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Free Transverse Vibration Analysis of Functionally Graded Beams Coupled to a Double Mass-Spring Element

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Abstract: **-**In this article, we study the vibratory motion of structures in bending within the framework of the Euler-Bernoulli theory, presenting a work that concerns the dynamic response of geometrically linear beams, through a very relevant case rarely seen in the literature. This is a continuous system containing three identical FGM beams coupled by an elastic system consisting of double elastic masses vertically connecting each beam to the other. The main objective of this paper is to determine the natural vibration frequencies of such a structure. The first part of this work focuses on determining the natural linear vibration frequencies of two homogeneous and isotropic beams connected by a double mass-spring system, in order to validate the results with those found in the literature. The second part consists of finding the natural frequencies of the free vibrations of three FGM beams elastically connected by the double mass-spring linear mechanical system. Using the boundary conditions and continuity conditions, we obtain a system of equations that will be solved numerically by the Newton Raphson method, whose eigenvalues are the natural frequencies of the structure under study. The effects of the volume fration index and the positions of the coupling systems and boundary conditions will be presented and discussed.

Keywords: *FGM beam, Euler-Bernoulli, Beam coupled, Bending vibration, Spring-mass.*

1. Introduction

The dynamics of structures is a vast and wide-ranging field, requiring in-depth study to find relevant ideas and information to provide experts in the field with the data they need to follow well-detailed recommendations to get rid of the damage that can occur or an unexpected problem if the structure has not been established according to the recommendations cited as a result of the wrong study.

In various aspects of our daily lives, a multitude of beam structures can be encountered, with elastically connected beam structures being prevalent across diverse fields. These include civil engineering applications like high-rise buildings and bridges, aeronautical uses such as aircraft wing manufacturing, marine engineering for ship construction, automotive part fabrication, medical devices, and even within advanced industries like the production of multi-layer carbon nanotubes.

The present study represents a thorough investigation informed by a comprehensive review of various scholarly articles concerning the dynamic behavior of elastically connected beam structures. Notably, the pioneering work of [1]deserves recognition for its exploration of the natural free vibration frequencies of multiple elastically connected beams using the AMDM method. Similarly, [2]contributed significantly by employing the classical section method to analyze the free vibration frequencies of two beams coupled by a double spring-mass elastic system. Building upon this, [3]replicated [2]'s findings but introduced a mathematical approach that streamlined the system and facilitated the determination of free vibration frequencies. Additionally, [4]utilized an exact stiffness method to accurately determine the linear vibration frequencies of three elastically connected beams. [5]specifically focused on the case of the E-FGM beam, employing an exact method to derive adimensional vibration frequencies. Furthermore, [6] investigated an FGM beam on a Winkler and Pasternak type foundation, utilizing the power law and variable separation method to identify natural adimensional vibration frequencies. In a separate context, [7]examined the FGM beam within the framework of Rayleigh theory, while [8]extended the analysis to encompass both Euler-Bernoulli theory and Timoshenko theory. Outassafte, O. [9]delved into the detection of cracks in circular arches by analyzing variations in natural frequencies alongside employing the firefly hybrid algorithm. Similarly, El Khouddar, Y. [10]investigated the impact of hygro-thermal effects on the nonlinear free and forced vibrations of piezoelectric functional gradient beams, considering beams with an arbitrary number of concentrated masses. Additionally, Outassafte, O. [11] examined both linear and geometrically nonlinear free in-plane vibration characteristics of circular arches with damages. Finally, El Khouddar, Y.E. [11]conducted a comprehensive analysis of nonlinear forced vibrations in piezoelectric functionally graded beams operating in thermal environments. These cited studies serve as pivotal prior research in the field of vibration analysis pertaining to elastically connected beams, laying the groundwork for our own investigation.Finally, it is worth noting that the methodology presented in this study can be extended to other cases previously investigated in the literature. For instance, it can be applied to tapered beams [12], where the cross-sectional dimensions vary along the beam's length, or even to other structural configurations such as arches[13], [14]. By adapting the analytical framework and computational techniques outlined in this study, researchers can explore the dynamic behavior of these diverse structural systems, offering broader insights into their vibrational characteristics and enhancing the applicability of the analytical approach across different engineering contexts.

