Properties of Semi Residuated Almost Distributive Lattices

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Abstract
In this paper, mainly we have proved some important properties of semi residuation ‘ ⊕ ’ and multiplication ‘ . ’ in a Semi Residuated Almost Distributive Lattice(SRADL) P and given an example of a semi residuated ADL P. For r, s ϵ P, we have proved that the characterization of r ⊕ s and r.s when s is a complemented element of a semi residuated ADL P.

Keywords: Almost Distributive Lattice (ADL), Semi Residuation, Multiplication, Semi Residuated ADL(SRADL) and Complemented ADL.

1.1 INTRODUCTION

A Boolean algebra known as the Almost Distributive Lattice (ADL) had a considerable generalization in 1981 because of the work of Swamy, U. M., and Rao, G. C. [5]. With the possible exception of commutativity of ∨, commutativity of ∧ and right distributivity of ∨ over ∧, an ADL satisfies nearly all of the conditions of a distributive lattice. Additionally, the class of ADLs can extend several concepts from the class of distributive lattices through the distributive lattice formed by its principal ideals. The idea of residuation goes back to Dedikind, who introduced it into ideal theory and the idea of an abstract study of residuation in multiplicative structures was initiated by Ward [6, 7]. Ward, M., and Dilworth, R.P., explored residuated lattices in [8, 9]. The concepts of Residuation and Multiplication in an Almost Distributive Lattice and the definition of a Residuated Almost Distributive Lattice were first described in [3] by Rao, G.C., and Raju, S.S. In the major body of this article, we have proved that the main important properties of semi residuation ‘⊕ ’ and multiplication ‘ . ’ in a Semi Residuated Almost Distributive Lattice(SRADL) P and also given an example of a semi residuated ADL P. For r, s ϵ P, we have proved that the characterization of r ⊕ s and r.s when s is a complemented element of a semi residuated ADL P. We reviewed the definition of an Almost Distributive Lattice (ADL) and some of its fundamental characteristics in section 2.1. These are from Rao, G.C. [2], Swamy, U.M., and Rao, G.C.[5]. We presented the idea of semi-residuation in an ADL P and defined a Semi Residuated ADL in section 3.1 by Rao, G.C., and Raju, S.S. In this section we have proved that the main important properties of semi residuation ‘ ⊕ ’ and multiplication ‘ . ’ in a Semi Residuated Almost Distributive

Lattice(SRADL) P and also given an example of a semi residuated ADL P. For r, s ϵ P, we have proved that the characterization of r $\oplus$ s and r.s when s is a complemented element of a semi residuated ADL P.

2.1 PRELIMINARIES

We have compiled a few important definitions and results which are well-known and shall be implemented extensively in the present work.

Definition 2.1.1 [2]: A relation R on P is said to as a partial order relation on P if it satisfies the conditions reflexive, antisymmetric and transitive. In general, partial orders are denoted with "≤"
We define (P, ≤) as a partly ordered set (Poset) if "≤" is a partial order on P.

Definition 2.1.2 [2]: A lattice is a poset (P, ≤) in which every subset of P with exactly two elements has supremum and infimum in P. (P, ≤) is a lattice r, s ∈ P ⟺{r, s} has supremum and infimum in P. If (P, ∨, ∧) be any lattice. Then
(i) A non-void set H of P is said to be a sub lattice of P, if r ∧ s, r∨ s∈ H, for all r, s ∈ H.
(ii) A sublattice H of P is said to be convex if r, s ∈ H, t ∈ P, r ≤ s, r ≤ t ≤ s ⇒ t ∈ H.

Definition 2.1.3 [2]: In a Poset (P, ≤) if for every r, s ∈ P, either r ≤ s or s ≤ r hold, then (P, ≤) is said to be a chain or simply it is said to be an ordered set. We observe that every chain is a lattice but not vice versa.

Definition 2.1.4 [2]: Lattice is an algebra (P, ∨, ∧) of type (2, 2), if it satisfies the following axioms:
1) a) r ∨ r = r, b) r ∧ r = r
2) a) r ∨ s = s∨ r, b) r ∧ s = s ∧ r
3) a) (r ∨ s) ∨ t = r ∨ (s ∨ t), b) (r ∧ s) ∧ t = r ∧ (s ∧ t)
4) a) (r ∨ s) ∧ s = s, b) (r ∧ s) ∨ s = s.

In any lattice (P, ∨, ∧) the following are equivalent:
r ∧ (s ∨ t) = (r ∧ s) ∨ (r ∧ t)
(r ∨ s) ∧ t = (r ∧ t) ∨ (s ∧ t)
r ∨ (s∧ t) = (r ∨ s) ∧ (r ∨ s)
(r ∧ s) ∨ t = (r ∨ t) ∧ (s ∨ t), for all r, s, t ∈ P.

Definition 2.1.5 [2]: A Lattice (P, ∨, ∧) is said to be a distributive lattice if it satisfies any of the above four inequalities.

Definition 2.1.6 [2]: In a lattice (P, ∨, ∧) an element 0∈L issaid to be a zero element or least element of P, if it satisfies 0 ∧ r = 0 ∀ r ∈ P and an element 1 of P is said to be 1 element or greatest element of P, if it satisfies r ∨ 1 = 1 ∀ r ∈ P. If suppose P has both 0 and 1, then P is said to be a bounded lattice.

