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Generalized Eccentricity k^{th} Power Sum Adjacency Energy of Graphs $(EGE^kSA(G))$

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Abstract

Let G be a finite, simple and undirected graph. For any integer $1 \leq k < \infty$, generalized eccentricity k^{th} power sum adjacency matrix of G is $m \times m$ matrix with its $(i,j)^{th}$ entry as $e(v_i)^k + e(v_j)^k$, $if\ v_i\ adjacent\ to\ v_j$ and zero otherwise, where $e(v)$ is the eccentricity of the vertex v of a graph G. In this paper, the new energy of graph under the name as generalized eccentricity k^{th} power sum adjacency energy of G $(EGE^kSA(G))$ has been introduced. Generalized eccentricity k^{th} power sum adjacency energy $EGE^kSA(G)$ of some standard graphs and regular graphs obtained by complete graph.

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1.Introduction

Let $G = (V(G), X(G))$ be a finite, simple and undirected graph with $|V(G)| = m$ and $|X(G)| = q$. The distance $d(u, v)$ between any two vertices u and v in a graph G is the length of the shortest $u - v$ path. Eccentricity of a vertex is defined as the maximum distance between a vertex to all other vertices [1]. In 1978, the concept energy of a graph G originated by I. Gutman [9].

In 2023, B. Fathima have defined the generalized eccentricity k^{th} power sum energy $EGE^kS(G)$ of G [8]. Motivated by these papers, the concept of generalized k^{th} power sum adjacency energy $EGE^kSA(G)$ of G.

Let G be a graph with m vertices and q edges. For any integer $1 \leq k < \infty$, a graph G whose matrix is denoted by $GE^kSA(G) = [ge^ksa_{ij}]$ is determined as $ge^ksa_{ij} = \begin{cases} e^k(v_i) + e^k(v_j), & if\ v_i\ adjacent\ to\ v_j \\ 0, & otherwise \end{cases}$.

The generalized eccentricity k^{th} power sum adjacency energy of G is indicated by $EGE^tSA(G) = \sum_{i=1}^{m} |\eta_i|$, where $\eta_1, \eta_2, \cdots, \eta_m$ are eigenvalues of $GE^kSA(G)$.

2. Preliminaries

Lemma 2.1 [5]

Let M, N, P and Q be matrices with M invertible. Then we have $\begin{vmatrix} M & N \\ P & Q \end{vmatrix} = |M||Q - PM^{-1}N|$

Lemma 2.2 [5]

Let M, N, P and Q be matrices. Let $S = \begin{pmatrix} M & N \\ P & Q \end{pmatrix}$ if M and P commutes. Then $|S| = |MQ - PN|$.

Lemma 2.3 [15]

If $A(K_p)$ is the adjacency matrix of K_p, then $A^2(K_P) = (p-2)A(K_p) + (p-1)I_p$.

3. Generalized eccentricity k^{th} power of sum adjacency energy of some graphs

Theorem 3.1

In a complete graph K_m $(m \geq 2)$, $EGE^kSA(K_m) = 4(m-1)$.

Proof:

Let K_m be the complete graph with m vertices for $m \geq 2$.

Since K_m is a connected graph with $e(v_i) = 1,\ 1 \leq k \leq m$, we get

$$ge^k sa_{ij}(K_m) = \begin{cases} 1^k + 1^k, & if\ v_i\ adjacent\ to\ v_j \\ 0, & otherwise \end{cases}.$$

and the generalized eccentricity k^{th} power sum adjacency eigenvalues of K_m are -2 of multiplicity $(m-1)$ and $2(m-1)$ of multiplicity 1 respectively. Hence $EGE^kSA(K_m) = 4(m-1)$.

Theorem 3.2

In a complete bipartite graph $K_{m,n}$ $(m, n \geq 2)$, $EGE^kSA(K_{m,n}) = 2^{k+2}(m)$.

Proof:

Let G be a complete bipartite graph of order $m+n$ and mn edges.