2. Objectives

In this study, we investigate the scenario involving three Functionally Graded Material (FGM) beams interconnected by two elastic mechanical systems, each comprising identical double spring-mass configurations linking one beam to another. Initially, we validate existing literature findings regarding the natural frequencies of free vibration for two isotropic beams coupled via a double spring-mass system. Subsequently, we delve into our primary objective of determining the natural frequencies of free vibration for the aforementioned configuration involving three FGM beams interconnected by the double spring-mass linear elastic system. Our analysis considers the material properties characteristic of FGM beams. Additionally, we generate plots illustrating the vibration modes and examine the impact of various parameters on the vibration frequencies. Through these investigations, we aim to deepen understanding of the vibrational characteristics and behavior of such interconnected FGM beam systems.

3. Methods

Problem formulation

The problem considered is shown in Fig. 1, and consists of three identical FGM beams made of ceramic and another material. This type of beam has' specific properties and is characterized by a material distribution law given in the rest of this work. The three beams are connected by a double spring-mass linear elastic mechanical system such that each beam is linked to the other by this double spring-mass system, with the stiffness constants

K_1 and K_2 and the masses M_1 and M_2 being the same. Each FGM beam in Fig. 1 has length L, cross-section A, width b and thickness h, clamped in all its ends.

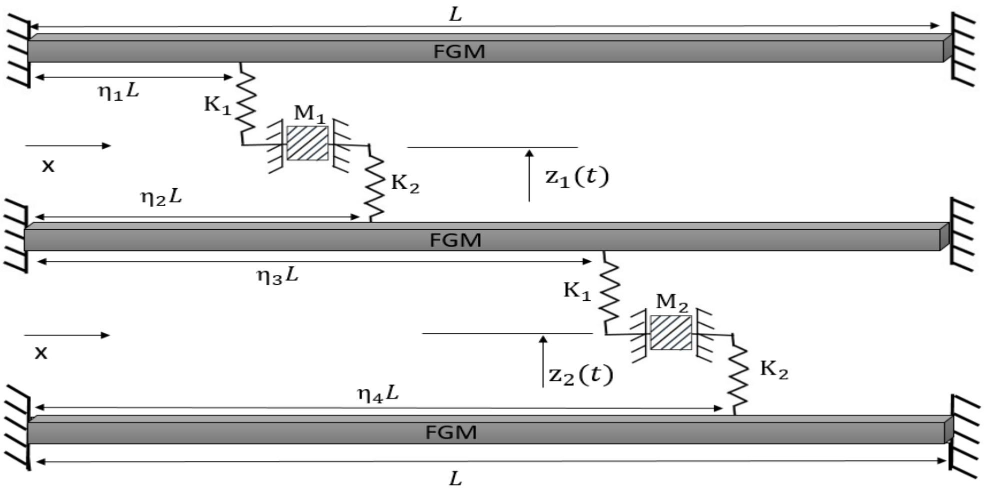

Fig.1. Three FGM beams connected by two double spring-mass spring systems

FGM beam theory

We consider an FGM beam with a rectangular cross-section of length L, width b and thickness h as shown in the following figure:

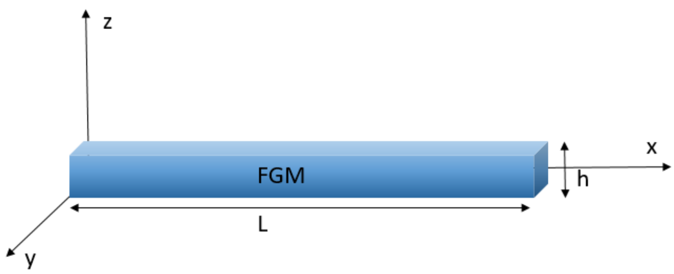

Fig.2. Rectangular cross-section FGM beam

FGM beams are manufactured from two components placed one on top of the other, ceramic and metal, the percentage of which in relation to the entire beam varies according to requirements. FGM beams are considered to have a Young's modulus and density that vary continuously along the thickness axis, while the fish coefficient is considered to be constant. All the effective properties of this type of beam are therefore given by:

$P= P_c V_c+P_m V_m$(1)

Where P_c, P_m, V_c and V_m are the ceramic and material properties of the FGM beam and their volume fractions.

And we also have the relationship that translates the distribution of the volume of ceramic and material in relation to the volume of the beam expressed by:

$V_c+V_m=1$ (2)

FGM beams are known by a ceramic distribution in the beam known by the distribution law its mass in the beam which can take different mathematical forms for this work we use the power law given by the expression:

$V_c = \left(\frac{z}{h} + \frac{1}{2}\right)^d$ (3)

With z the dimension and h the thickness of the beam and d a power factor varying between 0 and +∞, if d=0 i.e. the beam is entirely ceramic.

