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A Queueing System with First Essential Phase of Service Followed by Optional Second Phase of Service and Server's Choice for a Working Vacation

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Abstract: -We investigate the steady state behavior of a batch arrival single server queue in which the first phase of service with general service time is essential and the second phase of service with heterogeneous general service time is optional. We term such a two-phase service as generalized Coxian-2 service, which means that every customer may take only the first essential phase of service or the first essential phase of service followed by second optional phase of service. It is further assumed that after completion of service(s) selected by a customer,the server, with a certain probability, may opt to take a working vacation with exponentially distributed length of time. As soon as the server's working vacation time completes, it instantly stops the undergoing service and the customer with incomplete service joins the head of the queue. This customer whose service is interrupted, will be taken up by the server for the regular first essential phase of the Coxian-2 service. We obtain steady state probability generating functions for the queue size at a random epoch of time. Some particular cases of interest have been discussed.

Keywords: *Generalized Coxian-2 service, generating function, working vacation, queue size, steady state.*

1. Introduction

In real life, there are many queueing situations with one or the other kinds of interruptions during the period when the server is proving service. These interruptions may occur due to sudden breakdowns experienced by the server or due to vacations taken by the server. During vacations the server stops working but resumes work as soon as the vacation period ends. Most of the papers on vacations deal with non-working vacations. More recently, quite a few authors have worked on queueing systems with working vacations. In such kind of vacations, the server does not stop working completely but does provide service but with a different rateof service. As soon as the vacation period is over, the server instantly switches to the original rate. For queueing systems with general interruptions and breakdowns, we refer the reader to Fadhil, R et al [1], Federgruen and So [2], Madan et al [6[and Takine [7], for papers on non-working vacations, we refer the reader to Hur and Ahn [3], Ke [4], Madan and Abu-Rub [5] and Zhang et al [8] and for queueing systems with working vacations, we refer the reader to Tian et al [9], Zhang et al [10], Sun and Li [11, 12] and Li and Tian [13, 14]. In the present paper, we study a single server queueing system with arrivals in batches of variable size. The server provides first essential phase of service to all customers. On completion of the first phase of service, a customer has the option to ask for the second phase of service. We term such a combination of two phases of service as generalized Coxian-2 service. Both phases follow a general service time distribution with different service times. In addition, we assume that after completing the first phase or both phases of service chosen by a customer, the server has the option to take a working vacation of a random duration with an exponential service time and exponentially distributed vacation time. We determine steady state solution for the queue size in terms of generating functions for all states of the system. Some interesting particular cases have been derived.

2. The Mathematical Model

We describe the mathematical model of our study by the following underlying assumptions:

- Customers arrive at the system in batches of variable size in accordance with a compound Poisson process. Let $\lambda c_i dt$ ($i = 1, 2, 3, \ldots$) be the first order probability that a batch of i customers arrives at the system during a short interval of time $(t, t+dt]$, where $0 \le c_i \le 1$, $\sum_{i=1}^{\infty} c_i = 1$ and $\lambda > 0$ is the mean arrival rate of batches. The arriving batches wait in the queue in the order of their arrival. It is further assumed that customers with each batch are pre-ordered for the purpose of service.

- The system comprises of a single server who provides generalized Coxian-2 service which means essential first phase of service followed by optional second phase of service. The first phase of service is provided to all customers one by one on a first-come, first-served basis. Let $\boldsymbol{G_1(x)}$ and $\boldsymbol{g_1(x)}$ respectively be the distribution function and the density function of the first phase service time and let $\mu_1(x)dx$ be the conditional probability of completion of first phase service, given that the elapsed time is x, so that

