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Results on Restrained Certified Domination Number of Graphs

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Abstract: In this article, we have defined the concept of restrained certified domination number of graphs. For any connected graph G, a restrained dominating set S$\subseteq V(G)$ is said to be a restrained certified dominating set if for every $v \in S$ there exists either at least two neighbors in $V - S$ or no neighbors in $V - S$. The minimum cardinality of the restrained certified dominating set is called the restrained certified domination number and is denoted by $\gamma_{rcer}(G)$. A restrained certified dominating set of cardinality $\gamma_{rcer}(G)$ is called a $\gamma_{rcer} -$ set. Relation of $\gamma_{rcer}(G)$ with other graph theoretical parameters have been discussed. Also this paper includes the characterization of graphs. Nordhas – Gaddum type results have been studied for some values of n.

Keywords: *certified domination, restrained domination, restrained certified dominating set, restrained certified domination number.*

AMS classification: 05C12, 05C69.

1 Introduction:

In this article, we have defined the concept of restrained certified domination number of graphs. The concept of restrained domination was introduced by Telle [6] as vertex partitioning problem. In [4] the concept of certified domination was introduced. Domination nowadays is an emerging topic in graph theory. For detailed knowledge about domination parameters one can refer[7,8]. Motivated by the ideas mentioned above we are urged to define a new concept called restrained certified domination. The possible upper and lower bounds of γ_{rcer}(G) have been determined. The value of restrained certified domination never be $n-1$. Relation of $\gamma_{rcer}(G)$ with other graph theoretical parameters have been studied. The *corona product* $G \circ H$ of two graphs G and H is obtained by taking one copy of G and $|V(G)|$ copies of H and by joining each vertex of the i^{th} copy of H to the i^{th} vertex of G where $1 \leq i \leq |V(G)|$. The *friendship graph* F_n can be constructed by joining n copies of the cycle graph $C_3$with a common vertex, which becomes a universal vertex. The open neighborhood N(v) of the vertex v consists of the set of vertices adjacent to v, that is, N(v) = $\{w \in V : vw \in E\}$. For a set $S \subseteq V$, the open neighbourhood of S is defined to be $\bigcup_{v \in S} N(v)$. The *complement* $\bar{G}$ of a graph G = (V,E) is defined to be a simple graph with vertex set V in which two vertices u and v are adjacent if and only if they are not adjacent in G.

A set S⊆V(G) is said to be a *dominating set* if every vertex v∈V(G) is either an element of S or is adjacent to an element of S. The minimum cardinality taken over all dominating sets is called the *domination number* and is denoted by $\gamma(G)$. A set S ⊆ V(G) is called a *certified dominating set* of G if S is a dominating set of G and every vertex belonging to S has either zero or at least two neighbours in V(G) –S. The cardinality of the smallest certified dominating set is called the certified domination number of G and is denoted by γ_{cer}(G).

1.1 Definition

For any connected graph G, a restrained dominating set S$\subseteq V(G)$ is said to be a restrained certified dominating set if for every $v \in S$ there exists either at least two neighbours or no neighbours in $V - S$. The minimum cardinality of the restrained certified dominating set is called the restrained certified domination number and is denoted by $\gamma_{rcer}(G)$. A restrained certified dominating set of cardinality $\gamma_{rcer}(G)$ is called a $\gamma_{rcer}-$ set.

Example:

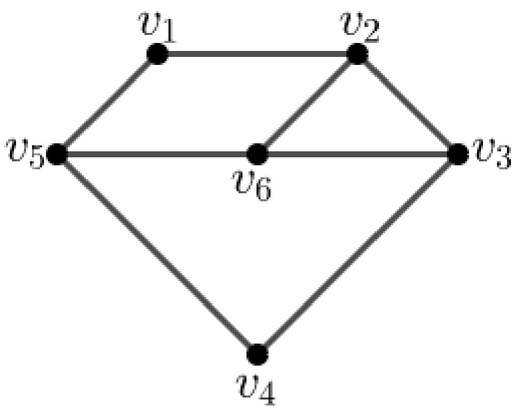


Fig. 1

In the above figure, take S = $\{v_2, v_4\}$. The vertex $v_1 \in V - S$ has a neighbour v_5 in $V - S$ as well as a neighbour v_2 in S. Again the vertex $v_5 \in V - S$ has neighbours in $V - S$ as well as in S. Also the vertices of S has at least two neighbours in $V - S$. Thus the set S = $\{v_2, v_4\}$ is a minimum restrained certified dominating set and hence the restrained certified domination number is $\gamma_{rcer}(G) = 2$.