The expressions for Young's modulus, density and poison coefficient are given by the following formulas:

Young's modulus of the FGM beam is: $E(z)=(E_c-E_m)V_c+E_m$. (4)

The density of the FGM beam is : $\rho(z)=(\rho_c-\rho_m)V_c+\rho_m$ (5)

The Poisson's ratio of the FGM beam is : $\nu(z)=(\nu_c-\nu_m)V_c+\nu_m$ (6)

And the expression for the density of the FGM beam per unit length is given by :

$$I_0 = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \rho(z)\, dA \quad (7)$$

The equation governing the transverse displacement of the FGM beam, we work according to the classical Euler-Bernoulli theory, still within the framework of linear elasticity. The displacement field at a point M of the FGM beam within the framework of the Euler-Bernoulli theory is given by the following form [5,6]:

$$U(M,t) = \begin{cases} w(x,z) = w(x,t) \\ u(x,z,t) = u(x,t) - z\frac{\partial w(x,t)}{\partial x} \end{cases} (8)$$

Where u(x,t) and w(x,t) are respectively the longitudinal and transverse displacements in the mid-plane of the beam, and t is time.

We also have the expressions for the normal strain and normal stress undergone by the beam given by :

$$\varepsilon_x = \frac{\partial u(x,t)}{\partial x} - z\frac{\partial^2 w}{\partial x^2} \quad (9)$$

$$\sigma_x = E(z)\varepsilon_x = E(z)\left(\frac{\partial u(x,t)}{\partial x} - z\frac{\partial^2 w(x,t)}{\partial x^2}\right) \quad (10)$$

This allows us to write the expressions concerning the resultant of the internal forces, i.e. the normal force, the bending moment and the transverse shear force, given by the following formulas:

$$N_x = \int_A \sigma_x dA = \int_A E(z)\left(\frac{\partial u(x,t)}{\partial x} - z\frac{\partial^2 w(x,t)}{\partial x^2}\right) dA = A_1\frac{\partial u(x,t)}{\partial x} - B_1\frac{\partial^2 w(x,t)}{\partial x^2} \quad (11)$$

$$M_x = \int_A \sigma_x z dA = \int_A E(z)z\left(\frac{\partial u(x,t)}{\partial x} - z\frac{\partial^2 w(x,t)}{\partial x^2}\right) dA = B_1\frac{\partial u(x,t)}{\partial x} - D_1\frac{\partial^2 w(x,t)}{\partial x^2} \quad (12)$$

$$Q_x = \frac{\partial M}{\partial x} = \frac{\partial}{\partial x}\left(B_1\frac{\partial u(x,t)}{\partial x} - D_1\frac{\partial^2 w(x,t)}{\partial x^2}\right) = B_1\frac{\partial^2 u(x,t)}{\partial x^2} - D_1\frac{\partial^3 w(x,t)}{\partial x^3} \quad (13)$$

Where: $\{A_1; B_1; D_1\} = \{\int_A (1; (z-z_0); (z-z_0)^2) E(z)dz\}$ (14)

Such that z_0 is the straight line defining the neutral axis of the FGM beam given by the formula :

$$z_0 = \frac{\int_{-h/2}^{h/2} zE(z)dz}{\int_{-h/2}^{h/2} E(z)dz} (15)$$

Considering an element of length dx of the FGM beam and applying to it the law of dynamics and the equilibrium of moments, taking into account the expressions given previously and after some simplifications we find the differential equation of free vibration of an FGM beam verified by the transverse displacement within the framework of the Euler-Bernoulli theory, which gives the following equation:

$$\chi\frac{\partial^4 w(x,t)}{\partial x^4} + I_0\frac{w(x,t)}{\partial t^2} = 0 (16)$$

Where: $\chi = (D_1 - \frac{B_1^2}{A_1})$ (17)