Definition 2.1.7 [2]: A lattice P is said to be a complemented bounded lattice, if for any r∈ P there is an element s ∈ P such that r ∨ s = 1 and r ∧ s = 0. A lattice P is said to be relatively complemented lattice, if for any u, v ∈ P such that u ≤ v and the bounded lattice [u, v] = {w ∈ P / u ≤ w ≤ v} is a complemented lattice.

A lattice P is known to be distributive if and only if the relative compliments of every element in every interval [u, v], u ≤ v are unique
Definition 2.1.8 [2]: A Boolean algebra is defined as a bounded distributive and complemented lattice.

Definition 2.1.9 [2]: A sub lattice I of P is said to be an ideal of P if i ∈ I, r ∈ P imply r ∧ i ∈ I.
If I = P, then an ideal I of P is said to be proper.
If an ideal I of P satisfies the following two properties:
1) r ∧ s ∈ I, r, s ∈ P ⇒ either r ∈ I or s ∈ I
2) I ≠ P,
then I is said to be a prime ideal of P.

If an ideal N of P satisfies the following two properties:
(i) N ≠ P (ii) If V is an ideal of P such that N ⊆ V ⊆ P ⇒ either N = V or V = P, then N is said to be maximal of P.

In the following, we have defined an ADL.

Definition 2.1.10 [2]: An algebra (P, ∨, ∧) of type (2, 2) is said to be an **Almost Distributive Lattice (ADL)** if it satisfies the following axioms:
(1) (l ∨ m) ∧ n = (l ∧ n) ∨ (m ∧ n)
(2) l ∧ (m ∨ n) = (l ∧ m) ∨ (l ∧ n)
(3) (l ∨ m) ∧ m = m
(4) (l ∨ m) ∧ l = l
(5) l ∨ (l∧ m) = l, for all l, m, n ∈ P.
From the above Definition, we have every distributive lattice is an ADL.
If suppose there exists an element 0 ∈ P such that 0 ∧ l = 0 for every l ∈ P, then (P, ∨, ∧, 0) of type (2, 2, 0) is said to be an Almost Distributive Lattice with an element 0 or simply it is said to be an ADL with 0.

Example 2.1.1 [2]: Suppose Z is a non-void set. We fix an element z_0∈ Z. For all p, q ∈ P,

$$p \wedge q = \begin{cases} z_0, & \text{if } p = z_0 \\ q, & \text{if } p \neq z_0 \end{cases} \qquad p \vee q = \begin{cases} q, & \text{if } p = z_0 \\ p, & \text{if } p \neq z_0. \end{cases}$$

Then (Z, ∨, ∧, z_0) is an Almost Distributive Lattice(ADL) with z_0 as its zero element. This ADL is said to be a **discrete ADL.**

For any two elements r, s ∈ P, r is said to be less than or equals to s and denote r ≤ s, if r ∧ s = r. Then binary relation “≤“ is said to be a partial ordering relation(or simply partial ordering) on P.

In any ADL P, the following results hold.

From here onwards by P we mean an ADL (P, ∨, ∧, 0).

Theorem 2.1.1 [2]: For any l, m, n ∈ P, we have
(1) l ∧ 0 = 0 and 0 ∨ l = l
(2) l ∧ l = l = l ∨ l
(3) (l ∧ m) ∨ m = m, l ∨ (m ∧ l) = l and l ∧ (l ∨ m) = l
(4) l ∧ m = l ⟺ l ∨ m = m and l ∧ m = m ⟺ l ∨ m = l
(5) l ∧ m = l ∧ m and l ∨ m = m ∨ l whenever l ≤ m
(6) l ∧ m ≤ m and l ≤ l ∨ m
(7) ∧ is associative in P
(8) l ∧ m ∧ n = m ∧ l ∧ n
(9) (l ∨ m) ∧ n = (m ∨ l) ∧ n

(10) $l \wedge m = 0 \Leftrightarrow m \wedge l = 0$
(11) $l \vee (m \vee l) = l \vee m$.

We have seen that an ADL P meets almost all of the characteristics of a distributive lattice, possibly with the exception of the right distributivity of ∨ over ∧, the commutativity of ∨, the commutativity of ∧ and the absorption law $(r \wedge s) \vee r = r$. Here an ADL P forms a distributive lattice by means of any one of these properties.

Theorem 2.1.2 [2]: Suppose (P, ∨, ∧, 0) is an ADL with an element 0. Then the following conditions are equivalent: (i) $(l \wedge m) \vee n = (l \vee n) \wedge (m \vee l)$ for all l, m, n ∈ P (ii) $l \wedge m = m \wedge l$ for all l, m ∈ P (iii) $l \vee m = m \vee l$ for all l, m ∈ P (iv) (P, ∨, ∧, 0) is distributive.

Proposition 2.1.1 [2]: Suppose (P, ∨, ∧, 0) is an Almost Distributive Lattice. Then for any l, m, n ∈ P with $l \leq m$, we have the following properties:
(1) $l \wedge n \leq m \wedge n$
(2) $n \wedge l \leq n \wedge m$
(3) $n \vee l \leq n \vee m$.

