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# Generalized$\alpha$-$\psi$ Contractive Type Mappings and Related Fixed Point Theorems with Applications

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***Abstract:***-In this paper, we introduce and study a class of mappings called generalized $\alpha$-$\psi$ contractive mappings, which are a generalization of the well-known $\alpha$-$\psi$ contractive mappings. We explore various fixed point theorems for such mappings in complete metric spaces, using the concept of $\alpha$ -admissible mappings. Additionally, we apply our main results to establish fixed point theorems for metric spaces with partial orders. Our results extend and improve upon a number of previously established results in the literature, and we provide illustrative examples to demonstrate the effectiveness of our approach.

***Keywords***: *Fixed point;α-admissible mapping;Contractive type mapping;Metric space*.

## 1. Introduction

Metric fixed point theory has been a key area of research in mathematics for over a century, and has had a profound impact on the development of functional analysis and other areas of mathematics. The concept of a metric space, introduced by French mathematician M. Frechet in 1906, has provided a framework for the study of abstract spaces and has inspired the development of new mathematical structures, such as complex-valued metric spaces and semi-metric spaces. By generalizing and extending the notion of a metric space, researchers have gained new insights into the nature of mathematical structures and have expanded the boundaries of functional analysis.In this direction several authors obtained further results [4,5,9,10,11,15].
In recent years, a number of mathematicians have obtained fixed point results for contraction typemappings in metric spaces equipped with a partial order. Some early results in this direction wereestablished by Ran and Reurings [19] who studied the existence of fixed points for certain mappings in partially ordered metric spaces and applied their results to matrix equations. Nieto and Lopez [12] built upon the results of Ran and Reurings [19] by extending the theorem to cover non-decreasing mappings and deriving solutions to certain partial differential equations with periodic boundary conditions.
The concept of mixed monotone property, which refers to contractive operators of the form $F: X \times X \to X$ that have both monotone increasing and monotone decreasing properties, was first introduced by Bhaskar and Lakshmikantham [16] for partially ordered metric spaces. Based on this concept, they derived a number of coupled fixed point theorems, which are related to fixed point theorems for a single operator that has one of

these monotone properties. In addition to the work of Bhaskar and Lakshmikantham [16], Chatterji [2] considered various contractive conditions for self-mappings in metric spaces, which are a special case of partially ordered metric spaces. Later, Dass and Gupta [3] built upon these ideas by exploring rational type contractions to find unique fixed points in complete metric spaces.
The work of Samet et al. [21] introduced α-ψ contractive type mappings as a new category of contractive mappings. The fixed point results obtained by Samet et al. [21] extended and generalized several fixed point results that exist in the literature, including the Banach contraction principle. In a further development, Karapinar and Samet [18] generalized the notion of α-ψ contractive type mappings and obtained various fixed point theorems for these mappings. Recently, Raji M. [24]presented a class of contractive type mappings calledgeneralized α-ψ contractive pair of mappings and study various coincidence fixedpoint theorems for such mappings in complete metric spaces and introduced the notion of α-admissible with respect to g mapping which in turn generalized theconcept of g-monotone mapping.
The main contribution of this paper is to derive fixed point results for generalized α-ψ contractive type mappings. These results unify and generalize the findings of Karapinar and Samet [18], Samet et al. [21], Ciri´c et al. [23] and other related results. Moreover, we obtain fixed point results for metric spaces with partial orderings as a direct consequence of our main results. Additionally, we provide illustrative examples to demonstrate the improved results obtained with our approach. An interesting results in this direction can be seen in ([25]-[41]).

Throughout this article, the standard notations and terminologies in nonlinear analysis are used.

## 2. Preliminaries

We start this section by presenting the notions of $\alpha$-$\psi$ contractive and $\alpha$-admissible mappings. Denote with $\Psi$ the family of nondecreasing functions $\Psi: [0,\infty) \to [0,\infty)$ such that $\sum_{n=1}^{+\infty} \psi^n(t) < \infty$ for each $t > 0$, where $\psi^n$ is the nth iterate of $\psi$.