Equations of motion and solutions

We now have fundamental relations that enable us to write the mathematical equations corresponding to the problem posed in Figure 1. The method of resolution used in this work for the search for natural frequencies of vibration is the tracer method as shown in Fig. 1. We use the method of separation of variables to solve the differential equation (16), which allows us to admit a general solution in the form:

$$\mathrm{w_{ij}(x,t) = W_{ij}(x)exp(i\omega t)} \quad (18)$$

where $\mathrm{W_{ij}(x)}$ is the transverse displacement of each section of three beams and ω is the temporal frequency of vibration of the entire system in s^{-1}. By replacing the latter expression in equations (16) and (19) we find the differential equation verified by W(x) for each section, which is:

For the upper FGM beam, the dynamic equations for each section are given by :

$$\chi\frac{\partial^4 \mathrm{W_{1j}(x)}}{\partial \mathrm{x}^4} - I_0\omega^2\mathrm{W_{1j}(x)} = 0 \qquad ;\ (j=1,2) \qquad (19)$$

For the FGM beam in the middle, the dynamic equations for each section are given by :

$$\chi\frac{\partial^4 \mathrm{W_{2j}(x,t)}}{\partial \mathrm{x}^4} - I_0\omega^2\mathrm{W_{2j}(x)} = 0 \qquad ;\ (j=1,2,3) \qquad (20)$$

For the lower FGM beam, the dynamic equations for each section are given by :

$$\chi\frac{\partial^4 \mathrm{W_{3j}(x,t)}}{\partial \mathrm{x}^4} - I_0\omega^2\mathrm{W_{3j}(x)} = 0 \qquad ;\ (j=1,2) \qquad (22)$$

From equations (19), (20) and (21) we write the dimensionless differential equations for each beam as follows, where $\mathrm{W(x) = hW^*(x^*)}$ and $\mathrm{x = Lx^*}$with h is thikness of beam :

$$\frac{\partial^4 W_{1j}^*(x^*)}{\partial {x^*}^4} - \bar{\beta}^4 W_{1j}^*(x^*) = 0 \quad ;\ (j=1,2) \qquad (23)$$

$$\frac{\partial^4 W_{2j}^*(x^*)}{\partial {x^*}^4} - \bar{\beta}^4 W_{2j}^*(x^*) = 0 \ ;\ (j=1,2,3) \qquad (24)$$

$$\frac{\partial^4 W_{3j}^*(\mathrm{x})}{\partial {x^*}^4} - \bar{\beta}^4 W_{3j}^*(x^*) = 0 \quad ;\ (j=1,2) \qquad (25)$$

Where $\bar{\beta}^4 = \frac{L^4 I_0 \omega^2}{\chi}$; $\bar{\beta}$ is the dimensionless wave number (26)

The differential equations (23), (24) and (25) express the vibratory behavior of the FGM beam, and their solutions can be written as follows:

$W_{1j}^*(x^*) = C_{1j}\sin\big(\bar{\beta}_n(x^* - \eta_1)\big) + C_{2j}\,cos\big(\bar{\beta}_n(x^* - \eta_1)\big) + C_{3j}\,sinh\big(\bar{\beta}_n(x^* - \eta_1)\big) + C_{4j}cosh\big(\bar{\beta}_n(x^* - \eta_1)\big)$; (j=1,2).

$W_{2j}^*(x^*) = C_{5j}\sin\big(\bar{\beta}_n(x^* - \eta_k)\big) + C_{6j}\,cos\big(\bar{\beta}_n(x^* - \eta_k)\big) + C_{7j}sinh\big(\bar{\beta}_n(x^* - \eta_k)\big) + C_{8j}cosh\big(\bar{\beta}_n(x^* - \eta_k)\big)$; (j=1,2) and (k=2,3).

$W_{3j}^*(x^*) = C_{9j}\sin\big(\bar{\beta}_n(x^* - \eta_4)\big) + C_{10j}cos\big(\bar{\beta}_n(x^* - \eta_4)\big) + C_{11j}sinh\big(\bar{\beta}_n(x^* - \eta_4)\big) + C_{12j}cosh\big(\bar{\beta}_n(x^* - \eta_4)\big)$; (j=1,2).