$$\mu_1(x) = \frac{g_1(x)}{1-G_1(x)} \tag{2.1}$$

and, therefore,

$$g_1(x) = \mu_1(x) e^{-\int_0^x \mu_1(t)dt} \quad . \tag{2.2}$$

- After completion of the first phase of service, the server provides second phase of service which is optional. A customer may take second phase of service with probability p or may leave the system with probability *1-p***.** Let $\boldsymbol{G_2(x)}$ and $\boldsymbol{g_2(x)}$ respectively be the distribution function and the density function of the first phase service time and let $\mu_2(x)dx$ be the conditional probability of completion of first phase service, given that the elapsed time is x , so that

$$\mu_2(x) = \frac{g_2(x)}{1-G_2(x)} \tag{2.3}$$

and, therefore,

$$g_2(x) = \mu_2(x) e^{-\int_0^x \mu_2(t)dt} \quad . \tag{2.4}$$

- On every completion of service(s) chosen by a customer, the server has a choice of taking a working vacation with probability α, or no vacation with probability $1-\alpha$. We further assume that as soon as a customer's service during vacation period is completeed, the server again has a choice to continue being on vacation with pobabilityα or to return to provide the first essential service with probability $1-\alpha$.

- We assume that the server's working vacation period has an exponential distribution with mean working vacation time $1/\gamma,\ \gamma > 0$ and therefore, γdt is the probability that a working vacation will terminate during the time interval $(t, t+dt]$

- It is further assumed that during a working vacation, the server serves the customers one by one following an exponential distribution with mean service time $1/\vartheta,\ \vartheta > 0$. Accordingly, ϑdt is the probability that during a working vacation a service will complete during the time interval $(t, t+dt]$

- As soon as either vacation is complete, the server instantly takes up the customer, if any, at the head of the queue and resumes providing first essential phase of service. If there is no customer waiting for service, the server still joins the queue and remains idle till a new batch of customers arrives.

- Various stochastic processes involved in the system are independent of each other.

3. Definitions and Equations

We assume that $W_n^{(j)}(x,t)$, $j=1, 2$ is the probability that at time t, there are $n \geq 0$ customers in the queue excluding one customer in j th phase service with elapsed service time x. Accordingly, $W_n^{(j)}(t) = \int_{x=0}^{\infty} W(x,t)\,dx$ denotes the probability that at time t, there are $n \geq 0$ customers in the queue excluding one customer in the j th phase service irrespective of the value of x. Further, we define $V_n(t)$ to be the probability that at time t there are $n \geq 0$ customers in the queue excluding one customer in service and the server is in the state of working vacation. Further, Q(t) be the probability that at time t, the queue is empty and the server is idle. Further, let $P_n(t) = \sum_{j=1}^{2} W(t) + V_n(t)$ denote the probability that at time t there are n (≥0) customers in the queue irrespective of whether the server is providing the first or the second phase of service or he is on the working vacation. Finally, let Q(t) be the probability that at time t, there is no customer in the system and the server is idle.

4. Steady State Equations Governing the System

Let

$$\underset{t\to\infty}{Lim}\, W_n^{(j)}(x,t) = W_n^{(j)}(x),\ \underset{t\to\infty}{Lim}\, W_n^{(j)}(t) = W,\ ,\ \underset{t\to\infty}{Lim}\, V_n(t) = V_n$$

$\underset{t\to\infty}{Lim} P_n(t) = \sum_{j=1}^{2} \underset{t\to\infty}{Lim}\, W_n^{(j)}(t) + \underset{t\to\infty}{Lim} V_n(t) = P_n$, $j=1, 2$, and $\underset{t\to\infty}{Lim}\ Q(t) = Q$ denote the corresponding steady state probabilities.