Definition 2.1.11 [2]: An element m ∈ P is said to be a maximal element (or maximal), if it is maximal as in the partially ordered set (or simply poset)(P, ≤). That is, for any n ∈ P, $m \leq n$ implies m = n.

Theorem 2.1.3 [2]: Suppose P is an ADL and m ∈ P. Then the following conditions are equivalent:
(i) m is maximal with respect to a binary relation ≤
(ii) $m \wedge r = r$, for any r ∈ P
(iii) $m \vee r = m$, for any r ∈ P.

Lemma 2.1.1 [2]: Suppose P is an ADL with maximal m and s, t ∈ P. If $s \wedge t = t$ and $t \wedge s = s$ then s is a maximal ⇔ if t is maximal. Also we have the following equivalent conditions: (i) $s \wedge t = t$ and $t \wedge s = s$ (ii) $s \wedge m = t \wedge m$.

Proposition 2.1.1 [2]: If (P, ∨, ∧, 0, m) is an ADL, then the set I(P) of all ideals of P is a complete lattice under set inclusion. In this lattice, for any I, J ∈ I(P), the l.u.b. and g.l.b. of I, J are given by the following
(i) $I \vee J = \{(i \vee j) \wedge m / i \in I, j \in J\}$
(ii) $I \wedge J = I \cap J$.

We have the set PI(P) = {(u] /u ∈ P} of all principal ideals of P forms a sublattice of I(P). (Since $(u] \vee (v] = (u \vee v]$ and $(u] \cap (v] = (u \wedge v]$).

Definition 2.1.13 [2]: An ADL P = (P, ∨, ∧, 0, m) with maximal m is said to be a complete ADL, if it satisfies the property that PI(P) is a complete sub lattice of the lattice I(P).

Theorem 2.1.4 [2]: Suppose P = (P, ∨, ∧, 0, m) is an ADL with maximal m. Then the lattice ([0, m], ∨, ∧) is a complete lattice ⇔ P is a complete ADL.

3.1 PROPERTIES OF SEMI RESIDUATED ALMOST DISTRIBUTIVE LATTICES(SRADL's)

In this section, we have given an example of a semi residuated ADL P. we have proved some important properties of semi residuation ' ⊕ ' and multiplication ' . ' in a Semi Residuated Almost Distributive Lattice(SRADL) P. For r, s

ϵ P, we have proved that the characterization of r ⊕ s and r.s when s is a complemented element of a semi residuated ADL P.

Initially, we start with the definition of **Semi Residuation.** It has taken from [4].

Definition 3.1.1:[4] Suppose P be an ADL with maximal m. A binary operation '⊕' on an ADL P is said to be a **Semi Residuation** over P if, for r, s, t ϵ P the following conditions are satisfied.
(R1) $r \wedge s = s$ if and only if $r \oplus s$ is maximal
(R2) $r \wedge s = s \Rightarrow$ (i) $(r \oplus t) \wedge (s \oplus t) = s \oplus t$ and (ii) $(t \oplus s) \wedge (t \oplus r) = t \oplus r$
(R3) $[(r \oplus s) \oplus t] \wedge m = [(r \oplus t) \oplus s] \wedge m$
(R4) $[(r \wedge s) \oplus t] \wedge m = (r \oplus t) \wedge (s \oplus t) \wedge m$

Definition 3.1.2:[3] Suppose P be an ADL with maximal m. A binary operation ' . ' on an ADL P is said to be a **Multiplication** over P if, for r, s, t ϵ P the following conditions are satisfied.
(M1) $(r.s) \wedge m = (s.r) \wedge m$
(M2) $[(r.s).t] \wedge m = [r.(s.t)] \wedge m$
(M3) $(r.m) \wedge m = r \wedge m$
(M4) $[r.(s \vee t)] \wedge m = [(r.s) \vee (r.t)] \wedge m$.

In the following, we have given the definition of a **Semi Residuated Almost Distributive Lattice**. It has taken from [3].
Definition 3.1.3:[3] An ADL P with maximal m is said to be a **Semi Residuated Almost Distributive Lattice (or Semi Residuated ADL)(or SR ADL)**, if there exists a semi residuation "⊕" and a multiplication ''•'' on P satisfying the following condition **(K)**.

(K) $(x \oplus r) \wedge s = s$ if and only if $x \wedge (r.s) = r.s$, for any x, r, s ϵ P.

In the following, we have given an example of a **Semi Residuated Almost Distributive Lattice (SRADL)**.

Example 3.1.1: Let P be a discrete ADL with 0.
Fix $m(\neq 0)$ ϵ P. Define two binary operations '⊕' and '•' on P by
$x \oplus y = x$, for all x, y ϵ P,
$x.y = m$, for all x, y ϵ P-{0}
and $x.y = 0$, if either $x = 0$ or $y = 0$ or both.
Then L is a semi resituated ADL.

First we have given the following Lemma, whose proof can be obtained from the definition of Semi Residuated ADL.

Lemma 3.1.1: Let P be a semi residuated ADL. Then
(1) $(r \oplus r) \oplus s$ is maximal, for all r, s ϵ P
(2) If an element m of P is maximal then $m \oplus r$ is maximal, for all r ϵ P.