These functions are known in the literature as (c)-comparison functions.

**Definition 2.1**[17] Let $(X,d)$ be a metric space and $T: X \to X$ be a given mapping. We say that $T$ is an $\alpha$-$\psi$-contractive if there exist a (c)-comparison functions $\psi \in \Psi$ and a function $\alpha: X \times X \to \mathbb{R}$ such that

$\alpha(x,y)d(Tx,Ty) \le \psi\big(d(x,y)\big)$, for all $x, y \in X$. (2.1)

Clearly, any contractive mapping, that is, a mapping satisfying Banach contraction, is an $\alpha$-$\psi$ contractive mapping with $\alpha(x,y) = 1$for all $x, y \in X$ and $\psi(t) = kt, k \in (0,1)$.

**Definition 2.2**[20] Let $X$ be a nonempty set, $T: X \to X$ and $\alpha: X \times X \to \mathbb{R}^+$, we say that $T$ is an $\alpha$-admisible mapping if

$$x, y \in X, \alpha(x,y) \ge 1 \Longrightarrow \alpha(Tx,Ty) \ge 1. \quad (2.2)$$

**Theorem 2.3** [21] Let $(X,d)$ be a complete metric space and $T: X \to X$ be an $\alpha$-$\psi$ contractive mapping. Suppose that

i$T$ is $\alpha$ admissible;

ii there exists $x_0 \in X$ such that $\alpha(Tx,Ty) \ge 1$;

iii $T$ is continuous.

Then there exists $u \in X$ such that $Tu = u$.

**Theorem 2.4** [21] Let $(X,d)$ be a complete metric space and $T: X \to X$ be an $\alpha$-$\psi$ contractive mapping. Suppose that

i$T$ is $\alpha$ admissible;

ii there exists $x_0 \in X$ such that $\alpha(Tx_0,Tx_0) \ge 1$;

iii if $\{x_n\}$ is a sequence in $X$ such that $\alpha(x_n,x_{n+1}) \ge 1$ for all $n$ and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n,x) \ge 1$ for all $n$.

Then there exists $u \in X$ such that $Tu = u$.

**Theorem 2.5**[21] Adding to the hypotheses of Theorem 1.3 (resp., Theorem 1.4) the condition, for all $x, y \in X$, there exists $z \in X$ such that $\alpha(x,z) \geq 1$ and$\alpha(y,z) \geq 1$, and one obtains uniqueness of the fixed point.

**Remark 2.6** Clearly, since $\psi$ is nondecreasing, every $\alpha$-$\psi$ contractive mapping is a generalized $\alpha$-$\psi$ contractive mapping.

Karapinar and Samet [18] introduced the following concept of generalized $\alpha$-$\psi$-contractive type mappings:

**Definition 2.7** [18] Let $(X,d)$ be a metric space and $T: X \to X$ be a given mapping. We say that $T$ is a generalized $\alpha$-$\psi$-contractive type mapping if there exist two functions $\alpha: X \times X \to [0,\infty)$ and $\psi \in \Psi$ such that for all $x, y \in$ X, we have

$$\alpha(x,y)d(Tx,Ty) \leq \psi\big(M(x,y)\big), (2.3)$$

where $M(x,y) = max\left\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$

Further, Karapinar and Samet [18] established fixed point theorems for this new class of contractive mappings. Also, they obtained fixed point theorems on metric spaces endowed with a partial order and fixed point theorems for cyclic contractive mappings.

## 3. Main Results

We introduce the concept of generalized $\alpha$-$\psi$ contractive type mappings as follows.