Applying the usual probability reasoning based on the underlying assumptions of the model, we obtain the following set of integra differential-difference forward equations:

$$\frac{d}{dx} W_n^{(1)}(x) + \big(\lambda + \mu_1(x)\big) W_n^{(1)}(x)$$

$$= \lambda \sum_1^n c_i\, W_{n-i}^{(1)}(x), \quad n \geq 1 \tag{4.1}$$

$$\frac{d}{dx} W_0^{(1)}(x) + (\lambda + \mu_1(x)) W_0^{(1)}(x,t) = 0 \tag{4.2}$$

$$\frac{d}{dx} W_n^{(2)}(x) + \big(\lambda + \mu_2(x)\big) W_n^{(2)}(x)$$

$$= \lambda \sum_1^n c_i\, W_{n-i}^{(2)}(x),\ n \geq 1 \tag{4.3}$$

$$\frac{d}{dx} W_0^{(2)}(x) + (\lambda + \mu_2(x)) W_0^{(2)}(x,t) = 0, \tag{4.4}$$

$$(\lambda + \vartheta + \gamma) V_n = \lambda \sum_{i=1}^{n} c_i\, V_{n-i} + \alpha(1-p) \int_0^{\infty} W_{n+1}^{(1)}(x) \mu_1(x) dx$$

$$+ \alpha \int_0^{\infty} W_{n+1}^{(2)}(x) \mu_2(x) dx + \alpha \vartheta V_{n+1} n \geq 1 \tag{4.5}$$

$$(\lambda + \vartheta + \gamma) V_0 = \alpha(1-p) \int_0^{\infty} W_1^{(1)}(x) \mu_1(x) dx + \alpha \int_0^{\infty} W_1^{(2)}(x) \mu_2(x) dx + \alpha \vartheta V_1 \tag{4.6}$$

$$\lambda Q = (1-p) \int_0^{\infty} W_0^{(1)}(x) \mu_1(x) dx + \int_0^{\infty} W_0^{(2)}(x) \mu_2(x) dx + \int_0^{\infty} W_0^{(2)}(x) \mu_2(x) dx + \vartheta V_0 \tag{4.7}$$

The above equations will be solved based on the following boundary conditions:

$$W_n^{(1)}(0) = (1-p)(1-\alpha) \int_0^{\infty} W_{n+1}^{(1)}(x) \mu_1(x) dx$$

$+(1-\alpha)\int_0^\infty W_{n+1}^{(2)}(x)\mu_2(x)dx$

$+(1-\alpha)\int_0^\infty W_{n+1}^{(2)}(x)\mu_2(x) + (1-\alpha)\vartheta V_{n+1} \ +\gamma V_n + \lambda c_{n+1}Q \qquad n\geq 0,$ (4.8)

$W_n^{(2)}(0) = p\int_0^\infty W_n^{(1)}(x)\mu_1(x)dx,\ n\geq 0,$ (4.9)

5. Steady State Queue Size at a Random Epoch

We define the following probability generating functions (pgf's):

$W^{(j)}(x,z) = \sum_{n=0}^\infty z^n W_n^{(j)}(x),\ A^{(j)}(z) = \sum_{n=0}^\infty z^n W_n^{(j)},\quad j=1,2,$

$V(z) = \sum_{n=0}^\infty z^n V_n\ ,$

$P(z) = \sum_{j=1}^2 W^{(j)}(z) + V(z)\ ,\ |z|\leq 1.$

$C(z) = \sum_{i=1}^\infty z^i c_i,$ (5.1)

Multiplying equation (4.1) by z^n, summing over n and adding the result to (4.2) and using (5.1) we get

$\frac{d}{dx}W^{(1)}(x,z) + \big(\lambda + \mu_1(x) - \lambda C(z)\big)W^{(1)}(x,z) = 0$ (5.2)

Similar operation on equations (4.3)- (4.4), (4.5) -(4.6) and (4.8) and (4.9) yield

$\frac{d}{dx}W^{(2)}(x,z) + (\lambda + \mu_2(x) - \lambda C(z))W^{(2)}(x,z) = 0$ (5.3)

$[(\lambda - \lambda C(z) + \vartheta + \gamma)z - \alpha\vartheta]V(z)$

$= \alpha(1-p)\int_0^\infty W^{(1)}(x,z)\mu_1(x)dx + \alpha\int_0^\infty W^{(2)}(x,z)\mu_2(x)dx \ -\!-\alpha\lambda Q$ (5.4)