The previous solutions present twenty-four unknowns, these are the integration constants, to which we add the vertical displacements z_1 and z_2 respectively of the masses M_1 and M_2, which are identical to the double spring-mass systems. We therefore have twenty-six unknowns in total, which are determined by the boundary conditions and continuity conditions, in fact we write down all the conditions of the system illustrated in figure 1 to completely solve the problem and find the natural vibration frequencies.

$$W_{j1}^{*}(0)=0\,;\ W_{j1}^{*'}(0)=0\,;\qquad (j=1,2,3)$$

$$W_{12}^{*}(1)=0\,;\ W_{12}^{*'}(1)=0\,;\ W_{23}^{*}(1)=0\,;$$

$$W_{23}^{*'}(1)=0\,;W_{32}^{*}(1)=0\,;\ W_{32}^{*'}(1)=0\,;$$

$$W_{j1}^{*}(\eta_j)-W_{j2}^{*}(\eta_j)=0\,;\ \ W_{j1}^{*'}(\eta_j)-\ W_{j2}^{*'}(\eta_j)=0;$$

$$W_{j1}^{*''}(\eta_j)-\ W_{j2}^{*''}(\eta_j)=0\,;(j=1,2)$$

$$W_{22}^{*}(\eta_3)-W_{32}^{*}(\eta_3)=0\,;\ \ W_{22}^{*'}(\eta_3)-\ W_{32}^{*'}(\eta_3)=0;$$

$$W_{22}^{*''}(\eta_3)-\ W_{23}^{*''}(\eta_3)=0\,;(27)$$

$$W_{31}^{*}(\eta_4)-W_{32}^{*}(\eta_4)=0\,;\ \ W_{31}^{*'}(\eta_4)-\ W_{32}^{*'}(\eta_4)=0;$$

$$W_{31}^{*''}(\eta_4)-\ W_{32}^{*''}(\eta_4)=0\,;$$

$$[W_{11}^{*'''}(\eta_1)-\ W_{12}^{*'''}(\eta_1)]-\alpha_k[W_{11}^{*}(\eta_1)-z_1^{*}]=0\,;$$

$$[W_{21}^{*'''}(\eta_2)-\ W_{22}^{*'''}(\eta_2)]+\alpha_k[z_1^{*}-W_{22}^{*}(\eta_2)]=0\,;$$

$$[W_{22}^{*'''}(\eta_1)-\ W_{23}^{*'''}(\eta_1)]-\alpha_k[W_{22}^{*}(\eta_3)-z_2^{*}]=0\,;$$

$$[W_{31}^{*'''}(\eta_3)-\ W_{32}^{*'''}(\eta_3)]+\alpha_k[z_2^{*}-W_{32}^{*}(\eta_4)]=0\,;$$

Where: $\alpha_k=\alpha_{k1}=\alpha_{k2}$, because: K_1=K_2=K, with : $\alpha_K=KL^3/EI_0$. (28)

The system of equations obtained by (27) can be written in the following matrix form:

$[R]\{C_{ij}\}=\{0\}$ (29)

The non-trivial solution of (28) allows us to find the natural vibration frequencies by solving the nonlinear equation given by:

det(R)=0 (30)

Using MatLab software, we solve the nonlinear equation (29) following the Newthon-Raphson algorithm to find the dimensionless natural frequencies $\bar{\beta}_n$.

4. Results and discussion

In this section, results are given in the form of examples, beginning with a comparison with results obtained from the literature, followed by results relating to the example shown in figure 1 and a study of the effect of some parameters.

Comparative study

We present in this work at the beginning a comparative study with references [2] and [3] which treated the case of two isotropic beams of same length and same section clamped in the two sides of the left and they are free on two sides of the right and supporting each respectively two masses M_1 and M_2 and they are linked by a system of double spring-mass such as η_1=η_2=0.5 and $\alpha_{k1}=\alpha_{k2}=1000$ and $\alpha_{m1}=\alpha_{m2}=2$ and $\alpha_m=1$.[2]
The Table 1 shows that the first eight values of natural frequency obtained in this article.

Table 1. The first 8 non-dimensional frequency parameters of the system consisting by two beams coupled by double spring-mass.

Mode No.	Dimensionless frequencies		
	Present	Ref. [2]	Ref. [3]
1	1.07026267	1.07026286	1.07026290
2	1.55878245	1.55878983	1.55878981
3	3.30958283	3.30959152	3.30959149
4	6.33425058	6.33425479	6.33425478
5	6.81982112	6.81983036	6.81983042
6	7.46937434	7.46938324	7.46938323
7	7.92595032	7.92595594	7.92595591
8	10.66672884	10.66673582	10.66673583

it is clear that the results obtained in present by using the method of dividing the domain into sections are identical to those already found in references [2] and [3].