**Definition 3.1** Let $(X,d)$ be a metric space and $T: X \to X$ be a given mapping. $T$ is a generalized $\alpha$-$\psi$ contractive type mapping if there exist two functions $\alpha: X \times X \to [0,\infty)$ and $\psi \in \Psi$ such that for all $x, y \in$ X, we have

$$\alpha(x,y)d(Tx,Ty) \leq \psi\big(M(x,y)\big), (3.1)$$

where $M(x,y) = max\left\{\begin{matrix} d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{d(x,y)}, \\ \frac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)}, \frac{d(x,Tx)d(x,Ty)+d(y,Tx)d(y,Ty)}{d(y,Tx)+d(x,Ty)} \end{matrix}\right\}$

**Theorem 3.2** Let $(X,d)$ be a complete metric space. Suppose that $T: X \to X$ is a generalized $\alpha$-$\psi$ contractive mapping and satisfies the following conditions:

i. $T$ is $\alpha$ admissible;

ii. there exists $x_0 \in X$ such that $\alpha(Tx_0, Tx_0) \geq 1$;

iii. $T$ is continuous.

Then there exists $u \in X$ such that $Tu = u$.

***Proof*** By (ii) there exists a point $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. We define a sequence $\{x_n\}$ in $X$ by $x_{n+1} = Tx_n$ for all $n \geq 0$. If $x_n = x_{n+1}$ for some $n \geq 0$, then $Tx_n = x_{n+1} = x_n$, so that $x_n$ is a fixed point of $T$ and the proof is finished.

Assume that $x_n \neq x_{n+1}$, for all $n \geq 0$. Since $T$ is an $\alpha$-admisible, we have

$$\alpha(x_0, Tx_0) = \alpha(x_0, x_1) \geq 1 \Longrightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1. (3.2)$$

Continuing this process, we have

$$\alpha(x_1, x_2) \geq 1 \Longrightarrow \alpha(Tx_1, Tx_2) = \alpha(x_2, x_3) \geq 1,$$

Inductively, we get

$\alpha(x_n, x_{n+1}) \geq 1$, for all $n \geq 0$. (3.3)

Applying (3.1) and (3.3), for all $n \geq 1$, we have

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq \alpha(x_n, x_{n-1}) d(Tx_n, Tx_{n-1}) \leq \psi\big(M(x_n, x_{n-1})\big) (3.4)$$

On the other hand, we have

$$M(x_n, x_{n-1}) = max \left\{ \begin{matrix} d(x_n, x_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_{n-1}), \frac{d(x_n, Tx_n) d(x_{n-1}, Tx_{n-1})}{d(x_n, x_{n-1})}, \\ \frac{d(x_n, Tx_n) d(x_{n-1}, Tx_{n-1})}{d(x_n, x_{n-1}) + d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_n)}, \frac{d(x_n, Tx_n) d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_n) d(x_{n-1}, Tx_{n-1})}{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})} \end{matrix} \right\}$$

$$= max\{d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_n), d(x_{n-1}, x_n)\}$$

$$\leq max\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\} (3.5)$$

From (3.4) and taking in consideration that $\psi$ is a nondecreasing function, we get

$$d(x_{n+1}, x_n) \leq \psi(max\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\}), (3.6)$$

for all $n \geq 1$.

If for some $n \geq 1$, we have $d(x_n, x_{n-1}) \leq d(x_n, x_{n+1})$, from (3.6), we get

$$d(x_{n+1}, x_n) \leq \psi\big(d(x_n, x_{n+1})\big) < d(x_n, x_{n+1}) (3.7)$$

a contraction. Thus, for all $n \geq 1$, we have

$$max\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\} = d(x_n, x_{n-1}) (3.8)$$

Using (3.6) and (3.8), we get

$$d(x_{n+1}, x_n) \leq \psi\big(d(x_n, x_{n-1})\big) \text{ for all } n \geq 1, (3.9)$$

Continuing this process, we get

$$d(x_{n+1}, x_n) \leq \psi^2 d(x_{n-1}, x_{n-2})$$

Using mathematical induction, we have

$d(x_{n+1}, x_n) \leq \psi^n d(x_1, x_0)$ for all $n \geq 1$. (3.10)

From (3.10) and using triangular inequality, for all $k \geq 1$, we have

$$d(x_n, x_{n+k}) \leq d(x_n, x_{n+1}) + \cdots + d(x_{n+k-1}, x_{n+k})$$

$$\leq \sum_{p=n}^{n+k-1} \psi^p d(x_1, x_0)$$

$$\leq \sum_{p=n}^{n+k-1} \psi^p d(x_1, x_0) \to 0,$$

as $p \to \infty$. (3.11)

Thus, $\{x_n\}$ is a Cauchy sequence in $X$.

