interior

1. Introduction

One of the earliest branches of mathematics which applied fuzzy set theory systematically is General Topology. Although fuzzy topology is a generalization of topology in classical mathematics, it has its own marked characteristics. Topology is one such branch of mathematics which places emphasis on the qualitative aspects of physical things and relegates their quantitative measure in the background. Fuzzy topology has found applications in upcoming fields such as digital topology and image processing. Chang, C.L introduced the concept of fuzzy topological spaces [5]. It was in the year 1968 that C. L. Chang [5] grafted the notion of a fuzzy set into general topology and for such spaces he attempted to develop the basic topological concepts like open set, closed set, neighbourhood, interior of a set, continuity, compactness etc. Azad [1] has introduced the concept of fuzzy semi-open sets in fuzzy topological spaces. Bin Shahana [3-4] has introduced the concept of fuzzy pre-open sets and fuzzy α-open sets in fuzzy topological space. Thakur [14] has introduced the concept of fuzzy semi pre-open sets in fuzzy topological spaces. Csaszar [6] introduced the notions of generalized topological spaces. Palani Cheety [9] introduced the concept of generalized fuzzy topology and investigates various properties.

In this paper, we have introduced the concept of $g_f -$topological space and $g_f -$ closure and $g_f -$ interior and verify the results with the help of some counter examples. Some require basic definitions, concepts of topological space and notations are discussed in Section 2. The section 3 has been headed by $g_f -$ topological space, in which we verified various results related it by giving suitable examples. In section 4, we study the concept of $g_f -$closure and $g_f -$interior in $g_f -$topological space and the Section 5 concludes the paper.

2. Preliminaries

Definition 2.1: Let X be a non-empty universal crisp set. A **fuzzy topology** on X is a non-empty collection τ of fuzzy sets on X satisfying the following conditions

i) Fuzzy sets 0 and 1 belong to τ
ii) Any arbitrary union of members of τ is in τ
iii) A finite intersection of members of τ is in τ

Here 0 and 1 represent the Zero Fuzzy Set and the Whole Fuzzy set on X, defined as, $0(x)=0, \forall x \in X$ $1(x)=1, \forall x \in X$ and the pair (X, τ) is called Fuzzy Topological Space on X. For Convenience, we shall denote the fuzzy topological space simply as X.

Definition 2.2: Let (X, τ) be fuzzy topological space. The members of the collection τ are called fuzzy open sets of fuzzy topological space X. The complement of a fuzzy open set of X is called a fuzzy closed set. Thus, a fuzzy set λ on X is a fuzzy closed set in (X, τ) if its complement λ^C is fuzzy open set in X with respect to fuzzy topology τ.

Definition 2.3: Let (X, τ) be a fuzzy topological space. For a fuzzy set A in X the closure of A, denoted by Cl (A) is defined as $Cl(A) = \inf\{K: A \subseteq K, K^C \in \tau\}$. Thus the fuzzy set Cl (A) is the smallest fuzzy closed set in X containing the fuzzy set A. From the definition, if follows that Cl (A) is the intersection of all fuzzy closed sets in X containing A.

Definition 2.4: Let (X, τ) be a fuzzy topological space. For a fuzzy set A in X, the interior of A, denoted by Int(A) is defined as $Int(A) = Sup\{Q : Q \subseteq A, Q \in \tau\}$. Thus the fuzzy set Int (A) is the largest fuzzy open set in X contained in the fuzzy set A. From definition, it follows that Int(A) is the union of all fuzzy open sets in X contained in A.

Proposition 2.1: Let (X, τ) be a fuzzy topological space. Then

i) Arbitrary Intersection of fuzzy closed sets is a fuzzy closed set.
ii) Finite union of fuzzy closed sets is a fuzzy closed set.

Proposition 2.2: Let(X, τ) be a fuzzy topological space and let A be a fuzzy set in X. Then

i) $Cl(A) = A$ if and only if A is a fuzzy closed set in X.
ii) $Int(A) = A$ if and only if A is a fuzzy open set in X.