Lemma 3.1.2: Let P be a semi residuated ADL with maximal m. For r, s, t, u ϵ P**,** the following hold in P.
(1) $r \wedge [r.(r \oplus s)] = r.(r \oplus s)$
(2) $r \wedge [s.(r \oplus s)] = s.(r \oplus s)$
(3) $(r \oplus s)] \wedge r = r$
(4) $[r \oplus (r \oplus s)] \wedge (r \vee s) = r \vee s$

(5) $[(r \oplus s) \oplus t)] \wedge [r \oplus (s.t)] = r \oplus (s.t)$
(6) $[r \oplus (s.t)] \wedge [(r \oplus s) \oplus t)] = (r \oplus s) \oplus t$
(7) $[(r \wedge s) \oplus s] \wedge (r \oplus s) = r \oplus s$
(8) $(r \oplus s) \wedge [(r \wedge s) \oplus s = (r \wedge s) \oplus s$
(9) $[(r \wedge s) \oplus r] \wedge m = (s \oplus r) \wedge m$
(10) $[t \oplus (r \wedge s)] \wedge [t \oplus r) V (t \oplus s)] = (t \oplus r) V (t \oplus s)$
(11) If $r \oplus s = r$ then $r \wedge (s.u) = s.u \Rightarrow r \wedge u = u$
(12) $\{r \oplus [r \oplus (r \oplus s)]\} \wedge (r \oplus s) = r \oplus s$
(13) $[(a V b) \oplus c] \wedge [(a \oplus c) V (b \oplus c)] = (a \oplus c) V (b \oplus c)$
(14) $r \wedge m \geq s \wedge m \Rightarrow (r \oplus t) \wedge m \geq (s \oplus t) \wedge m$
(15) $(r \oplus s) \wedge \{r \oplus [r \oplus (r \oplus s)]\} = r \oplus [r \oplus (r \oplus s)]$
(16) $r \wedge s = s \Rightarrow (r.t) \wedge (s.t) = s.t$
(17) $r \wedge s \wedge (r.s) = r.s$
(18) $[r \oplus (s.t)] \wedge [r \oplus (s \wedge t)] = r \oplus (s \wedge t)$
(19) $[(r.s) \oplus r] \wedge s = s$
(20) $(r.s) \wedge [(r \wedge s).(r V s)] = (r \wedge s).(r V s)$
(21) $r V s$ is maximal $\Rightarrow (r.s) \wedge r \wedge s = r \wedge s$
(22) $(x_1 V x_2)^{n+1} \wedge m = [x_1^{n+1} V (x_1^n.x_2) V(x_1^{n-1}.x_2^2) V\ldots\ldots V (x_1.x_2^n) V x_2^{n+1}] \wedge m$, for any $x_1, x_2 \in L$, $n \in z^+$.
(23) $(x_1 V x_2)^{k1+k2} \wedge m \leq (x_1^{k1} V x_2^{k2}) \wedge m$, for any $x_1, x_2 \in L$, $K_1, K_2 \in z^+$.

Proof : Let $r, s, t, u \in P$.

(1) Since $r \wedge r = r$, by R1, we get that $r \oplus r$ is a maximal element of P.
Then $(r \oplus r) \wedge (r \oplus s) = r \oplus s$.
Now, by condition (K) of Definition 3.1.3, we get that
$r \wedge [r.(r \oplus s)]= r.(r \oplus s)$.

(2) We have $(r \oplus s) \wedge (r \oplus s) = r \oplus s$.
By condition (k) of Definition 3.1.3, we get that
$r \wedge [s.(r \oplus s)] = s.(r \oplus s)$.

(3) By condition (1) of Lemma 3.1.1, $(r \oplus r) \oplus s$ is maximal of P.
So that $[(r \oplus r) \oplus s] \wedge x = x$, for all $x \in P$.
Now, by R3, we get that $[(r \oplus s) \oplus r] \wedge x = x$, for all $x \in L$.
Therefore, $(r \oplus s) \oplus r$ is maximal.
Now by R1, we get that $(r \oplus s) \wedge r = r$.

(4) $[(r \oplus s). (r V s)] \wedge m = [(r \oplus s).(r V s)] \wedge m$
$\Rightarrow [\{(r \oplus s).r\} V \{(r \oplus s).s\}] \wedge m = [(r \oplus s).(r V s)] \wedge m$
$\Rightarrow [\{(r.(r \oplus s) \} V \{s.(r \oplus s)\}] \wedge m = [(r \oplus s).(r V s)] \wedge m.$
$\Rightarrow [\{r \wedge (r.(r \oplus s)\} V \{r \wedge (s.(r \oplus s)\}] \wedge m = [(r \oplus s).(r V s)] \wedge m$
(By (1) and (2))
$\Rightarrow [r \wedge \{r.(r \oplus s)\} \wedge m] V [r \wedge \{ s.(r \oplus s)\}] \wedge m] = [(r \oplus s).(r V s)] \wedge m$
$\Rightarrow [r \wedge \{ (r \oplus s).r\} \wedge m] V [r \wedge \{(r \oplus s).s\}] \wedge m] = [(r \oplus s).(r V s)]$
$\Rightarrow [\{r \wedge ((r \oplus s).r)\} V \{r \wedge ((r \oplus s).s)\}] \wedge m = [(r \oplus s).(r V s)]$
$\Rightarrow [r \wedge \{ (r \oplus s).r) V ((r \oplus s).r)\}] \wedge m = [(r \oplus s).(r V s)] \wedge m$
$\Rightarrow r \wedge [(r \oplus s).(r V s)] \wedge m = [(r \oplus s).(r V s)] \wedge m$
$\Rightarrow r \wedge [(r \oplus s).(r V s)] = [(r \oplus s).(r V s)]$
$\Rightarrow [r \oplus (r \oplus s) \wedge (r V s)] = (r V s)$