Since complete there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$. By continuity of $T$, we have

$$Tu = T\left(\lim_{n\to\infty} x_n\right)$$

$$= \lim_{n\to\infty} Tx_n$$

$$= \lim_{n\to\infty} x_{n+1}$$

$= u.$ (3.12)

Hence, $u$ is a fixed point of $T$.

The next theorem does not require the continuity assumption of $T$.

**Theorem 3.3** Let $(X,d)$ be a complete metric space. Suppose that $T: X \to X$ is a generalized $\alpha$-$\psi$ contractive mapping and satisfies the following conditions:

i. $T$ is $\alpha$ admissible;

ii. there exists $x_0 \in X$ such that $\alpha(Tx_0, Tx_0) \geq 1$;

iii. if $\{x_n\}$ is a sequence in $X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n$ and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that$\alpha(x_{n(k)}, x) \geq 1$ for all $k$.

Then there exists $u \in X$ such that $Tu = u$.

***Proof*** From the proof of Theorem 3.2, we know that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \geq 0$, converges for some $u \in X$. From (3.3) and condition (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, u) \geq 1$ for all $k$. Applying (3.1), for all $k$, we get

$$d(x_{n(k)+1}, Tu) = d(Tx_{n(k)}, Tu) \leq \alpha d(x_{n(k)}, u) d(Tx_{n(k)}, Tu) \leq \psi\left(M(x_{n(k)}, u)\right). \quad (3.13)$$

On the other hand, we have

$$M(x_{n(k)}, u) = max\left\{\begin{matrix} d(x_{n(k)}, u), d(x_{n(k)}, x_{n(k)+1}), d(u, Tu), \dfrac{d(x_{n(k)}, x_{n(k)+1}) d(u, Tu)}{d(x_{n(k)}, u)}, \\ \dfrac{d(x_{n(k)}, x_{n(k)+1}) d(u, Tu)}{d(x_{n(k)}, u) + d(x_{n(k)}, Tu) + d(u, x_{n(k)+1})}, \dfrac{d(x_{n(k)}, x_{n(k)+1}) d(x_{n(k)}, Tu) + d(u, x_{n(k)+1}) d(u, Tu)}{d(u, x_{n(k)+1}) + d(x_{n(k)}, Tu)} \end{matrix}\right\} \quad (3.14)$$

Letting $k \to \infty$ in (3.14), we get

$$\lim_{k\to\infty} M(x_{n(k)}, u) = d(u, Tu) \quad (3.15)$$

Suppose that $d(u, Tu) > 0$. From (3.15), for $k$ large enough, we have $M(x_{n(k)}, u) > 0$, which implies that $\psi\left(M(x_{n(k)}, u)\right) < M(x_{n(k)}, u)$. Thus, from (3.13), we get

$$d(x_{n(k)+1}, Tu) < M(x_{n(k)}, u) \quad (3.16)$$

Letting $k \to \infty$ in (3.16),and using (3.15), we get

$$d(u, Tu) \leq d(u, Tu) \quad (3.17)$$

a contradiction. Thus, we have $d(u, Tu) = 0$, that is, $u = Tu$. Hence, we have a fixed point $u$ of $T$.

**Example 3.4** Let $X = \{(1,0), (0,1)\} \subseteq \mathbb{R}^2$, and define the Euclidean distance as

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$d\big((x,y),(u,v)\big) = |x-u| + |y-v|$ for all $(x,y),(u,v) \in X$. (3.18) Clearly, $d(X,d)$ is a complete metric space. The mapping $T(x,y) = (x,y)$ is trivially continuous and satisfies for any $\psi \in \Psi$.