Since $ge^k sa_{ij}(K_{m,n}) = \begin{cases} 2^{k+1}, & if\ v_i\ adjacent\ to\ v_j \\ 0, & otherwise \end{cases}$, we get

$GE^kSA(K_{m,n}) = \begin{bmatrix} 0 & 2^{k+1}J \\ 2^{k+1}J & 0 \end{bmatrix}$ and where $J = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$

$$\begin{aligned} PGE^kSA(K_{m,n})(\eta) &= |\eta I_m - GE^kSA(K_{m,n})| \\ &= \begin{vmatrix} \eta I_m & -2^{k+1}J \\ -2^{k+1}J & \eta I_m \end{vmatrix} \\ &= (\eta I_m)^2 - (2^{k+1}J)^2 \\ &= (\eta I_m - 2^{k+1}J)\,(\eta I_m + 2^{k+1}J) \\ &= (\eta I_m - 2^{k+1}m)\,(\eta I_m - 2^{k+1}m)\eta^{2m-2} \end{aligned}$$

Hence $S_p(GE^kSA(K_{m,n})) = \begin{pmatrix} 2^{k+1}m & 2^{k+1}m & 0 \\ 1 & 1 & 2m-2 \end{pmatrix}$ and

$EGE^kSA(K_{m.n}) = 2^{k+2}(m)$.

Theorem 3.3

In a star graph $K_{1,m}$ $(m \geq 2)$, $EGE^kSA(K_{1,m}) = 2(2^k+1)\sqrt{m}$.

Proof:

Let G be a star graph of order m+1 and m edges.

Since $ge^k sa_{ij}(K_{1,m}) = \begin{cases} 2^k+1, & if\ v_i\ adjacent\ to\ v_j \\ 0, & otherwise \end{cases}$, we get

$$GE^kSA(K_{1,m}) = \begin{bmatrix} 0 & 2^k+1 & 2^k+1 & \cdots & 2^k+1 \\ 2^k+1 & 0 & 0 & \cdots & 0 \\ 2^k+1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2^k+1 & 0 & 0 & \cdots & 0 \end{bmatrix} \text{ and}$$

$$PGE^kSA(K_{m,n})(\eta) = |\eta I_m - GE^kSA(K_{1,m})|$$

$$= \begin{vmatrix} \eta I_m & -(2^k+1) & -(2^k+1) & \cdots & -(2^k+1) \\ -(2^k+1) & \eta I_m & 0 & \cdots & 0 \\ -(2^k+1) & 0 & \eta I_m & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -(2^k+1) & 0 & 0 & \cdots & \eta I_m \end{vmatrix}$$

$$= \eta^{m-1}(\eta^2 - (2^k+1)^2 m)$$

Hence $S_p(GE^kSA(K_{1,m})) = \begin{pmatrix} 2^{k+1}\sqrt{m} & 2^{k+1}\sqrt{m} & 0 \\ 1 & 1 & m-1 \end{pmatrix}$ and

$EGE^kSA(K_{1,m}) = 2(2^k+1)\sqrt{m}$.

4. Generalized eccentricity k^{th} power sum adjacency energy of some regular graphs obtained by complete graph

Theorem 4.1

Let $D_1(K_{2m})$ be the edge deleting graph 1 of K_{2m}. Then $EGE^kSA(D_1(K_{2m})) = 2^{k+3}(m-1)$.

Proof:

Let G be a edge deleting graph 1 of order $2m$, $m = 1,2,\cdots,n$ and $2m(m-1)$ edges. Since $ge^k sa_{ij}(D_1(K_{2m})) = \begin{cases} 2^{k+1}, & if\ v_i\ adjacent\ to\ v_j \\ 0, & otherwise \end{cases}$,

we get $GE^kSA(D_1(K_{2m})) = \begin{bmatrix} 2^{k+1}A(K_m) & 2^{k+1}A(K_m) \\ 2^{k+1}A(K_m) & 2^{k+1}A(K_m) \end{bmatrix}$ and

$$PGE^kSA(D_1(K_{2m}))(\eta) = |\eta I_m - GE^kSA(D_1(K_{2m}))|$$

$$= \begin{vmatrix} \eta I_m - 2^{k+1}A(K_m) & -2^{k+1}A(K_m) \\ -2^{k+1}A(K_m) & \eta I_m - 2^{k+1}A(K_m) \end{vmatrix}$$

$$= |(\eta I_m - 2^{k+1}A(K_m))^2 - (2^{k+1}A(K_m))^2|$$

$$= |\eta^2 I_m - 2\eta(2^{k+1}A(K_m))|$$

$$= (2\eta)^m \left|\frac{\eta^2}{2\eta} I_m - 2^{k+1}A(K_m)\right|$$

$$= (2\eta)^m (\tfrac{\eta}{2} - 2^{k+1}(m-1))(\tfrac{\eta}{2} + 2^{k+1})^{m-1}$$

$$= \eta^m(\eta - 2^{k+2}(m-1))(\eta + 2^{k+2})^{m-1}$$

Hence $S_p(GE^kSA(D_1(K_{2m})) = \begin{pmatrix} 2^{k+2}(m-1) & -2^{k+2} & 0 \\ 1 & m-1 & m \end{pmatrix}$ and

$EGE^kSA\big(D_1(K_{2m})\big) = 2^{k+3}(m-1)$.