3. g_f −Topological Space

Definition 3.1: Let X be a crisp set and let "μ" be a collection of fuzzy sets on X. Then "μ" is called $g_f -$ topologyon X if it satisfies following conditions

i) The fuzzy sets 0 and 1 are in μ where $0,1: X \to I$ are defined as $0(x) = 0$ and $1(x) = 1$ for all $x \in X$
ii) If $\{\lambda_j\}, j \in J$ is any family of fuzzy sets on X where $\lambda_j \in \mu$ then $\cup_{j\in J} \lambda_j \in \mu$

The pair (X, μ) is called $g_f -$topological space.

Definition 3.2: Let (X, μ) be $g_f -$topological space. The members of the collection "μ" are called $g_f -$open set in $g_f -$topological space. The complement of $g_f -$open set in X is called $g_f -$close set.

Example 3.1: Let $X = \{x_1, x_2\}$, and we consider fuzzy sets $A = \{(x_1, 0.4), (x_2, 0.7)\}$, $B = \{(x_1, 0.6), (x_2, 0.5)\}$ and $C = \{(x_1, 0.6), (x_2, 0.7)\}$ on X. Then clearly $\mu = \{0, A, B, C, 1\}$ is $g_f -$ topology on X, but not fuzzy topology on X.

Remark 3.1: Arbitrary union of $g_f -$open sets is $g_f -$open set

Proposition 3.1: Let (X, μ) be $g_f -$topological space. Then:

i) 0 and 1 are fuzzy $g_f -$closed sets in X.
ii) Arbitrary intersection of $g_f -$closed sets in X is $g_f -$closed set in X.

Proof (i): Since 0 and 1 are $g_f -$open sets in X therefore their complement 1 and 0 are $g_f -$closed sets in X.

(ii): Let $\{\lambda_j\}_{j\in J}$ be a collection of $g_f -$closed sets in X, where J is index set. Then $\{\lambda_j^c\}_{j\in J}$ is a collection of $g_f -$open sets in X. This implies $\cup_{j\in J} \lambda_j^c$ is $g_f -$open sets in X. Hence $(\cup_{j\in J} \lambda_j^c)^c = \cap_{j\in J} \lambda_j$ is $g_f -$closed set in X.

Remark 3.2: Since arbitrary union of $g_f -$open set is $g_f -$open set, $I_\mu(\lambda)$ is $g_f -$open set in X. Further since arbitrary intersection of $g_f -$closed set is $g_f -$closed set. $Cl_\mu(\lambda)$ is $g_f -$closed set in X. Intersection of two $g_f -$open sets may not $g_f -$open set and therefore union of two $g_f -$closed sets may not $g_f -$closed set in X. In Example 3.1 we see that $A \cap B = \{(x_1, 0.4), (x_2, 0.5)$ is not $g_f -$ open set in X and $A \cup B = \{(x_1, 0.6), (x_2, 0.7)\}$ is not $g_f -$closed set in X.

Proposition 3.2: let $\{\mu_j\}_{j\in J}$ be a collection of g_f −topologies on X, where J is an index set then their intersection $\cap_{j\in J} \mu_j$ is also a g_f −topology on X.

Proof: let $\{\mu_j\}_{j\in J}$ be a collection of g_f −topologies on X, where J is an arbitrary index set be a collection of g_f −topologies on X. Then $0,1 \in \{\mu_j\}_{j\in J}$ for all $j \in J$. This implies $0,1 \in \cap_{j\in J} \{\mu_j\}$. Now let $A_\alpha \in \cap_{j\in J} \{\mu_j\}$ for $\alpha \in J_1$ where J_1 is an arbitrary index set. Then $A_\alpha \in \{\mu_j\}_{j\in J}$ for all $j \in J$ and for all $\alpha \in J_1$. Since each $\{\mu_j\}_{j\in J}$ be a collection of g_f −topologies on X. it follow that $\cup_{\alpha\in J_1} A_\alpha \in \mu_j$ for all $j \in J$. Hence $\cup_{\alpha\in J_1} A_\alpha \in \cap_{j\in J} \mu_j$. Thus $\cap_{j\in J} \{\mu_j\}$ is g_f −topology on X. However collection of g_f −topology on X is not closed set under the operation of union i.e. union of two g_f −topologies is not necessarily g_f −topology.