$ZW^{(1)}(0,z) = (1-p)(1-\alpha)\int_0^\infty W^{(1)}(x,z)\mu_1(x)dx$

$+(1-\alpha)\int_0^\infty W^{(2)}(x,z)\mu_2(x)d + (1-\alpha)\vartheta V(z) + \gamma zV(z) + [\lambda C(z) - \lambda(1-\alpha)]Q,$ (5.5)

$W^{(2)}(0,z) = p\int_0^\infty W^{(1)}(x,z)\mu_1(x)dx,$ (5.6)

Next, we integrate (5.2) and (5.3) between the limits 0 and x to get

$W^{(1)}(x,z) = W^{(1)}(0,z)\,exp\begin{bmatrix}-\big(\lambda - \lambda C(z)\big)x \\ -\int_0^x \mu_1(t)dt\end{bmatrix}$ (5.7)

$W^{(2)}(x,z) = W^{(2)}(0,z)\,exp\begin{bmatrix}-\big(\lambda - \lambda C(z)\big)x \\ -\int_0^x \mu_2(t)dt\end{bmatrix}$ (5.8)

Where $W^{(1)}(0,z)$ and $W^{(2)}(0,z)$ are given respectively in (5.5) and (5.6).

We further integrate equations (5.7) and (5.8) with respect to x by parts to get

$W^{(1)}(z) = W^{(1)}(0,z)\left(\frac{1-B_1(\lambda-\lambda C(z))}{(\lambda-\lambda C(z))}\right)$ (5.9)

$W^{(2)}(z) = W^{(2)}(0,z)\left(\frac{1-B_2(\lambda-\lambda C(z))}{(\lambda-\lambda C(z))}\right)$ (5.10)

Where

$B_j\big(\lambda - \lambda C(z)\big) = \int_0^\infty e^{-(\lambda-\lambda C(z))x}dA^{(j)}(x),\ \ j=1,2$ is the Laplace-Steiltjes transform of the jth phase service time.

Next, to find out the values of the integrals $\int_0^\infty W^{(1)}(x,z)\mu_1(x)dx$ and $\int_0^\infty W^{(2)}(x,z)\mu_2(x)dx$ appearing in equations (5.4), (5.5) and (5.6), we proceed as follows:

We multiply equations (5.7) and (5.8) by $\mu_1(x)$ and $\mu_2(x)$ respectively to obtain

$\int_0^\infty W^{(1)}(x,z)\mu_1(x)dx = W^{(1)}(0,z)\bar{B}^{(1)}[\lambda - \lambda C(z)]$ (5.11)

$$\int_0^\infty W^{(2)}(x,z)\mu_2(x)dx = A^{(2)}(0,z)B^{(2)}[\lambda - \lambda C(z)] \quad (5.12)$$

We use equations (5.11) and (5.12) into equations (5.4), (5.5) and (5.6) and on simplifying, we get

$$[(\lambda - \lambda C(z) + \vartheta + \gamma)z - \alpha\vartheta]V(z)$$
$$= \alpha(1-p)W^{(1)}(0,z)\bar{B}^{(1)}[\lambda - \lambda C(z)]$$
$$+\alpha W^{(2)}(0,z)\bar{A}^{(2)}[\lambda - \lambda C(z)] - \alpha\lambda Q \quad (5.13)$$

$$\mathrm{Z}W^{(1)}(0,z) = (1-p)(1-\alpha)W^{(1)}(0,z)\bar{B}^{(1)}[\lambda - \lambda C(z)]$$
$$+(1-\alpha)W^{(2)}(0,z)\bar{B}^{(2)}[\lambda - \lambda C(z)$$
$$+(1-\alpha)\vartheta V(z) \quad +\gamma\, zV(z)$$
$$+[\lambda C(z) - \lambda(1-\alpha)]Q \quad (5.14)$$

$$W^{(2)}(0,z) = pW^{(1)}(0,z)\bar{B}^{(1)}[\lambda - \lambda C(z)], \quad (5.15)$$