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(5) We have $[r \oplus (s.t)] \wedge [r \oplus (s.t)] = r \oplus (s.t)$

$\Rightarrow r \wedge [(s.t).\{r \oplus (s.t\}] = (s.t) [(r \oplus (s.t)]$

$\Rightarrow r \wedge [s.\{t.\{r \oplus (s.t)\}\}] = s.[t. \{r \oplus (s.t)\}]$

$\Rightarrow (r \oplus s) \wedge [t.\{r \oplus (s.t)\}] = t.[r \oplus (s.t)]$

$\Rightarrow [(r \oplus s) \oplus t] \wedge [r \oplus (s.t)] = r \oplus (s.t)$

(6) We have $[(r \oplus s) \oplus t] \wedge [(r \oplus s) \oplus t)] = (r \oplus s) \oplus t$

$\Rightarrow (r \oplus s) \wedge [t.\{(r \oplus s) \oplus t)\}] = t.[(r \oplus s) \oplus t)]$

$\Rightarrow r \wedge (s.\{t.[(r \oplus s) \oplus t)\}] = s.\{t. [(r \oplus s) \oplus t\}]$

$\Rightarrow r \wedge [(s. t).\{(r \oplus s) \oplus t)\}] = (s. t) [(r \oplus s) \oplus t]$

$\Rightarrow [r \oplus (s. t)] \wedge [(r \oplus s) \oplus t)] = (r \oplus s) \oplus t.$

(7) $[(r \wedge s) \oplus t] \wedge (r \oplus s) = [((r \wedge s) \oplus s] \wedge m \wedge (r \oplus s)$

$= [(r \oplus s) \wedge (s \oplus s)] \wedge m \wedge (r \oplus s)$

$= [(r \oplus s) \wedge (s \oplus s)] \wedge (r \oplus s)$

$= [(s \oplus s) \wedge (r \oplus s)] \wedge (r \oplus s)$

$= (r \oplus s) \wedge (r \oplus s)$ (Since $s \oplus s$ is maximal)

$= r \oplus s.$

(8) We have $(r \oplus s) \wedge (r \oplus s) \wedge (s \oplus s) \wedge m = (r \oplus s) \wedge (s \oplus s)$

$\Rightarrow (r \oplus s) \oplus [(r \oplus s) \wedge (s \oplus s)] \wedge m]$ is maximal.

$\Rightarrow (r \oplus s) \oplus [\{(r \wedge s) \oplus s \} \wedge m]$ is maximal.

$\Rightarrow (r \oplus s) \wedge [(r \wedge s) \oplus s] \wedge m] = [(r \wedge s) \oplus s] \wedge m$

$\Rightarrow (r \oplus s) \wedge [(r \wedge s) \oplus s] = (r \wedge s) \oplus s.$

(9) $[(r \wedge s) \oplus r] \wedge m = (r \oplus r) \wedge (s \oplus r) \wedge m$

$= (s \oplus r) \wedge m$ (Since $r \oplus r$ is maximal)

(10) We have $r \wedge r \wedge s = r \wedge s$ and $s \wedge r \wedge s = r \wedge s$, by R2 (ii) of Definition 3.1.1, we get that

$[t \oplus (r \wedge s)] \wedge (t \oplus r) = t \oplus r$ and $[t \oplus (r \wedge s)] \wedge (t \oplus s) = t \oplus s.$

Therefore, $[t \oplus (r \wedge s)] \wedge [(t \oplus r) \vee (t \oplus s)]$

$= [\{t \oplus (r \wedge s)\} \wedge (t \oplus r)] \vee [\{t \oplus (r \wedge s)\} \wedge (t \oplus r)]$

$= (t \oplus r) \vee (t \oplus s)$.

(11) Assume that $r \oplus s = r$.
Suppose $r \wedge (s.u) = s.u$.
By condition (k) of Definition 3.1.3, we get that $(r \oplus s) \wedge u = u$.
Hence $r \wedge u = u$ (Since $r \oplus s = r$).

(12) $[r \oplus (r \oplus s)] \wedge [r \oplus (r \oplus s)] = r \oplus (r \oplus s)$

$\Rightarrow r \wedge [(r \oplus s).\{r \oplus (r \oplus s)\}] = (r \oplus s).[r \oplus (r \oplus s)]$

$\Rightarrow r \wedge [(r \oplus s).\{r \oplus (r \oplus s)\}] \wedge m = [(r \oplus s).\{r \oplus (r \oplus s)\} \wedge m$

$\Rightarrow r \wedge [\{r \oplus (r \oplus s)\}.(r \oplus s)] \wedge m = [\{r \oplus (r \oplus s)\}.(r \oplus s)] \wedge m$

$\Rightarrow r \wedge [\{r \oplus (r \oplus s)\}.(r \oplus s)] = [\{r \oplus (r \oplus s)\}.(r \oplus s)]$

$\Rightarrow [r \oplus \{r \oplus (r \oplus s)\}] \wedge (r \oplus s)] = r \oplus s$

(13) We have $(r \vee s) \wedge r = r$ and $(r \vee s) \wedge s = s$.