4. g_f −Closure and g_f −Interior

Definition 4.1: Let (X, μ) be g_f −topological space. For a fuzzy set A in X the g_f −closure of A is defined as $Cl_\mu(A) = \inf\{K : A \subseteq K, K^C \in \mu\}$. Thus $Cl_\mu(A)$ is the smallest g_f −closed set in X containing the fuzzy set A. From the definition, if follows that $Cl_\mu(A)$ is the intersection of all g_f −closed sets in X containing A.

Definition 4.2: Let (X, μ) be g_f −topological space. For a fuzzy set A in X, the g_f −interior of A, is defined as $I_\mu(A) = \mathrm{Sup}\{Q : Q \subseteq A, Q \in \mu\}$. Thus $I_\mu(A)$ is the largest g_f −open set in X contained in the fuzzy set A. From the definition, if follows that $I_\mu(A)$ is the union of all g_f −open sets in X contained in A.

Proposition 4.1: Let (X, μ) be g_f −topological space and let A be a fuzzy set in X. Then A is g_f −closed set if and only if $Cl_\mu(A) = A$.

Proof: Suppose that λ is g_f −closed set in X. then clearly the smallest g_f −closed set containing A is itself A. Hence $Cl_\mu(A) = A$. Conversely suppose $Cl_\mu(A) = A$ then by definition of g_f −closure it follow that $Cl_\mu(A)$ is g_f −closed set.

Proposition 4.2: Let (X, μ) be g_f −topological space and let A and B are two g_f −open set on X. Then following properties holds.

i) $Cl_\mu(0) = 0$.
ii) $Cl_\mu(1) = 1$.
iii) If $A \subseteq B$ then $Cl_\mu(A) \subseteq Cl_\mu(B)$.
iv) $Cl_\mu(A) \cup Cl_\mu(B) \subseteq Cl_\mu(A \cup B)$.
v) $Cl_\mu\left(Cl_\mu(A)\right) = Cl_\mu(A)$

Proof: Since 0 and 1 are g_f −closed set from let (X, μ) be g_f −topological space and let A be g_f −open set in X. then A is g_f −closed set if and only if $Cl_\mu(A) = A$ we have $Cl_\mu(0) = 0$ and $Cl_\mu(1) = 1$. Suppose $A \subseteq B$ in X. Since $B \subseteq Cl_\mu(B)$ and $A \subseteq B$ we have $A \subseteq CI_\mu(B)$. Now $Cl_\mu(B)$ is g_f −closed set we have, $Cl_\mu(A) \subseteq Cl_\mu(B)$ because $Cl_\mu(A)$ is the smallest g_f −closed set containing A. As $A \subseteq A \cup B$, $B \subseteq A \cup B$ we have $Cl_\mu(A) \subseteq Cl_\mu(A \cup B)$ and $Cl_\mu(B) \subseteq Cl_\mu(A \cup B)$ this implies $Cl_\mu(A) \cup Cl_\mu(B) \subseteq Cl_\mu(A \cup B)$. Since $Cl_\mu(A)$ is g_f −closed set in X. if follow that $Cl_\mu\left(Cl_\mu(A)\right) = Cl_\mu(A)$.

Proposition 4.3: Let X be g_f −topological space and $\{A_j\}_{j\in J}$ be a family of fuzzy subsets of X. Then

i) $\cup_{j\in J} Cl_\mu(A_j) \subseteq Cl_\mu(\cup_{j\in J} A_j)$.
ii) $Cl_\mu(\cap_{j\in J} A_j) \subseteq \cap_{j\in J} Cl_\mu(A_j)$.