2 γ_{rcer} values of some standard graphs

Observations 2.1.

1. For any complete graph G,γ_{rcer}(G)= 1.
2. Let G be a connected wheel graph. Then $\gamma_{rcer}(G) = 1, n \geq 4$.
3. For the star graph G, $\gamma_{rcer}(G) = n, n \geq 2$.
4. For the path graph G, $\gamma_{rcer}(G)$=n.

Theorem 2.2.

For the cycle graph C_n where $n \geq 6$,$\gamma_{rcer}(G) = \begin{cases} \frac{n}{3}, if \text{ n} \equiv 0 \text{ (mod 3)} \\ n, if\ n \equiv 1,2(mod\ 3) \end{cases}$

Proof.

We prove this theorem by considering the following cases.

$Case(i)$: $n \equiv 0\ (mod\ 3)$

Let $V(C_n)$ =$\{v_1, v_2, \ldots, v_n\}$ be the vertices of C_n, where $n \geq 6$ and $n \equiv 0\ (mod\ 3)$. Let $S = \{v_1, v_4, \ldots, v_{n-2}\}$. Consider some $v_i \in S$. Since $\delta(G) = 2$, $|N(v_i) \cap (V - S)| = 2$ for all $v_i \in S$. Also $|N(v_j) \cap (V - S)| = 1$ for $v_j \in V - S$. Thus the set S is minimum restrained certified dominating set. Therefore $\gamma_{rcer}(G) = \frac{n}{3}$.

$Case(ii)$: $n \equiv 1(mod\ 3)$

Let S = $\{v_1, v_4, v_7, v_{10}, v_{13}, \ldots, v_{n-3}, v_{n-1}\}$. Since $\delta(G) = 2, deg(v_i) = 2$ for all $v_i \in S$. Therefore vertices of S has exactly two neighbours in $V - S$. Thus S is a certified dominating set.

Consider $v_n \in V - S$. The adjacent vertices of v_n are v_1 and v_{n-1} where $v_1 \in S$ and $v_{n-1} \in S$. That is

v_n has no neighbours in $V-S$, so include v_n in S. Again, consider $v_{n-2} \in V-S$. The neighbourhoods of v_{n-2} are $N(v_{n-2}) = \{v_{n-1}, v_{n-3}\} \subseteq S$. Here v_{n-2} has no neighbours in $V-S$. In this way if we check the vertices of $V-S$, they have no neighbours in $V-S$. Proceeding like this, finally we arrive at $S = V(C_n)$. Hence $\gamma_{rcer}(G) = n$.

$Case(iii)$: $n \equiv 2 \ (mod\ 3)$

Let $S = \{v_1, v_4, v_7, v_{10}, \ldots., v_{n-4}, v_{n-1}\}$. Clearly all the vertices in S are non adjacent vertices. Hence every member in S has two neighbours in $V-S$. Now consider $v_n \in V-S$. The vertex v_n has no neighbours in $V-S$ because v_1 and v_{n-1} belong to S. Therefore v_n also belongs to S.

Consider v_{n-1}. Neighbours of v_{n-1} are v_n and v_{n-2} where $v_n \in S$ and $v_{n-2} \in V-S$. By the definition of restrained certified domination, v_{n-1} should have either atleast two neighbours or no neighbours at all in $V-S$. But v_{n-1} has only one neighbour in $V-S$. Hence v_{n-1} also belongs to S. Proceeding the same way, we get $S = \{v_1, v_2, v_3, \ldots, v_{n-1}, v_n\}$. Thus $\gamma_{rcer}(C_n) = n, n \geq 6, n \equiv 2(mod\ 3)$.

Note 2.3.

For the cycle graph C_n, $\gamma_{rcer}(G) = \begin{cases} 1 \ if \ n = 3 \\ n \ if \ n = 4,5 \end{cases}$.

Theorem 2.4.

The pendant vertices of a graph belongs to the restrained certified dominating set.

Proof.

Let G be a connected graph and S be the restrained certified dominating set. By the definition of restrained certified dominating set if a vertex is not in S, then it should be adjacent to a vertex in S and to a vertex in $V-S$. But each pendant vertex is of degree one. Therefore the pendant vertices belong to S.