Next, we use (5.15) into (5.13) and (5.14), simplify to obtain

$$[(\lambda - \lambda C(z) + \vartheta + \gamma)z - \alpha\vartheta]V(z)$$
$$= \begin{Bmatrix} \alpha(1-p)\bar{B}^{(1)}[\lambda - \lambda C(z)] \\ +\alpha pW^{(1)}(0,z)\bar{B}^{(2)}[\lambda - \lambda C(z)] \end{Bmatrix} A^{(1)}(0,z - \alpha\lambda Q \quad (5.16)$$

$$\begin{bmatrix} z - (1-p)(1-\alpha)B^{(1)}[\lambda - \lambda C(z)] \\ -(1-\alpha)p\bar{B}^{(1)}[\lambda - \lambda C(z)]\bar{B}^{(2)}[\lambda - \lambda C(z)] \end{bmatrix} W^{(1)}(0,z)$$
$$= [(1-\alpha)\vartheta + \gamma z\,]V(z) + [\lambda C(z) - \lambda(1-\alpha)]Q, \quad (5.17)$$

On solving (5.16) and (5.17) for $\mathrm{W}^{(1)}(0,\mathrm{z}))$ and V(z), we obtain

$$W^{(1)}(0,z) = \frac{\begin{matrix}[(\lambda - \lambda C(z) + \vartheta + \gamma)z - \alpha\vartheta][\lambda C(z) - \lambda(1-\alpha)]Q \\ -[(1-\alpha)\vartheta + \gamma z\,]\alpha\lambda Q\end{matrix}}{\begin{matrix}\begin{bmatrix} z - (1-p)(1-\alpha)\bar{B}^{(1)}[\lambda - \lambda C(z)] \\ -(1-\alpha)p\bar{B}^{(1)}[\lambda - \lambda C(z)]\bar{B}^{(2)}[\lambda - \lambda C(z)] \end{bmatrix}[(\lambda - \lambda C(z) + \vartheta + \gamma)z - \alpha\vartheta] \\ -\begin{Bmatrix} \alpha(1-p)\bar{B}^{(1)}[\lambda - \lambda C(z)] \\ +\alpha p\bar{B}^{(2)}[\lambda - \lambda C(z)] \end{Bmatrix}[(1-\alpha)\vartheta + \gamma z\,]\end{matrix}} \quad (5.18)$$

$$V(z) = \frac{\begin{matrix}\left[z-(1-p)(1-\alpha)\bar{A}^{(1)}[\lambda-\lambda C(z)]-(1-\alpha)p\bar{A}^{(1)}[\lambda-\lambda C(z)]\bar{A}^{(2)}[\lambda-\lambda C(z)]\right][-\alpha\lambda Q] \\ +\{\alpha(1-p)\bar{A}^{(1)}[\lambda-\lambda C(z)]+\alpha pA^{(1)}(0,z)\bar{A}^{(2)}[\lambda-\lambda C(z)]\}[\lambda C(z)-\lambda(1-\alpha)]Q\end{matrix}}{\begin{matrix}\begin{bmatrix} z-(1-p)(1-\alpha)\bar{B}^{(1)}[\lambda-\lambda C(z)] \\ -(1-\alpha)p\bar{B}^{(1)}[\lambda-\lambda C(z)]\bar{B}^{(2)}[\lambda-\lambda C(z)] \end{bmatrix}[(\lambda-\lambda C(z)+\vartheta+\gamma)z-\alpha\vartheta] \\ -\begin{Bmatrix} \alpha(1-p)\bar{B}^{(1)}[\lambda-\lambda C(z)] \\ +\alpha p\bar{B}^{(2)}[\lambda-\lambda C(z)] \end{Bmatrix}[(1-\alpha)\vartheta+\gamma z\,]\end{matrix}} \quad (5.19)$$