$\alpha\big((x,y),(u,v)\big)d\big(T(x,y),T(u,v)\big) \le \psi\Big(M\big((x,y),(u,v)\big)\Big)$, (3.19)

for all $(x,y),(u,v) \in X$, were

$$\alpha\big((x,y),(u,v)\big) = \begin{cases} 1 & if\ (x,y) = (u,v), \\ 0 & if\ (x,y) \ne (u,v). \end{cases} \tag{3.20}$$

Thus, $T$ is a generalized $\alpha$-$\psi$ contractive mapping. On the other hand, for all $(x,y),(u,v) \in X$, we have

$$\alpha\big((x,y),(u,v)\big) \ge 1 \longrightarrow (x,y) = (u,v) \longrightarrow T(x,y) = T(u,v) \longrightarrow \alpha\big(T(x,y),T(u,v)\big) \ge 1. \tag{3.21}$$

Thus, $T$ is $\alpha$ admissible. Moreover, for all $(x,y) \in X$, we have $\alpha\big((x,y),(u,v)\big) \ge 1$. Then the assumptions of Theorem 3.2 are satisfied. Note that the assumptions of Theorem 3.3 are also satisfied; indeed if $\{(x_n,y_n)\}$ is a sequence in $X$ that converges to some point $(x,y) \in X$ with $\alpha\big((x_n,y_n),(x_{n+1},y_{n+1})\big) \ge 1$ for all $n$, then, from the definition of $\alpha$, we have $(x_n,y_n) = (x,y)$ for all $n$, which implies that $\alpha\big((x_n,y_n),(x,y)\big) = 1$ for all $n$. However, in this case, $T$ has two fixed points in $X$.

To assure the uniqueness of a fixed point of a generalized $\alpha$-$\psi$ contractive mapping, we will consider the following hypothesis.

For all $x,y \in Fix(T)$, there exists $z \in X$ such that $\alpha(x,z) \ge 1$ and $\alpha(y,z) \ge 1$. (3.22)

**Theorem 3.5** Adding condition (3.22) to the hypotheses of Theorem 3.2 (resp., Theorem 3.3), we obtain that $u$ is the unique fixed point of $T$.

***Proof*** Suppose that $v$ is another fixed point of $T$. From (3.22), there exists $z \in X$ such that

$$\alpha(u,z) \ge 1 \text{ and } \alpha(v,z) \ge 1. \tag{3.23}$$

Since $T$is $\alpha$ admissible, from (3.23), we have

$$\alpha(u,T^n z) \ge 1 \text{ and } \alpha(v,T^n z) \ge 1, \forall n. \tag{3.24}$$

Define the sequence $\{z_n\}$ in $X$ by $z_{n+1} = Tz_n$for all $n \ge 0$ and $z_0 = z$. From (3.24), for all $n$, we have

$$d(u,z_{n+1}) = d(Tu,Tz_n) \le \alpha(u,z_n)d(Tu,Tz_n) \le \psi\big(M(u,z_n)\big) \tag{3.25}$$

On the other hand, we have

$$M(u,z_n) = max\left\{\begin{matrix} d(u,z_n), d(u,u), d(z_n,z_{n+1}), \dfrac{d(u,u)d(z_n,z_{n+1})}{d(u,z_n)}, \\ \dfrac{d(u,u)d(z_n,z_{n+1})}{d(u,z_n)+d(u,z_{n+1})+d(z_n,u)}, \dfrac{d(u,u)d(u,z_{n+1})+d(z_n,u)d(z_n,z_{n+1})}{d(z_n,u)+d(u,z_{n+1})} \end{matrix}\right\}$$
$$\le max\{d(u,z_n), d(u,z_{n+1})\} \tag{3.26}$$

Using (3.25), (3.26) and the monotone property of $\psi$, we get

$$d(u,z_{n+1}) \le \psi(max\{d(u,z_n), d(u,z_{n+1})\}), \tag{3.27}$$

for all $n$. Without restriction to the generality, we can suppose that $d(u, z_n) > 0$ for all $n$. If $max\{d(u, z_n), d(u, z_{n+1})\} = d(u, z_{n+1})$, we get from (3.27)that

$$d(u, z_{n+1}) \leq \psi(d(u, z_{n+1})) < d(u, z_{n+1}) \tag{3.28}$$

a contradiction. Thus, we have $max\{d(u, z_n), d(u, z_{n+1})\} = d(u, z_n)$, and

$$d(u, z_{n+1}) \leq \psi(d(u, z_n)) \tag{3.29}$$

for all $n$.