Example:

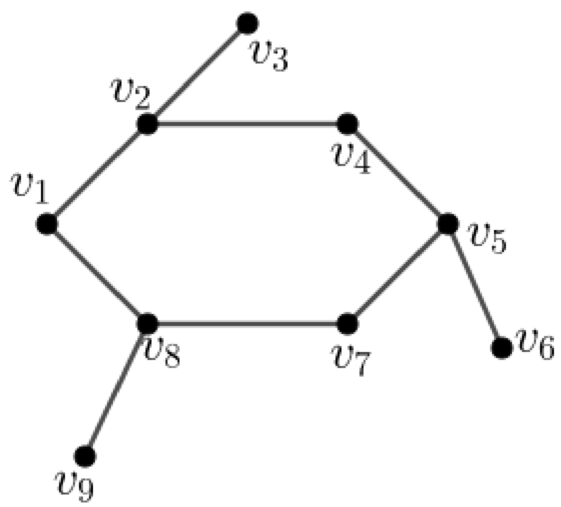


Fig.2

Here the pendant vertices are $\{v_3, v_6, v_9\}$. The restrained certified dominating set of the graph is $\{v_2, v_3, v_5, v_6, v_8, v_9\}$ which contains the pendant vertices of the graph.

Theorem 2.5.

Super set of a γ_{rcer} -set need not be a γ_{rcer}-set.

Proof.

Let S be the γ_{rcer}-set. Without loss of generality assume $S \cup \{u\}$ forms a super set of S, where $u \in V(G)$. If a vertex in $V-(S \cup \{u\})$ has no neighbour in $V-(S \cup \{u\})$ and $S \cup \{u\}$, then $S \cup \{u\}$ is not a γ_{rcer}-set. Therefore, super set of S need not be γ_{rcer}-set.

Example:

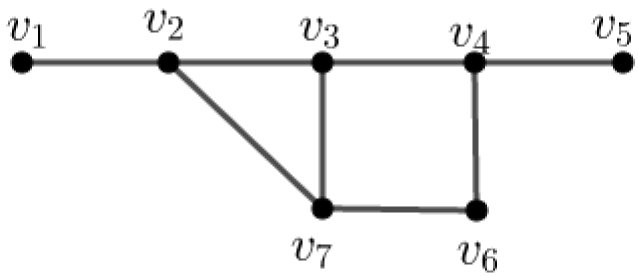


Fig. 3

Let $S=\{v_1, v_2, v_4, v_5\}$ be the restrained certified dominating set of the above graph. Suppose we add a vertex v_3 to S. The vertex v_3 in S has only one neighbour in $V-S$. Thus $S \cup \{v_3\}$ is not a restrained certified dominating set.

Observation 2.6.

A restrained dominating set need not be a restrained certified dominating set.

Example:

For example consider the cycle C_7. Let $S=\{v_1, v_4, v_5, v_6, v_7\}$ be the restrained dominating set. Now, consider the vertex v_1 of degree 2 in S. The vertex v_1 is adjacent to a vertex is S and to a vertex in $V-S$. But by the definition of restrained certified dominating set a vertex in S should be adjacent to at least 2 vertices in $V-S$. Thus S is not a restrained certified dominating set.

Note: $\gamma_{rcer}(G)$ never be n-1.

Theorem 2.7.

Let G be a connected graph of order n. If the pendant vertices of G belongs to the $\gamma_{rcer}-set$ then the support vertices also belongs to $\gamma_{rcer}-set$.

Proof.

Let S be the $\gamma_{rcer}-set$. Suppose there exists a pendant vertex which belongs to the $\gamma_{rcer}-set$ say v_1 . By the definition of restrained certified dominating set, $v_1 \in S$ should have atleast two neighbours. But v_1 is a pendant vertex implies it's support also belongs to S. Hence if a pendant vertex belongs to S, its support also belongs to S.

2.8. General Bound

The value of $\gamma_{rcer}(G)$ ranges over 1 $to\ n$. Sharpness in lower bound is attained for complete graph and wheel graph. Maximum bound is for Path graph and Star graph.

3 Characterisation of $\gamma_{rcer}(G)=1$

Theorem 3.1.

Let G be a connected graph of order n≥ 3 with no leaves. Then $\gamma_{rcer}(G)=1$ if and only if there exists a vertex of degree n-1.

Proof.

Assume that $\gamma_{rcer}(G) = 1$. Let $S = \{v\}$ be the restrained certified dominating set of G. Then $\{v\}$ dominates all the other vertices implies that degree of $\{v\}$ is maximum. Clearly the maximum degree of a graph is $n - 1$. Thus there exists a vertex $\{v\}$ of degree $n - 1$.