Using (5.19) in (5.15), we get

$$W^{(2)}(0,z) = \frac{p\bar{A}^{(1)}[\lambda-\lambda C(z)]\left\langle \begin{matrix}[(\lambda-\lambda C(z)+\vartheta+\gamma)z-\alpha\vartheta][\lambda C(z)-\lambda(1-\alpha)]Q \\ -[(1-\alpha)\vartheta+\gamma z\,]\alpha\lambda Q\end{matrix}\right\rangle}{\begin{bmatrix} z-(1-p)(1-\alpha)\bar{B}^{(1)}[\lambda-\lambda C(z)] \\ -(1-\alpha)p\bar{B}^{(1)}[\lambda-\lambda C(z)]\bar{B}^{(2)}[\lambda-\lambda C(z)]\end{bmatrix}[(\lambda-\lambda C(z)+\vartheta+\gamma)z-\alpha\vartheta] - \begin{Bmatrix}\alpha(1-p)\bar{B}^{(1)}[\lambda-\lambda C(z)] \\ +\alpha p\bar{B}^{(2)}[\lambda-\lambda C(z)]\end{Bmatrix}[(1-\alpha)\vartheta+\gamma z\,]} \quad (5.20)$$

Next, using (5.18) in (5.9) and (5.20) in (5.10), we get

$$W^{(1)}(\,z) = \frac{[(\lambda-\lambda C(z)+\vartheta+\gamma)z-\alpha\vartheta][\lambda C(z)-\lambda(1-\alpha)]Q-[(1-\alpha)\vartheta+\gamma z\,]\alpha\lambda Q\left(\frac{1-G_1(\lambda-\lambda C(z))}{(\lambda-\lambda C(z))}\right)}{\begin{bmatrix} z-(1-p)(1-\alpha)\bar{B}^{(1)}[\lambda-\lambda C(z)] \\ -(1-\alpha)p\bar{B}^{(1)}[\lambda-\lambda C(z)]\bar{B}^{(2)}[\lambda-\lambda C(z)]\end{bmatrix}[(\lambda-\lambda C(z)+\vartheta+\gamma)z-\alpha\vartheta] - \begin{Bmatrix}\alpha(1-p)\bar{B}^{(1)}[\lambda-\lambda C(z)] \\ +\alpha p\bar{B}^{(2)}[\lambda-\lambda C(z)]\end{Bmatrix}[(1-\alpha)\vartheta+\gamma z\,]}$$

(5.21)

$$W^{(2)}(z) = \frac{p\bar{A}^{(1)}[\lambda-\lambda C(z)]\langle[(\lambda-\lambda C(z)+\vartheta+\gamma)z-\alpha\vartheta][\lambda C(z)-\lambda(1-\alpha)]Q-[(1-\alpha)\vartheta+\gamma z\,]\alpha\lambda Q\rangle\left(\frac{1-G_2(\lambda-\lambda C(z))}{(\lambda-\lambda C(z))}\right)}{\begin{bmatrix} z-(1-p)(1-\alpha)\bar{B}^{(1)}[\lambda-\lambda C(z)] \\ -(1-\alpha)p\bar{B}^{(1)}[\lambda-\lambda C(z)]\bar{B}^{(2)}[\lambda-\lambda C(z)]\end{bmatrix}[(\lambda-\lambda C(z)+\vartheta+\gamma)z-\alpha\vartheta] - \begin{Bmatrix}\alpha(1-p)\bar{B}^{(1)}[\lambda-\lambda C(z)] \\ +\alpha p\bar{B}^{(2)}[\lambda-\lambda C(z)]\end{Bmatrix}[(1-\alpha)\vartheta+\gamma z\,]}$$

(5.22)

We have thus determined all three generating functions V(z), $W^{(1)}(\,z)$and $W^{(2)}(z)$ in the above equations (5.19),(5.21) and (5.22).