$\Rightarrow [(r \vee s) \oplus t] \wedge (r \oplus t) = r \oplus t$ and $[(r \vee s) \oplus t] \wedge (s \oplus t) = s$.

$\Rightarrow [\{(r \vee s) \oplus t\} \wedge (r \oplus t)] \vee [\{(r \vee s) \oplus t\} \wedge (s \oplus t)] = (r \oplus t) \vee (s \oplus t)$

$\Rightarrow [(r \vee s) \oplus t] \wedge [(r \oplus t) \vee (s \oplus t)] = (r \oplus t) \vee (s \oplus t)$.

(14) Suppose $r \wedge m \geq s \wedge m$

then $s \wedge m \wedge r \wedge m = s \wedge m$

$\Rightarrow r \wedge s = s$.

Now, $(s \oplus t) \wedge m \wedge (s \oplus t) \wedge m = (r \oplus t) \wedge (s \oplus t) \wedge m$

$= [(r \wedge s) \oplus t] \wedge m$

$= (s \oplus t) \wedge m$ (Since $r \wedge s = s$).

Therefore, $(r \oplus t) \wedge m \geq (s \oplus t) \wedge m$.

(15) By property (4) above, we have

$\Rightarrow [r \oplus \{r \oplus \{r \oplus (r \oplus s)\}\}] \wedge m \geq [r \vee \{r \oplus (r \oplus s)] \wedge m$

$\Rightarrow [\{r \oplus \{r \oplus \{r \oplus (r \oplus s)\}\}\} \oplus s] \wedge m \geq [(r \vee [r \oplus (r \oplus s)] \oplus s] \wedge m$

(By property (14) above)

Now, $[(r \oplus s) \oplus \{r \oplus \{r \oplus (r \oplus s)\}\} \wedge m$

$= [\{r \oplus \{r \oplus \{r \oplus (r \oplus s)\}\}\} \oplus s] \wedge m$

$\geq$ [(r $\vee$ [r $\oplus$ (r $\oplus$ s)) $\oplus$ s}] $\wedge$ m

$\geq$ [(r $\oplus$ s) $\vee$ {(r $\oplus$ (r $\oplus$ s)) $\oplus$ s} $\wedge$ m (By (13))

= [{(r $\oplus$ (r $\oplus$ s)) $\oplus$ s} $\vee$ (r $\oplus$ s) $\wedge$ m

$\geq$ {(r $\oplus$ (r $\oplus$ s)) $\oplus$ s} $\wedge$ m

= [(r $\oplus$ s) $\oplus$ (r $\oplus$ s)] $\wedge$ m

= m.

Therefore, [(r $\oplus$ s) $\oplus$ {r $\oplus$ (r $\oplus$ (r $\oplus$ s))}] $\wedge$ m = m.

⇨ (r $\oplus$ s) $\oplus$ {r $\oplus$ (r $\oplus$ (r $\oplus$ s))} is maximal.

By condition R1 of Definition 3.1.1, we get that

(r $\oplus$ s) $\wedge$ [r $\oplus$ (r $\oplus$ s)] = r $\oplus$ {r $\oplus$ (r $\oplus$ s)}.

(16) Suppose r $\wedge$ s = s.

Then [(r.t) $\wedge$ m] $\vee$ [(s.t) $\wedge$ m]

= [(t.r) $\vee$ (t.s)] $\wedge$ m

= [t.(r $\vee$ s)] $\wedge$ m (By M4 of Definition 3.1.2)

= (t.r) $\wedge$ m (Since r $\vee$ s = r)

= (r.t) $\wedge$ m (By M1 of Definition 3.1.2)

⇨ (r.t) $\wedge$ m $\wedge$ (s.t) $\wedge$ m = (s.t) $\wedge$ m

⇨ (r.t) $\wedge$ (s.t) = s.t.

(17) [(r $\wedge$ s) $\oplus$ r] $\wedge$ s = [(r $\wedge$ s) $\oplus$ r] $\wedge$ m $\wedge$ s

= (r $\oplus$ r) $\wedge$ (s $\oplus$ r) $\wedge$ m $\wedge$ s.

= (s $\oplus$ r) $\wedge$ s = s (By property (3), above)

⇨ r $\wedge$ s $\wedge$ (r.s) = r.s (By condition (k) of Definition 3.1.3).

(18) By property (17), above we have s $\wedge$ t $\wedge$ (s.t) = s.t.

By R2(ii) of Definition 3.1.1, we get that

[r $\oplus$ (s.t)] $\wedge$ [r $\oplus$ (s $\wedge$ t)] = r $\oplus$ (s $\wedge$ t).