Conversely, assume that there exists a vertex of degree $n - 1$. Let $\{w\}$ be the vertex which is of degree $n - 1$. Suppose that $\{u, w\}$ is the restrained certified dominating set, where u is any vertex of G. Now $\deg(w) = n - 1$ implies w dominates $V(G)$ satisfying the condition of certified domination.

Also every vertex in $V - \{w\}$ has a neighbor in $V - \{w\}$ as well as adjaent to w. Thus the set $\{u, w\}$ is not the minimum restrained certified dominating set. $S = \{w\}$ is a minimum restrained certified dominating set. This implies $\gamma_{rcer}(G) = 1$.

Theorem 3.2.

Let G be a connected graph with no pendant vertices and $n \geq 3$. When $\gamma_{cer}(G) = \gamma_r(G) = 1$, then $\gamma_{rcer}(G) = 1$.

Proof.

Since $\gamma_{cer}(G) = 1$, let $S = \{v\}$ be the certified dominating set. Then S is a dominating set implies all the vertices of G are dominated by the vertex v. Since G has no pendant vertices, δ(G)≥ 2. Now consider a vertex in $V - S$. That vertex is adjacent to v and to a vertex in $V - S$.

Thus each vertex in $V - S$ satisfies the restrained condition. Hence $S = \{v\}$ itself is a restrained certified dominating set. Thus $\gamma_{rcer}(G) = 1$.

The following two theorems give the characterization for 2- regular graphs of order 4 and square of cycle graph.

Theorem 3.3.

Let G be a $2-$ regular graph with 4 vertices. Then $\gamma_{rcer}(G) = 4$.

Proof.

Let $V(G) = \{u, v, w, x\}$ be the vertices of G assigned in the clockwise direction. Since G is $2-$ regular, $deg(v) = 2$ where $v \in V(G)$. Now $n = 4$ and $deg(v) = 2$ implies γ_{rcer}(G) ≠ 1. Then $\gamma_{rcer}(G) = 2,3 \ or\ 4$.

$Case\ (i)$: $\gamma_{rcer}(G) = 2$.

$Subcase\ (i)$: The two vertices in $\gamma_{rcer}(G) -$ set are adjacent.

Let $S = \{u, v\}$ be the restrained certified dominating set where u and v are adjacent vertices in G. Now $|N(u) \cap (V - S)|$ =1 and $|N\ (v) \cap\ (V - S)| = 1$ which means S does not satisfies certified domination. Therefore the set S is not a $\gamma_{rcer} -$ set.

$Subcase\ (ii)$: The two vertices in $\gamma_{rcer} -$ set are non adjacent.

Now $|N(u) \cap\ (V - S)| = |N(w) \cap (V - S| = 2$ where $u, w\ \in S$. But $v\ \in\ V - S$ is adjacent to two vertices in S, which is a contradiction to every vertex in $V - S$ is adjacent to a vertex in S and $V - S$. Therefore the set of two vertices does not form a $\gamma_{rcer} -$ set of G. Thus $\gamma_{rcer}(G) \neq 2$.

$Case\ (ii)$: $\gamma_{rcer}(G) = 3$

Since $n = 4$ and $\gamma_{rcer}(G)$ never be $n-1$ implies that $\gamma_{rcer}(G) \neq 3$. Thus clearly from the above cases, we come to the conclusion that $\gamma_{rcer}(G) = 4$.

Theorem 3.4.

Let G be the square of a cycle graph. Then $\gamma_{cer}(G) = \left\lceil \frac{n}{5} \right\rceil, n \geq 5$.

Proof.

The square of a cycle graph G is a 4 - regular graph. Then $\delta(G) = \Delta(G) = 4$ for all $v \in V(G)$. Let $V(G) = \{v_1, v_2, \dots v_n\}$ be the vertices of G. Let $S = \{v_1, v_6, v_9, v_{12}, v_{17}, \dots, v_{n-4}\}$. Since G is 4 – regular each vertex dominates 4 vertices of G. Thus S is clearly a dominating set. Also, $v_i \in S$ has four neighbours in S and $v_j \in V - S$ has neighbours both in $S \ and \ V - S$. Thus S is a minimum restrained certified dominating set. Therefore, $\gamma_{rcer}(G) = \left\lceil \frac{n}{5} \right\rceil$.

Theorem 3.5.