This implies that

$$d(u, z_n) \leq \psi^n(d(u, z_0)), \ \forall n \geq 1. \tag{3.30}$$

Letting $n \to \infty$ in (3.30), we obtain that

$$\lim_{n\to\infty} d(z_n, u) = 0. \tag{3.31}$$

Similarly, we have

$$\lim_{n\to\infty} d(z_n, v) = 0. \tag{3.32}$$

From (3.31)and (3.32), it follows that $u = v$. Thus, $u$ is the unique fixed point of $T$.

**Example 3.6** Let $X = [0,1]$ be endowed with metric $d(x, y) = |x - y|$ for all $x, y \in X$.

Clearly, $d(X, d)$ is a complete metric space. Define the mapping $T: X \to X$ by

$$Tx = \begin{cases} \frac{1}{4}, & if\ x \in [0,1), \\ 0, & if\ x = 1. \end{cases} \tag{3.33}$$

In this case, T is not continuous. Define the mapping $\alpha: X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & if\ (x, y) \in \left(\left[0, \frac{1}{4}\right] \times \left[\frac{1}{4}, 1\right]\right) \cup \left(\left[\frac{1}{4}, 1\right] \times \left[0, \frac{1}{4}\right]\right), \\ 0, & otherwise. \end{cases} \tag{3.34}$$

Prove that

i. $T: X \to X$ is a generalized $\alpha$-$\psi$ contractive mapping, where $\psi(t) = \frac{t}{2}$ for all $t \geq 0$;

ii. $T$ is $\alpha$ admissible;

iii. there exists $x_0 \in X$ such that $\alpha(Tx_0, Tx_0) \geq 1$;

iv. if $\{x_n\}$ is a sequence in $X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n$ and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that$\alpha(x_{n(k)}, x) \geq 1$ for all $k$;

v. condition (3.22) is satisfied.

***Proof*** To show (i), we have to prove that (3.1) is satisfied for every $x, y \in X$. If $x \in [0, {}^1/_4]$ and $y = 1$, we have

$$\alpha(x,y)d(Tx,Ty) = d(Tx,Ty) = \left|\frac{1}{4} - 0\right| = \frac{1}{4}d(y,Ty) \le \psi\big(M(x,y)\big)$$

Then (3.1) holds. If $x = 1$ and $y \in [0, {}^{1}/_{4}]$, we have

$$\alpha(x,y)d(Tx,Ty) = d(Tx,Ty) = \left|0 - \frac{1}{4}\right| = \frac{1}{4}d(x,Tx) \le \psi\big(M(x,y)\big)$$

Then (3.1) holds also in this case. The other cases are trivial. Thus, (3.1) is satisfied for every $x, y \in X$.

To show (ii). Let $(x,y) \in X \times X$ such that $\alpha(x,y) \ge 1$. From the definition of $\alpha$, we have two cases.

**Case 1** if $(x,y) \in [0, {}^{1}/_{4}] \times [{}^{1}/_{4}, 1]$. In this case, we have $(Tx,Ty) \in [{}^{1}/_{4}, 1] \times [0, {}^{1}/_{4}]$, which implies that $\alpha(Tx,Ty) = 1$.

**Case 2** if $(x,y) \in [{}^{1}/_{4}, 1] \times [0, {}^{1}/_{4}]$. In this case, we have $(Tx,Ty) \in [0, {}^{1}/_{4}] \times [{}^{1}/_{4}, 1]$, which implies that $\alpha(Tx,Ty) = 1$.

Combining Case 1 and Case 2, we have $\alpha(Tx,Ty) \ge 1$. Thus, $T$ is $\alpha$ admissible.

To show (iii) Taking $x_0 = 0$, we have $\alpha(x_0, Tx_0) = \alpha\big(0, {}^{1}/_{4}\big) = 1$.