Theorem 4.2

Let $D_3(K_{2m})$ be the edge deleting graph 3 of K_{2m}. Then $EGE^kSA\big(D_1(K_{2m})\big) = 8(3^k)(m-1)$, where $(m \geq 3)$.

Proof:

Let G be a edge deleting graph 3 of K_{2m} order $2m$, $m = 3,4,\cdots,n$ and $m(m-1)$ edges.

Since $ge^ksa_{ij}\,(D_3(K_{2m})) = \begin{cases} 2(3^k), & if\ v_i\ adjacent\ to\ v_j \\ 0, & otherwise \end{cases}$,

we get $GE^kSA(D_3(K_{2m})) = \begin{bmatrix} 0 & 2(3)^kA(K_m) \\ (3^k)A(K_m) & 0 \end{bmatrix}$ and

$$PGE^kSA(D_3(K_{2m}))(\eta) = |\eta I_m - GE^kSA(D_3(K_{2m}))|$$

$$= \begin{vmatrix} \eta I_m & -2(3^k)A(K_m) \\ -2(3^k)A(K_m) & \eta I_m \end{vmatrix}$$

$$= |\eta I_m||\eta I_m - (2(3^k)A(K_m))^2|$$

$$= \eta^m \left|\eta I_m - 4(3^{2k})(\tfrac{(m-2)A(K_m)+(m-1)I_m}{\eta})\right|$$

$$= |\eta^2 I_m - 4(3^{2k})(m-2)A(K_m) - 4(3^{2k})(m-1)I_m|$$

$$= (m-2)^m \left|\left(\tfrac{\eta^2-4(3^{2k})(m-1)}{m-2}\right) I_m - 4(3^{2k})A(K_m)\right|$$

$$=(m-2)^m \left(\tfrac{\eta^2-4(3^{2k})(m-1)}{m-2} - 4(3^{2k})(m-1)\right)$$

$$(\frac{\eta^2 - 4(3^{2k})(m-1)}{m-2} + 4(3^{2k}))^{m-1}$$

$$= (\eta^2 - 4(3^{2k})(m-1)^2)(\eta^2 - 4(3^{2k}))^{m-1}$$

Hence $S_p(\mathrm{GE^kSA}(D_3(K_{2m})) = \begin{pmatrix} -2(3^k)(m-1) & 2(3^k)(m-1) & -2(3^k) & 2(3^k) \\ 1 & 1 & m-1 & m-1 \end{pmatrix}$ and $EGE^kSA\big(D_3(K_{2m})\big) = 8(3^k)(m-1)$.

Theorem 4.3

Let $J{K_m}^m$ be the join of complete graph. Then $EGE^kSA(J{K_m}^m) = 2^{k+3}(m-1)$.

Proof:

Let G be a join of complete graph of order $2m$ and m^2 edges.

Since $ge^ksa_{ij}\,(J{K_m}^m) = \begin{cases} 2^{k+1}, & if\ v_i\ adjacent\ to\ v_j \\ 0, & otherwise \end{cases}$,

we get $GE^kSA(J{K_m}^m) = \begin{bmatrix} 2^{k+1}A(K_m) & 2^{k+1}(I_m) \\ 2^{k+1}(I_m) & 2^{k+1}A(K_m) \end{bmatrix}$ and

$$PGE^kSA(J{K_m}^m)(\eta) = |\eta I_m - GE^kSA(J{K_m}^m)|$$

$$= \begin{vmatrix} \eta I_m - 2^{k+1}A(K_m) & -2^{k+1}(I_m) \\ -2^{k+1}(I_m) & \eta I_m - 2^{k+1}A(K_m) \end{vmatrix}$$

$$= (\eta I_m - 2^{k+1}A(K_m))^2 - (2^{k+1}(I_m))^2$$

$$= \left((\eta - 2^{k+1})I_m - 2^{k+1}(m-1)\right)\left((\eta - 2^{k+1})I_m + 2^{k+1}\right)^{m-1}$$

$$((\eta + 2^{t+1})I_m - 2^{k+1}(m-1))((\eta + 2^{k+1})I_m + 2^{k+1})^{m-1}$$

$$= \eta^{m-1}\left(\eta - 2^{k+1}(m)\right)(\eta + 2^{k+2} - 2^{k+1}(m))(\eta + 2^{k+2})^{m-1}$$

Hence $S_p(GE^K SA(JK_m{}^m)) = \begin{pmatrix} -2^{k+1}(m) & 2^{k+1}(m) - 2^{k+2} & -2^{k+2} & 0 \\ 1 & 1 & m-1 & m-1 \end{pmatrix}$

and $EGE^k SA(JK_m{}^m) = 2^{k+3}(m-1)$.