Let G be a connected graph with no pendant vertices. If $\gamma_r(G) = \gamma_{cer}(G)$ then $\gamma_{cer}(G) = \gamma_{rcer}(G)$. The result does not holds for C_4.

Proof.

Since G is a connected graph with no pendant vertices, $\delta \geq 2$. Assume $\gamma_r(G) = \gamma_{cer}(G)$. Let S be the restrained dominating set. $\gamma_r(G) = \gamma_{cer}(G)$ implies S is a certified dominating set. Now, every vertex in S has atleast two neighbours in S and each vertex in $V - S$ has neighbours in $V - S$ and S.Which means S itself is a restrained certified dominating set. Thus $\gamma_{cer}(G) = \gamma_{rcer}(G)$. In the case of $C_4, \gamma_r(G) = \gamma_{cer}(G) = 2$. But $\gamma_{rcer}(G) = 4$.

4 Relations with graph theoretical parameters:

Theorem 4.1.

Let G be a connected graph of order n. Then $\gamma_{rcer}(G) + \Delta(G) \leq 2n - 1$.

Proof.

For any graph, $\Delta(G) \leq n - 1$. The upper bound of $\gamma_{rcer}(G)$ is found to be n. Therefore $\gamma_{rcer}(G) + \Delta(G) \leq n - 1 + n = 2n - 1$.

Theorem 4.2.

Let G be a graph that is connected of order n. Then $\gamma_{rcer}(G) + \kappa(G) \leq 2n - 1$.

Proof.

Clearly $\kappa(G) \leq n - 1$. Also $\gamma_{rcer}(G) \leq n$. Implies $\gamma_{rcer}(G) + \kappa(G) \leq 2n - 1$.

5 Nordhas Gaddum results

The following theorems provide some values on nordhas - gaddum type results.

Theorem 5.1.

If G is a complete graph on n vertices the nordhas - gaddum result is as follows:

$\gamma_{rcer}(G) + \gamma_{rcer}(\overline{G}) = n + 1$ and $\gamma_{rcer}(G).\ \gamma_{rcer}(\overline{G}) = n$.

Proof.

Since $\Delta(G) = n - 1$, by theorem, $\gamma_{rcer}(G) = 1$. $\overline{G}$ is the complement of G in which there will be no edges because G is a complete graph. Therefore $\gamma_{rcer}(\overline{G}) = n$. Thus $\gamma_{rcer}(G) + \gamma_{rcer}(\overline{G}) = 1 + n$ and

$\gamma_{rcer}(G).\ \gamma_{rcer}(\overline{G}) = n.$

Theorem 5.2.

Let G be a cycle graph. Then $\gamma_{rcer}(G) + \gamma_{rcer}(\overline{G}) \leq n + 2$ and $\gamma_{rcer}(G).\ \gamma_{rcer}(\overline{G}) \leq 2n$ where $n \geq 6$.

Proof.

By theorem 2.2, $\gamma_{rcer}(G) \leq n$. Since G is 2-regular, $\overline{G}$ is n-3 regular. That is $\delta(G) = \Delta(G) = n - 3$. Therefore $\gamma_{rcer}(\overline{G}) = 2$, $n \geq 6$. Hence $\gamma_{rcer}(G) + \gamma_{rcer}(\overline{G}) \leq n + 2$, and $\gamma_{rcer}(G).\ \gamma_{rcer}(\overline{G}) \leq 2n$, where $n \geq 6$.

Theorem 5.3.

When $n = 2$, for any graph G

$\gamma_{rcer}(G) + \gamma_{rcer}(\overline{G}) = 4$ and $\gamma_{rcer}(G)\ .\ \gamma_{rcer}(\overline{G}) = 4$.

Proof.

If $G \cong K_2$, then by theorem 3.1, $\gamma_{rcer}(G) = 2$.

$G \cong K_2$ implies $\overline{G} \cong 2k_1$. We have $\gamma_{rcer}(k_1) = 1$. Therefore $\gamma_{rcer}(\overline{G}) = \gamma_{rcer}(2k_1) = 2$. Thus $\gamma_{rcer}(G) + \gamma_{rcer}(\overline{G}) = 4$ and $\gamma_{rcer}(G)\ .\ \gamma_{rcer}(\overline{G}) = 4$. Similarly if $\overline{G} \cong k_2$, we get the result.

Theorem 5.4.