To show (iv). Let $\{x_n\}$ be a sequence in $X$ such that $\alpha(x_n, x_{n+1}) \ge 1$ for all $n$ and $x_n \to x \in X$ as $n \to \infty$, for some $x \in X$. From the definition of $\alpha$, for all $n$, we have

$$(x_n, x_{n+1}) \in \left(\left[0, \frac{1}{4}\right] \times \left[\frac{1}{4}, 1\right]\right) \cup \left(\left[\frac{1}{4}, 1\right] \times \left[0, \frac{1}{4}\right]\right).$$

Since $\big([0, {}^{1}/_{4}] \times [{}^{1}/_{4}, 1]\big) \cup \big([{}^{1}/_{4}, 1] \times [0, {}^{1}/_{4}]\big)$ is a closed set with respect to the Euclidean metric, we get that

$$(x,x) \in \left(\left[0, \frac{1}{4}\right] \times \left[\frac{1}{4}, 1\right]\right) \cup \left(\left[\frac{1}{4}, 1\right] \times \left[0, \frac{1}{4}\right]\right),$$

which implies that $x = {}^{1}/_{4}$. Thus, we have $\alpha(x_n, x_{n+1}) \ge 1$ for all $n$.

To show (v). Let $(x,y) \in X \times X$. It is easy to show that, for $z = {}^{1}/_{4}$, we have $\alpha(x,z)\alpha(y,z) = 1$. So, condition (3.22) is satisfied.

Conclusion. Now, all the hypotheses of Theorem 3.5 are satisfied; thus, T has a unique fixed point $u \in X$. In this case, we have $u = {}^{1}/_{4}$.

**4 Consequences**

In this section we give some consequences of the main results presented above. Specifically, we apply our results to generalized metric spaces endowed with a partial order.

**Definition 4.1** [1, 8,15] Let $(X, \preccurlyeq)$ be a partially ordered set and $T: X \to X$ be a mapping. $T$ is nondecreasing with respect to $\preccurlyeq$ if

$$x, y \in X, x \preccurlyeq y \implies Tx \preccurlyeq Ty. \tag{4.1}$$

**Definition 4.2** [1, 6,22] Let $(X, \preccurlyeq)$ be a partially ordered set. A sequence $\{x_n\} \subset X$ is said to be nondecreasing with respect $\preccurlyeq$ if $x_n \preccurlyeq x_{n+1}$ for all $n$.

**Definition 4.3** [7, 13] Let $(X, \preccurlyeq)$ be a partially ordered set and $d$ be a metric on $X$. $(X, \preccurlyeq, d)$is regular if for every nondecreasing sequence $\{x_n\}$ in $X$ such that $x_n \to x \in X$ as $n \to \infty$, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that$x_{n(k)} \preccurlyeq x$ for all $k$.

**Corollary 4.4** Let $(X, \preccurlyeq)$be a partially ordered complete generalized metric space and $T: X \to X$ be a nondecreasing self mapping. Suppose that the following conditions are satisfied:

i. there exists a function $\psi \in \Psi$ for which

$$d(Tx, Ty) \leq \psi\big(M(x, y)\big), (4.2)$$

for all $x, y \in$ X with x ≼ y;

ii. there exists $x_0 \in$ X such that $x_0 \preccurlyeq Tx_0$;

iii. either $T$ is continuous, or $X$ is regular.

Then $T$ has a fixed point $u \in$ X.

***Proof*** Define a mapping $\alpha: X \times X \to [0, \infty)$ as follows.

$$\alpha(x, y) = \begin{cases} 1, & if\ x \preccurlyeq y\ or\ y \preccurlyeq x, \\ 0, & otherwise. \end{cases}$$

Then, the existence conditions of Theorem 3.2 hold and hence $T$ has a fixed point.

The following results are immediate consequences of Corollary 4.4.