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(19) We have $(r.s) \wedge (r.s) = r.s.$

By condition (k) of Definition 3.1.3, we get that $[(r.s) \oplus r] \wedge s = s.$

(20) We know that $s \wedge r \wedge s = r \wedge s$

$\Rightarrow (s.r) \wedge [(r \wedge s).r] = (r \wedge s).r$

$\Rightarrow (r.s) \wedge [(r \wedge s).r] = (r \wedge s).r$

Similarly, $(r.s) \wedge [(r \wedge s).s] = (r \wedge s).s$

Now, $(r.s) \wedge [(r \wedge s).(r \vee s)] \wedge m$

$= (r.s) \wedge [\{(r \wedge s).r\} \vee \{(r \wedge s).s\}] \wedge m$

$= [\{(r.s) \wedge [\{(r \wedge s).r\}\} \vee \{(r.s) \wedge \{(r \wedge s).s\}\}] \wedge m$

$= [\{(r \wedge s).r\}\} \vee \{(r \wedge s).s\}\}] \wedge m$

$= [(r \wedge s).(r \vee s)] \wedge m$

Therefore, $(r.s) \wedge [(r \wedge s).(r \vee s)] = (r \wedge s).(r \vee s).$

(21) Suppose $r \vee s$ is maximal.

By (19), above, we have $(r.s) \wedge [(r \wedge s).(r \vee s)] = (r \vee s).$

$\Rightarrow (r.s) \wedge [(r \wedge s).(r \vee s)] \wedge m = [(r \wedge s).(r \vee s)] \wedge m$

$\Rightarrow (r.s) \wedge (r \wedge s) \wedge m = (r \wedge s) \wedge m$

$\Rightarrow (r.s) \wedge r \wedge s = r \wedge s.$

(22) Let $x_1, x_2 \in L$.

This result is proved by Mathematical Induction on n.

Put n = 1, then

$[(x_1 \vee x_2)^2 \wedge m = [(x_1 \vee x_2).(x_1 \vee x_2)] \wedge m$

$[(x_1 \vee x_2)^{2]} \wedge m = [x_1^2 \vee (x_1.x_2) \vee (x_2.x_1) \vee x_2^2] \wedge m$

$= [x_1^2 \vee (x_1.x_2) \vee x_2^2] \wedge m.$

Therefore, the result is true for n = 1.

Assume that the result is true for n = k.

That is $[(x_1 \vee x_2)^{k+1}] \wedge m = [x_1^{k+1} \vee (x_1^k.x_2) \vee (x_1^{k-1} . x_2^2) \vee \ldots\ldots \vee (x_1.x_2^k) \vee x_2^{k+1}] \wedge m.$

Now, $[(x_1 \vee x_2)^{k+2}] \wedge m = [(x_1 \vee x_2)^{k+1} . (x_1 \vee x_2) \wedge m.$

Hence $[(x_1 \vee x_2)^{k+2}] \wedge m = [\{x_1^{k+1} \vee (x_1^k.x_2) \vee (x_1^{k-1} . x_2^2) \vee \ldots\vee (x_1.x_2^k) \vee x_2^{k+1}\}.(x_1 \vee x_2)] \wedge m$

$= [\{x_1^{k+2} \vee (x_1^{k+1}.x_2) \vee (x_1^k. x_2^2) \vee\ldots\vee (x_1^2.x_2^k) \vee x_2^{k+2}\} \vee \{(x_1^{k+1}.x_2)$

$\vee (x_1^k .x_2^2) \vee (x_1^{k-1}.x_2^3) \vee\ldots\vee (x_1.x_2^{k+1}) \vee x_2^{k+2}\}] \wedge m$

$= [x_1^{k+2} \vee (x_1^{k+1}.x_2) \vee (x_1^k . x_2^2) \vee\ldots\vee (x_1.x_2^{k+1}) \vee x_2^{k+2}] \wedge m.$

Therefore the result is true for $n = k+1$.

Hence $(x_1 \vee x_2)^{k+1} \wedge m = [x_1^{k+1} \vee (x_1^k.x_2) \vee (x_1^{k-1} . x_2^2) \vee\ldots\vee (x_1.x_2^k) \vee x_2^{k+1}] \wedge m$,

for any $x_1, x_2 \in L$, $n \in Z^+$.

(23) Let $x_1, x_2 \in L$, $k_1, k_2 \in Z^+$.

Then $(x_1 \vee x_2)^{k1+k2} \wedge m = [(x_1 \vee x_2)^{k1} .(x_1 \vee x_2)^{k2}] \wedge m.$

By property (22), above we get that

$(x_1 \vee x_2)^{k1+k2} \wedge m = [\{x_1^{k1} \vee (x_1^{k1-1}.x_2) \vee (x_1^{k1-2}.x_1^2) \vee\ldots\vee x_2^{k1}\}.\{x_1^{k2} \vee (x_1^{k2-1}).x_1$

$= [\{x_1^{k1+k2} \vee (x_1^{k1+k2-1}.x_2) \vee\ldots\vee (x_1^{k1}.x_2^{k2})\} \vee (x_1^{k1+k2-1}.x_2) \vee (x_1^{k1+k2-1}.x_2^2) \vee\ldots$

$\vee (x_1^{k1-1}.x_2^{k2+1})\} \vee\ldots\vee [(x_1^{k2}.x_2^{k1}) \vee (x_1^{k2-1}. x_2^{k2+1}) \vee\ldots\vee x_2^{k1+k2}\}] \wedge m.$

$= [\{x_1^{k1+k2} \vee (x_1^{k1+k2-1}.x_2) \vee\ldots\vee (x_1^{k1}.x_2^{k2})\} \vee \{x_1^{k1-1}.x_2^{k2+1}) \vee\ldots\vee$

$(x_1^0.x_2^{k1+k2})\}] \wedge m.$

$\leq (x_1^{k1} \wedge m) \vee (x_2^{k2} \wedge m)$

$= (x_1^{k1} \vee x_2^{k2}) \wedge m.$

Hence $(x_1 \vee x_2)^{k1+k2} \wedge m \leq (x_1^{k1} \vee x_2^{k2}) \wedge m$, for any $x_1, x_2 \in P$, $k_1, k_2 \in Z^+$.