Proposition 4.4: Let A be g_f −open set in g_f −topological space (X, μ). Then A is g_f −open set if and only if $I_\mu(A) = A$

Proof: Suppose that A is g_f −open set in X. Since $I_\mu(A)$ is the union of all g_f −open set in X contained in A and $A \subseteq A$ follows that $A \subseteq I_\mu(A)$. As we know that $I_\mu(A) \subseteq A$, we find that $I_\mu(A) = A$.

Conversely, suppose that $I_\mu(A) = A$. Then by the definition of g_f −interior of g_f −open set it follows that $I_\mu(A)$ is g_f −open set. Thus A is g_f −open set in X.

Proposition 4.5: Let (X, μ) be g_f −topological space and A, B are two fuzzy $\mathcal{S}$ets in X. Then

i) $I_\mu(0) = 0$.
ii) $I_\mu(1) = 1$.
iii) If $A \subseteq B$ then $I_\mu(A) \subseteq I_\mu(B)$.
iv) $I_\mu(A \cup B) = I_\mu(A) \cup I_\mu(B)$.
v) $I_\mu(A \cap B) \subseteq I_\mu(A) \cap I_\mu(B)$.
vi) $I_\mu\left(I_\mu(A)\right) = I_\mu(A)$.

Proof: Since 0 and 1 are g_f −open sets in g_f −topological space (X, μ) and let A be g_f −open set in X. Then A is g_f −open set if and only if $I_\mu(A) = A$ we have $I_\mu(0) = 0$ and $I_\mu(1) = 1$. Suppose $A \subseteq B$ in X. Since $I_\mu(A) \subseteq A$ and $A \subseteq B$ we have $I_\mu(A) \subseteq B$. Now $I_\mu(B)$ is g_f −open set we have $I_\mu(A) \subseteq I_\mu(B)$ because $I_\mu(B)$ is the largest g_f −open set contained in B. As $A \subseteq A \cup B$, $B \subseteq A \cup B$ we have $I_\mu(A) \subseteq I_\mu A \cup B)$ and $I_\mu(B) \subseteq I_\mu(A \cup B)$. This implies $I_\mu(A) \cup I_\mu(B) \subseteq I_\mu(A \cup B)$. Since $I_\mu(A)$ is g_f −open set in X, it follow that $I_\mu\left(I_\mu(A)\right) = I_\mu(A)$.

Proposition 4.6: Let X be g_f −topological space and $\{A_j\}_{j\epsilon J}$ be a family of subsets of X. Then

i) $\cup_{j\in J} I_\mu(A_j) \subseteq I_\mu(\cup_{j\in J} A_j)$.
ii) $I_\mu(\cap_{j\in J} A_j) \subseteq \cap_{j\in J} I_\mu(A_j)$.

Proposition 4.7: Let (X, μ) be g_f −topological space and λ be a fuzzy set in X. Then

i) $Cl_\mu(1 - A) = 1 - I_\mu(A)$.
ii) $I_\mu(1 - A) = 1 - Cl_\mu(A)$.

Proof (i): We have $I_\mu(A) = \cup_j A_j$ where A_j are g_f −open sets in X. and $A_j \subseteq A$ for all $j \in J$. This implies $1 - I_\mu(A) = 1 - \cup_j A_j = \cap_j A_j^c$, where $\{A_j^c\}$ is the family of g_f −closed sets containing $1 - A$. Hence, by definition of g_f −closure of fuzzy set we have $Cl_\mu(1 - A) = 1 - I_\mu(A)$.

(ii): Further, we have $Cl_\mu(A) = Cl_\mu\left(1 - (1 - A)\right) = Cl_\mu\left(1 - A_j^c\right) = 1 - I_\mu\left(A_j^c\right)$. This implies $I_\mu(1 - A) = 1 - Cl_\mu(A)$.

5. Conclusion

In this Paper we have studied a new concept of generalized fuzzy topological spaces in which many important results have been obtained with the help of some suitable examples. Further we have studied the concept of generalized fuzzy closure and generalized fuzzy interior and verify the results with the help of some examples.

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