**Corollary 4.5** Let $(X, \preccurlyeq)$be a partially ordered complete generalized metric space and $T: X \to X$ be a nondecreasing self mapping. Suppose that the following conditions are satisfied:

i. there exists a function $\psi \in \Psi$ for which

$$d(Tx, Ty) \leq \psi\big(d(x, y)\big), (4.3)$$

for all $x, y \in$ X with x ≼ y;

ii. there exists $x_0 \in$ X such that $x_0 \preccurlyeq Tx_0$;

iii. either $T$ is continuous, or $X$ is regular.

Then $T$ has a fixed point $u \in$ X.

***Proof*** Define a mapping $\alpha: X \times X \to [0, \infty)$ as follows.

$$\alpha(x, y) = \begin{cases} 1, & if\ x \preccurlyeq y\ or\ y \preccurlyeq x, \\ 0, & otherwise, \end{cases}$$

observe that the existence conditions of Theorem 3.3 hold and hence $T$ has a fixed point.

Particular case of the above results can be presented in the form of corollary as:

**Corollary 4.6** Let $(X, \preccurlyeq)$be a partially ordered complete generalized metric space and $T: X \to X$ be a nondecreasing self mapping. Suppose that the following conditions are satisfied:

i. there exists a function $\psi \in \Psi$ for which

$$d(Tx, Ty) \leq \psi\big(M(x, y)\big), (4.4)$$

where $M(x, y) = max \left\{ \begin{matrix} d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Tx)d(y,Ty)}{d(x,y)}, \\ \frac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)}, \frac{d(x,Tx)d(x,Ty)+d(y,Tx)d(y,Ty)}{d(y,Tx)+d(x,Ty)} \end{matrix} \right\}$

for all $x, y \in$ X with x ≼ y;

ii. there exists $x_0 \in$ X such that $x_0 \preccurlyeq Tx_0$;

iii. either $T$ is continuous, or $X$ is regular.

Then $T$ has a fixed point $u \in$ X.

***Proof*** Define a mapping $\alpha: X \times X \to [0, \infty)$ as follows.

$$\alpha(x, y) = \begin{cases} 1, & if\ x \preccurlyeq y\ or\ y \preccurlyeq x, \\ 0, & otherwise. \end{cases}$$

Then, the existence conditions of Theorem 3.2 hold and hence $T$ has a fixed point.

The following results are immediate consequences of Corollary 4.6.

**Corollary 4.7** Let $(X, \preccurlyeq)$be a partially ordered complete generalized metric space and $T: X \to X$ be a nondecreasing self mapping. Suppose that the following conditions are satisfied:

i. there exists a function $\psi \in \Psi$ for which

$$d(Tx, Ty) \leq \psi\big(d(x, y)\big), (4.5)$$

where $M(x, y) = max \left\{ \begin{matrix} d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Tx)d(y,Ty)}{d(x,y)}, \\ \frac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)}, \frac{d(x,Tx)d(x,Ty)+d(y,Tx)d(y,Ty)}{d(y,Tx)+d(x,Ty)} \end{matrix} \right\}$

for all $x, y \in$ X with x $\preccurlyeq$ y;

ii. there exists $x_0 \in$ X such that $x_0 \preccurlyeq Tx_0$;

iii. either $T$ is continuous, or $X$ is regular.

Then $T$ has a fixed point $u \in$ X.

***Proof*** Define a mapping $\alpha: X \times X \to [0, \infty)$ as follows.

$$\alpha(x, y) = \begin{cases} 1, & if\ x \preccurlyeq y\ or\ y \preccurlyeq x, \\ 0, & otherwise, \end{cases}$$

we observe that the existence conditions of Theorem 3.3 hold and hence $T$ has a fixed point.

**5 Conclusion**

In this paper, we introduced and studied a class of mappings called generalized $\alpha$-$\psi$ contractive mappings, which are a generalization of $\alpha$-$\psi$ contractive mappings. We consider various fixed point theorems for such mappings in complete metric spaces with the help of $\alpha$-admissible mappings. Furthermore, we establish fixed point theorems for metric spaces endowed with partial orders as an application of our main results. These theorems extend and improve upon existing results in the literature, and are accompanied by illustrative examples that demonstrate the benefits of our approach.

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