Now, we conclude this paper with the characterization of $r \oplus s$ and $r.s$ when s is a complemented element of a semi residuated ADL P.

First we have given the following definition.

Definition 3.1.4: Suppose P is an ADL and $r \in P$. An element $r^1 \in P$ is said to be a complement of r in P, if $r \wedge r^1 = 0$ and $r \vee r^1$ is maximal. In this case, we say that r is a complemented element of P. If each element of P is complemented, then P is called a complemented ADL.

Theorem 3.1.1: Suppose P is a semi residuated ADL with maximal m and $r, s \in P$. If an element s^1 is a complement of s in P, then $(r \oplus s) \wedge m = (r \vee s^1) \wedge m$.

Proof: Suppose an element s^1 is a complement of s in P.

Then $\{(r \oplus s) \oplus (r \vee s^1)\} \wedge m$

$= [\{r \oplus s) \oplus r\} \wedge \{(r \oplus s) \oplus s^1\}] \wedge m$

$= [(r \oplus r) \oplus s] \wedge [(r \oplus s) \oplus s^1\}] \wedge m$

$= \{(r \oplus s) \oplus s^1\}] \wedge m$ (Since $(r \oplus r) \oplus s$ is maximal)

$= [r \oplus (s.s^1)] \wedge m$

Therefore, $[(r \oplus s) \oplus (r \vee s^1)] \wedge m = m$ and

hence $(r \oplus s) \oplus (r \vee s^1)$ is a maximal element of L.

Thus by R1, $(r \oplus s) \wedge (r \vee s^1) = r \vee s^1$.

Now, $[(r \vee s^1) \oplus (r \oplus s) \wedge m$

$\geq [\{r \oplus (r \oplus s)\} \vee \{ s^1 \oplus (r \oplus s)\}] \wedge m$

(By property (13) of Lemma 3.1.2)

$= [(s \vee r) \wedge m] \vee [\{s^1 \oplus (r \oplus s)\} \wedge m$

(By property (4) of Lemma 3.1.2)

$\geq (s \vee s^1) \wedge m = m$ (Since $s \vee s^1$ is maximal).

Thus $[(r \vee s^1) \oplus (r \oplus s)] \wedge m = m$.

So that $(r \vee s^1) \oplus (r \oplus s)$ is maximal.

Hence, by R1, we get that $(r \vee s^1) \wedge (r \oplus s) = r \oplus s$.

Thus $(r \oplus s) \wedge m = (r \vee s^1) \wedge m$.

Theorem 3.1.2: Let P be a semi residuated ADL with a maximal element m and $r, s \in P$. If s is a complemented element of P, then $r \wedge s \wedge m = (r.s) \wedge m$. **Proof:** Suppose s is a complemented element of P and s^1 is a complement of s in P. Then $s \wedge s^1 = 0$ and $s \vee s^1$ is maximal.

We have $[r.(s \vee s^1)] \wedge (s \vee s^1) = r \wedge (s \vee s^1)$

$\Rightarrow [r.(s \vee s^1)] \wedge (s \vee s^1) \wedge m = r \wedge (s \vee s^1) \wedge m$

$\Rightarrow [r.(s \vee s^1)] \wedge m = r \wedge m$ (Since $s \vee s^1$ is maximal)

$\Rightarrow r \wedge m = [(r. s) \vee (r.s^1)] \wedge m$

$\Rightarrow r \wedge m =[(r.s) \vee (s^1.r)] \wedge m$ (By M1of Definition 3.1.2)

$\leq [(r.s) \vee s^1] \wedge m$

Thus $r \wedge m \leq [(r.s) \vee s^1] \wedge m \rightarrow (1)$.

From above (1), we get that

$s \wedge r \wedge m \leq s \wedge [(r.s) \vee s^1] \wedge m$

$= [\{s \wedge (r.s)\} \vee (s \wedge s^1)] \wedge m$

$= [s \wedge (r.s)] \wedge m$

$= [s \wedge (s.r)] \wedge m$ (By M1of Definition 3.1.2)

$= (s.r) \wedge m$

Now, $r \wedge s \wedge m = s \wedge r \wedge m$.

$\leq (s.r) \wedge m$

$= (r.s) \wedge m$.

Also, we have $r \wedge s \wedge (r.s) = r.s$.

So that $r \wedge s \wedge m \geq (r.s) \wedge m$.

Hence $r \wedge s \wedge m = (r.s) \wedge m$.

CONCLUSION

In this paper, we conclude that for r, s ϵ P, the characterization of $r \oplus s$ and r.s when s is a complemented element of a semi residuated ADL P.

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