Let G be a connected graph of order $n = 3$. Then

i) $\gamma_{rcer}(G) + \gamma_{rcer}(\overline{G})$ = 4 and $\gamma_{rcer}(G)\ .\ \gamma_{rcer}(\overline{G}) = 3$ if either G or $\overline{G}$ is isomorphic to K_3.

ii) $\gamma_{rcer}(G) + \gamma_{rcer}(\overline{G}) = 6$ and $\gamma_{rcer}(G)\ .\ \gamma_{rcer}(\overline{G}) = 9$ if either G or $\overline{G}$ is isomorphic to P_3.

Proof.

Let $\{v_1, v_2, v_3\}$ be the vertices of G.

i) Let G be isomorphic to K_3. Then by theorem, $\gamma_{rcer}(G) = 1$. Since $G \cong K_3$, $\overline{G}$ becomes a disconnected graph on 3 vertices. Therefore, $\gamma_{rcer}(\overline{G}) = 3$. Thus we can see that if $G \cong K_3$, then $\gamma_{rcer}(G) + \gamma_{rcer}(\overline{G})$ = 1+3 =4 and $\gamma_{rcer}(G)\ .\ \gamma_{rcer}(\overline{G})$ =1.3 =3. Similarly if $\overline{G} \cong K_3$, the result holds.

ii) Let G be a connected graph that is isomorphic to P_3. Then by the theorem on path graphs, $\gamma_{rcer}(G) = 3$. Since $G \cong P_3$, $\overline{G} \cong P_2 \cup K_1$. That is $\gamma_{rcer}(\overline{G}) = \gamma_{rcer}(P_2 \cup K_1) = 3$. Therefore $\gamma_{rcer}(G) + \gamma_{rcer}(\overline{G}) = 6$ and $\gamma_{rcer}(G)\ .\ \gamma_{rcer}(\overline{G})$ = 9.

Theorem 5.5.

For any graph on 4 vertices, $\gamma_{rcer}(G) + \gamma_{rcer}(\overline{G}) = 5$ and $\gamma_{rcer}(G)\ .\ \gamma_{rcer}(\overline{G}) = 4$ only if $i)$ G or $\overline{G} \cong K_4$ $ii)$ G or $\overline{G} \cong K_4 - e$.

Proof:

$i)$ Let $V(G) = \{v_1, v_2, v_3, v_4\}$ be the vertices of G. Suppose assume that G is isomorphic to K_4. Then v_1 is adjacent to v_2, v_3, v_4. Also v_2 is adjacent to every other v_i. Thus if we consider any pair of vertices in V(G) we can find an edge. Therefore $\Delta(G) = n - 1$. By theorem 3.1 we have $\gamma_{rcer}(G) = 1$.

Now in $\overline{G}$, due to adjacency between every pair of vertices in G, there will be no edge in between vertices of $\overline{G}$. The graph $\overline{G}$ will be a disconnected graph with 4 vertices. Therefore $\gamma_{rcer}(\overline{G}) = 4$.

Thus $\gamma_{rcer}(G) + \gamma_{rcer}(\overline{G}) = 5$ and $\gamma_{rcer}(G)\ .\ \gamma_{rcer}(\overline{G})$ = 4. Similarly we can prove the result if $\overline{G} \cong$

K_4.

$ii)$ Suppose G $\cong$ $K_4 - e$. Removing an edge e from K_4 leads to decrease in the degree of two vertices. The remaining vertices will have a maximum degree n-1. Therefore, by theorem 3.1 $\gamma_{rcer}(G) = 1$.

In $\bar{G}$, the vertices with degree n-1 in G, will be isolated. The removed edge e in G alone appear in $\bar{G}$. That is $\bar{G} \cong 2P_1 \cup P_2$. This implies $\gamma_{rcer}(2P_1 \cup P_2) = 2 \Longrightarrow 2\,\gamma_{rcer}(P_1) + \gamma_{rcer}(P_2) = 2 + 2 = 4$. Thus $\gamma_{rcer}(G) + \gamma_{rcer}(\overline{G}) = 5$ *and* $\gamma_{rcer}(G)\,.\,\gamma_{rcer}(\overline{G}) = 4$.

Theorem 5.6.

Let G belongs to any one of the graphs $G_1, G_2, \ldots, G_7$ of order $n = 4$. Then $\gamma_{rcer}(G) + \gamma_{rcer}(\bar{G}) = 6$ *or* 8, *and* $\gamma_{rcer}(G).\gamma_{rcer}(\bar{G}) = 8$ *or* 16.

Proof.