Note that the unknown probability Qcan be determined by the normalizing condition

V (1) + $A^{(1)}(1)$+ $A^{(2)}(1)$+Q=1 (5.23)

6. Particular Cases

CASE 1. No Second Phase Service (Only First Phase Service with Optional Working Vacations

In this case, we substitute p=0 in the main results (5.19), (5.21) and (5.22). Thus, we obtain

$$W^{(1)}(\,z) = \frac{[(\lambda-\lambda C(z)+\vartheta+\gamma)z-\alpha\vartheta][\lambda C(z)-\lambda(1-\alpha)]Q-[(1-\alpha)\vartheta+\gamma z\,]\alpha\lambda Q\left(\frac{1-G_1(\lambda-\lambda C(z))}{(\lambda-\lambda C(z))}\right)}{[z-(1-\alpha)\bar{B}^{(1)}[\lambda-\lambda C(z)]][(\lambda-\lambda C(z)+\vartheta+\gamma)z-\alpha\vartheta] -[\,z-(1-\alpha)\bar{B}^{(1)}[\lambda-\lambda C(z)]][(1-\alpha)\vartheta+\gamma z\,]} \quad (6.1)$$

$$W^{(2)}(\,z) = 0 \quad (6.2)$$

$$V(z) = \frac{\left[z-(1-\alpha)\bar{A}^{(1)}[\lambda-\lambda C(z)]\right][-\alpha\lambda Q] +\{\alpha\bar{A}^{(1)}[\lambda-\lambda C(z)\}\,[\,][\lambda C(z)-\lambda(1-\alpha)]Q}{[z-(1-\alpha)B^{(1)}[\lambda-\lambda C(z)]][(\lambda-\lambda C(z)+\vartheta+\gamma)z-\alpha\vartheta] -[\,z-(1-\alpha)\bar{A}^{(1)}[\lambda-\lambda C(z)]][(1-\alpha)\vartheta+\gamma z\,]} \quad (6.3)$$

CASE 2. First Phase Essential Service followed by Optional Second Phase Service Without Optional Working Vacations

In this case, we substitute $\alpha = 0$in the main results to obtain

$$W^{(1)}(\,z) = \frac{[(\lambda-\lambda C(z)+\vartheta+\gamma)z][\lambda C(z)-\lambda]Q}{\begin{bmatrix} z-(1-p)\bar{B}^{(1)}[\lambda-\lambda C(z)] \\ -pB^{(1)}[\lambda-\lambda C(z)]\bar{A}^{(2)}[\lambda-\lambda C(z)]\end{bmatrix}[(\lambda-\lambda C(z)+\vartheta+\gamma)z]} \quad (6.4)$$

$$W^{(2)}(z) = \frac{p\bar{A}^{(1)}[\lambda-\lambda C(z)]\langle[(\lambda-\lambda C(z)+\vartheta+\gamma)z-\alpha\vartheta][\lambda C(z)-\lambda(1-\alpha)]Q\rangle}{\begin{bmatrix} z-(1-p)\bar{B}^{(1)}[\lambda-\lambda C(z)] \\ -p\bar{B}^{(1)}[\lambda-\lambda C(z)]\bar{A}^{(2)}[\lambda-\lambda C(z)] \end{bmatrix}[(\lambda-\lambda C(z)+\vartheta+\gamma)z]} \quad (6.5)$$

$$V(z) = 0 \quad (6.6)$$

CASE 3. No Optional Second Phase Service and No Optional Working Vacations

In this case, we let $p = 0, \alpha = 0$, $\vartheta = 0$ and $\gamma=0$ in the main results found above and get

$$W^{(1)}(\ z) = \frac{[\lambda C(z)-\lambda]Q}{[z-\bar{B}^{(1)}[\lambda-\lambda C(z)]]} \quad (6.7)$$

$$W^{(2)}(\ z) = 0 \quad (6.8)$$

$$V(z) = 0 \quad (6.9)$$

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