Numerical examples

The results concerning the case of three FGM beams coupled by a double spring-mass system as illustrated in figure 1 are given in the following tables, considering the following values of the physical quantities used in this work. The metal is Aluminium (Al) and the ceramic is Alumina (Al_2O_3). [5,6]

$$E_m = 70 \times 10^9 Pa\,; E_c = 380 \times 10^9 Pa\,;\ \rho_m = 2702\,Kg/m^3\,;\ \rho_c = 3960\,Kg/m^3\ \ ;\ \nu_m = \nu_c = 0{,}3$$

The dimensionless natural vibration frequencies ϖ of the system shown in Figure 1 are presented in Table 2 and are determined using its two relationships :

$$\omega = (\beta)^2 \sqrt{\frac{I_0}{\chi}} = \left(\frac{\beta}{L}\right)^2 \sqrt{\frac{I_0}{\chi}} \quad \text{and} \quad \varpi = \omega \frac{L^2}{h} \sqrt{\frac{\rho_m}{E_m}}$$

With β is the wavenumber in m^{-1} determined by solving the nonlinear equation (29) and ϖ the natural vibration frequency of the FGM beam.

The results we are about to present are all considered for positions $\eta_1 = \eta_2 = 1/3\ et\ \eta_3 = \eta_4 = 2/3$for the two double springs-mass.

In the second table we give the natural vibration frequencies for the structure like in Figure 1 for three isotropic beams elastically connected.

Table.2.The ten non-dimensional frequencies $\overline{\beta}$ of the system shown in Figure 1 for the following values of the dimensionless quantities η_1=η_2=1/3 η_3=η_4=2/3 and $\alpha_{k1} = \alpha_{k2} = 1000$.

Mode No.	Dimensionless frequencies $\overline{\beta}$
	present
1	3.863917808814821
2	4.633618680454015

3	6.251420340544633
4	6.401255999098443
5	8.001052208261319
6	9.025631484016790
7	9.123146475862988
8	9.552382882374978
9	11.069919128156249
10	14.330354590471110

By comparing with the results of the previous table which includes only two coupled beams, the vibration frequencies obtained for three coupled beams are higher than those of table 1.

We present in Table 3 the dimensionless natural frequencies ϖ for a single FGM beam clamped in both ends using the data previously for different values of d.

Mode No. result	Dimensionless frequencies for ϖ				
	Al_2O_3	d=1	d=2	d=10	Al
1 present	12.4299	9.5683	8.7315	8.0675	6.4993
[6]	12.43	9.569	8.732	8.068	6.459
2 present	34.2624	26.3745	24.0679	22.2377	17.9150
[6]	34.264	26.376	24.069	22.239	17.803
3 present	67.1763	51.7110	47.1885	43.6001	35.0154
[6]	67.172	51.707	47.185	43.597	34.902

The results obtained in the present are the same as those obtained in the reference [6], and it is noted that the natural frequencies decrease as d increases and achieve a minimum value when the beam is made only of metal,so that we know what percentage we need to build an FGM beam according to our needs and requirements.

We have in the table 4 the result obtained for searching the dimensionless frequencies of the figure 1 for the values of position $\eta_1 = \eta_2 = 1/3$ and $\eta_3 = \eta_4 = 2/3$.

Table.4. The non-dimensional frequencies of the system shown in Fig. 1 for different values of geometric form factor d.

Mode No.	Dimensionless frequencies for ϖ			
	d=0	d=1	d=10	d=100
1	8.29474413	6.38513306	5.38362422	4.55305258
2	11.92855443	9.18236971	7.74211398	6.54768063
3	21.71224033	16.71366125	14.09212159	11.91802547

4	22.76552270	17.52445757	14.77574442	12.49618074
5	35.56652568	27.37842124	23.08411277	19.52275548
6	45.25873690	34.83929734	29.37474961	24.84288912

As before, the natural vibration frequencies of FGM beams decrease as the geometrical factor d increases, i.e. as the metal percentage becomes greater than the ceramic one.

The table below gives the dimensionless vibration frequencies of the system shown in Figure 1 for the value d=2 for different values of position of the elastic system.

Table.5. Values ofdimensionless frequencies $\bar{\omega}$ for different values of position of system double spring-mass.