$i)$ $\gamma_{rcer}(G_1) = 4$. Then $\gamma_{rcer}(\overline{G_1}) = 2$,

we get $\gamma_{rcer}(G_1) + \gamma_{rcer}(\overline{G_1}) = 6$ *and* $\gamma_{rcer}(G_1).\gamma_{rcer}(\overline{G_1}) = 8$

$ii)$ $\gamma_{rcer}(G_2) = 4$. Then $\gamma_{rcer}(\overline{G_2}) = 4$,

we get $\gamma_{rcer}(G_2) + \gamma_{rcer}(\overline{G_2}) = 8$ *and* $\gamma_{rcer}(G_2).\gamma_{rcer}(\overline{G_2}) = 16$

$iii)$ $\gamma_{rcer}(G_3) = 4$. Then $\gamma_{rcer}(\overline{G_3}) = 4$,

we get $\gamma_{rcer}(G_3) + \gamma_{rcer}(\overline{G_3}) = 8$ *and* $\gamma_{rcer}(G_3).\gamma_{rcer}(\overline{G_3}) = 16$.

$iv)$ $\gamma_{rcer}(G_4) = 2$. Then $\gamma_{rcer}(\overline{G_4}) = 4$,

we get $\gamma_{rcer}(G_4) + \gamma_{rcer}(\overline{G_4}) = 6$ *and* $\gamma_{rcer}(G_4).\gamma_{rcer}(\overline{G_4}) = 8$.

$v)$ $\gamma_{rcer}(G_5) = 4$. Then $\gamma_{rcer}(\overline{G_5}) = 2$,

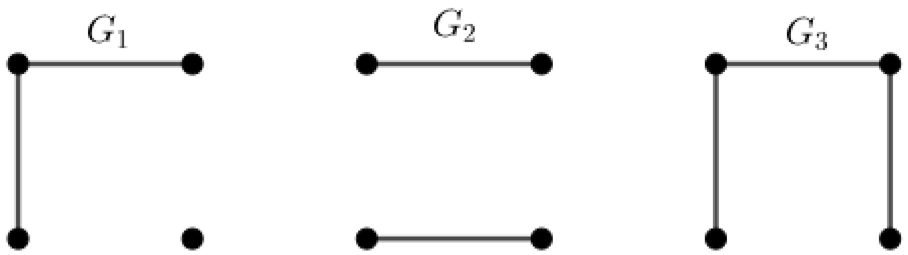
we get $\gamma_{rcer}(G_5) + \gamma_{rcer}(\overline{G_5}) = 6$ *and* $\gamma_{rcer}(G_5).\gamma_{rcer}(\overline{G_5}) = 8$.

$vi)$ $\gamma_{rcer}(G_6) = 4$. Then $\gamma_{rcer}(\overline{G_6}) = 4$,

we get $\gamma_{rcer}(G_6) + \gamma_{rcer}(\overline{G_6}) = 8$ *and* $\gamma_{rcer}(G_6).\gamma_{rcer}(\overline{G_6}) = 16$.

$vii)$ $\gamma_{rcer}(G_7) = 2$. Then $\gamma_{rcer}(\overline{G_7}) = 4$,

we get $\gamma_{rcer}(G_7) + \gamma_{rcer}(\overline{G_7}) = 6$ *and* $\gamma_{rcer}(G_7).\gamma_{rcer}(\overline{G_7}) = 8$.



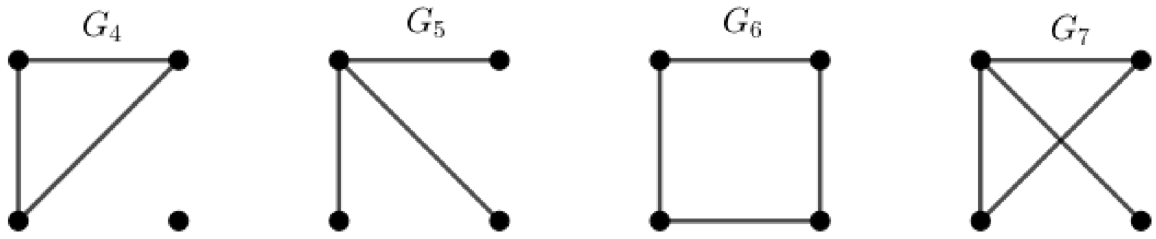


Fig. 4

Result 5.7.

Let G be a connected graph of order $n \geq 5$. Then $\gamma_{rcer}(G) + \gamma_{rcer}(\overline{G}) \leq n + 3$ and $\gamma_{rcer}(G) \,.\, \gamma_{rcer}(\overline{G}) \leq 3n$.