5. Generalized eccentricity k^{th} power sum adjacency energy complement of regular graph obtained from complete graph

Theorem 5.1

Let $D_2(K_{2m})$ be the complement of edge deleting graph 2 of K_{2m}. Then $EGE^k SA\left(\overline{D_2(K_{2m})}\right) = 2^{k+2}(m)$.

Proof:

Since $A\left(D_2(K_{2m})\right) = \begin{bmatrix} A(K_m) & 0 \\ 0 & A(K_m) \end{bmatrix}$ [by theorem 4.1] and $\bar{A} = J - I - A$, where $\bar{A}$ is the adjacency matrix of complement graph, where $J = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$

Therefore, we get $GE^k SA\left(\overline{D_2(K_{2m})}\right) = \begin{bmatrix} 0 & 2^{k+1}(J) \\ 2^{k+1}(J) & 0 \end{bmatrix}$ and

$$PGE^k SA\left(\overline{D_2(K_{2m})}\right) = \left|\eta I_m - GE^k SA\left(\overline{D_2(K_{2m})}\right)\right|$$

$$= \begin{vmatrix} \eta I_m & -2^{k+1}(J) \\ -2^{k+1}(J) & \eta I_m \end{vmatrix}$$

Thus the characteristic root of $GE^k SA\left(\overline{D_2(K_{2m})}\right)$ are $\pm 2^{k+1}(m)$ of multiplicity, zero of multiplicity $2m-2$ respectively and hence $EGE^k SA\left(\overline{D_2(K_{2m})}\right) = 2^{k+2}(m)$.

Theorem 5.2

Let $D_3(K_{2m})$ be the complement of edge deleting graph 3 of K_{2m}. Then $EGE^k SA\left(\overline{D_3(K_{2m})}\right) = 2^{k+3}(m-1)$.

Proof:

Since $GE^k SA(D_3(K_{2m})) = \begin{bmatrix} 0 & 2(3)^k A(K_m) \\ (3^k)A(K_m) & 0 \end{bmatrix}$ (by theorem 4.2) and $\bar{A} = J - I - A$, where $\bar{A}$ is the adjacency matrix of complement graph, where $J = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$,.

we get $GE^k SA\left(\overline{D_3(K_{2m})}\right) = \begin{bmatrix} 2^{k+1}A(K_m) & 2^{k+1}I_m \\ 2^{k+1}I_m & 2^{k+1}A(K_m) \end{bmatrix}$

$= GE^k SA(JK_m{}^m)$ (by theorem 4.3)

Since $EGE^k SA(JK_m{}^m) = 2^{k+3}(m-1)$.

Hence we get $EGE^k SA\left(\overline{D_3(K_{2m})}\right) = 2^{k+3}(m-1)$.

Theorem 5.3

Let $(JK_m{}^m)$ be the complement of join of complete graph. Then $EGE^kSA(\overline{JK_m{}^m}) = 8(3^k)(m-1)$, where $m \geq 3$.

Proof:

Since $GE^kSA(JK_m{}^m) = \begin{bmatrix} 2^{k+1}A(K_m) & 2^{k+1}(I_m) \\ 2^{k+1}(I_m) & 2^{k+1}A(K_m) \end{bmatrix}$ (by theorem 4.3) and $\bar{A} = J - I - A$, where $\bar{A}$ is the adjacency matrix of complement graph, where $J = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$

we get $GE^kSA(\overline{JK_m{}^m}) = \begin{bmatrix} 0 & 2(3)^kA(K_m) \\ (3^k)A(K_m) & 0 \end{bmatrix}$

$= GE^kSA(D_3(K_{2m}))$ (by theorem 4.2)

Since $EGE^kSA(D_3(K_{2m})) = 8(3^k)(m-1)$.

Hence $GE^kSA(\overline{JK_m{}^m}) = 8(3^k)(m-1)$.

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