In the following theorems we gave the characterization of corona product of graphs.

Theorem 5.8.

Let G be $C_m \circ P_n$ of order mn. Then $\gamma_{rcer}(C_m \circ P_n)$ = m.

Proof.

Let $\{v_1, v_2, \ldots, v_m\} \cup \{U_i\}$ be the vertices of the corona graph G where each U_i is a collection of the vertices of the path $P_n, 1 \leq i \leq m$. Let S=$\{v_1, v_2, \ldots, v_m\}$ be such that the vertices of S are of degree n+2. In G we have $\delta(G) = 2$.

Consider $v_1 \in S$. In V-S, the vertex v_1 has atleast two neighbours. Similiarly each vertex in S has atleast two neighbours in V-S. Thus S is a certified dominating set. Now, the vertices in V-S has exactly one neighbour in S as well as V-S. This implies that S is a minimum restrained certified dominating set. Thus $\gamma_{rcer}(C_m \circ P_n) = m$.

Observation 5.9.

$\gamma_{rcer}(K_m \circ P_n) = m, \ n \geq 2$.

Note:

$\gamma_{rcer}(G)$ has a beautiful property over corona product of graphs. The corona product of path with (cycle) complete graph is independent of the path we take.

Theorem 5.10.

Let G be the connected corona graph $C_n \circ K_1$. Then $\gamma_{rcer}(G) = 2n$.

Proof.

Let $\{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$ be the vertices of the corona graph $C_n \circ K_1$ such that $|V(C_n \circ K_1)| = 2n$. Let S be the $\gamma_{rcer} - set$. The vertices $u_i's$ are of degree one and $v_i's$ are of degree 3. By theorem that pendant vertices belongs to the $\gamma_{rcer} - set$, we have $u_i's, 1 \leq i \leq n$ belongs to the $\gamma_{rcer} - set$. Consider $u_1 \in S$. Now u_1 has only one neighbour in V-S which implies the neighbour of u_1 will also belongs to S. Similarly for each vertex u_i in S, there exists exactly one neighbour in V-S. Hence each v_i belongs to S. Thus S=$\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ be the $\gamma_{rcer} - set$. This implies $|S| = n + n = 2n$

$\therefore \gamma_{rcer}(G) = 2n$

Note: i) $\gamma_r(K_n) = \gamma_{cer}(K_n) = \gamma_{rcer}(K_n) = 1$

ii) $\gamma_r(G) = \gamma_{rcer}(G)$ for star graph.

Theorem 5.11.

Let G be a connected corona graph $K_{1,n} \circ K_1$. Then $\gamma_{rcer}(G)$=2(n+1).

Proof:

Let S be the $\gamma_{rcer} - set$. G has n+1 vertices of degree one and n+1 vertices of degree ≥ 2. By theorem 2.4 and 2.7 n+1 vertices belongs to $S\ and\ n + 1$ support vertex belong to S. Thus $\gamma_{rcer}(G) = 2(n + 1)$.

Theorem 5.12.

Let G be a connected corona graph $P_n \circ K_1$. Then $\gamma_{rcer}(G) = 2n$.

Proof.

Let $\{v_1, v_2, \ldots, v_{2n}\}$ be the vertices of G such that v_i is of degree one if i is odd and v_i is of degree ≥ 2 if i is even. Let S be the $\gamma_{rcer} - set$. By theorem pendant vertices belongs to $\gamma_{rcer} - set$, we have v_i with i is odd also belongs to $\gamma_{rcer} - set$. Now v_i with odd i has only one neighbour in V-S which does not satisfy the restrained certified domination condition. Hence v_i with i even also belongs to $\gamma_{rcer} - set$. Hence $\gamma_{rcer}(G) = 2n$.

Theorem 5.13.

Let G be the friendship graph. Then $\gamma_{rcer}(G) = 1$.

Proof.

G is a graph constructed by joining n copies of cycle graph C_3 with a common vertex and this common vertex becomes a universal vertex. Therefore, $\gamma_{rcer}(G) = 1$.

6 Conclusion

In this paper we have discussed the restrained certified domination number of graphs. Upper and lower bounds were found. Also we have characterised the graphs with $\gamma_{rcer}(G) = 1$. Nordhas gaddum results on several graphs have been studied. Together with this the $\gamma_{rcer}(G)$ values of some corona products are calculated.

7 References

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