Mode No.	Dimensionless frequencies for $\bar{\omega}$			
	$\eta_1 = \eta_2 = 0.25$ $\eta_3 = \eta_4 = 0.50$	$\eta_1 = \eta_2 = 0.25$ $\eta_3 = \eta_4 = 0.75$	$\eta_1 = \eta_2 = 0.5$ $\eta_3 = \eta_4 = 0.5$	$\eta_1 = \eta_2 = 1/3$ $\eta_3 = \eta_4 = 2/3$
1	5.63841241	6.74620948	5.12768316	5.82670600
2	8.97566240	8.47723221	11.311417997	8.37930363
3	14.15840596	12.64060873	21.49534961	15.25192808
4	20.59739538	14.06965940	23.29228514	15.99181428
5	24.06915365	18.93941921	24.06723773	24.98397601
6	24.56025047	30.10618833	27.07404103	31.79234337

Table 5 gives some ideas on the best positions for the double-spring mass system according to design requirements.

From the table, the best positions for low frequencies are $\eta_1 = \eta_2 = \eta_3 = \eta_4 = 0.5$, and the best positions for high frequencies are $\eta_1 = \eta_2 = 1/3 \; et \; \eta_3 = \eta_4 = 2/3$.

The table below gives six first dimensional vibration frequencies of the system in figure 1 for different boundary conditions and d=5 and $\eta_1 = \eta_2 = \eta_3 = \eta_4 = 0.5$.

Table.6. Values ofdimensionless frequencies $\bar{\omega}$ for different boundary conditions

Mode No.	Dimensionless frequencies $\bar{\omega}$			
	CCC-CCC	SSS-SSS	CCC-FFF	SSS-FFF
1	4.88226804	2.37927226	1.13808926	3.12762800
2	10.77004423	9.18542684	3.24221563	3.28984961
3	20.46656451	14.66843081	3.51343387	4.45232598
4	22.17749724	16.08369693	5.57681099	9.57625871
5	22.91535996	18.93196007	10.63112016	15.56159199
6	25.77825518	20.49629926	20.09252800	17.14833961

The previous table gives the variations of the natural frequencies for an FGM beam for different boundary conditions, it can be seen from these results that the CCC-CCC conditions which will be sensitive to high frequencies and the CCC-FFF structureswhich will be sensitive to low frequencies.

The figure 3 shows the vibration modes for the first six values of the natural frequencies normalized to the problem posed in figure 1 for the following positions $\eta_1 = \eta_2 = 1/3$ et $\eta_3 = \eta_4 = 2/3$ for double spring-mass systems.

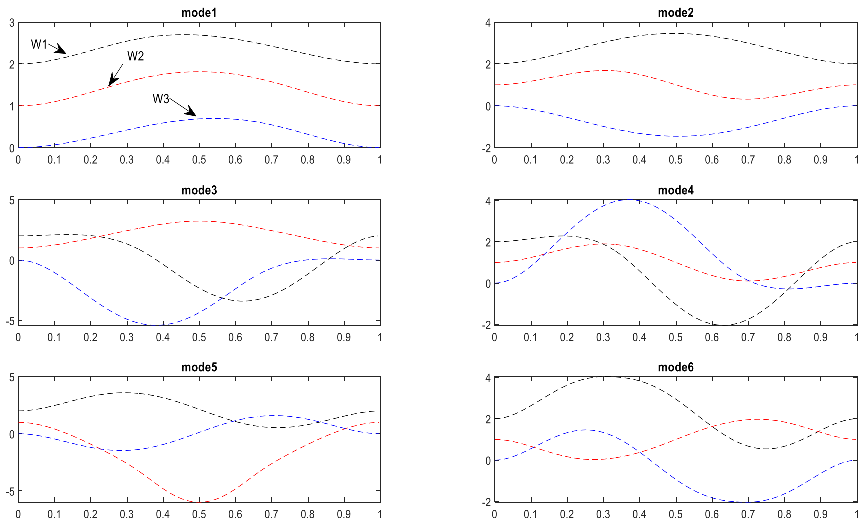

Fig.3. The first six modes of vibrations of system illustrated in figure 1 with $\eta_1 = \eta_2 = 1/3$ et $\eta_1 = \eta_2 = 2/3$

we have in figure 3 the first six modes of vibration of three FGM beams coupled by the mass-double spring system, which present the different ways of vibrating according to the resonance frequency, it is clear that these modes are in agreement with the boundary conditions.

The figure 4 shows the variations of the first four dimensionless natural frequencies $\overline{\omega}$ of the problem of figure 1 for different values of the geometrical factor d for the same position of the coupling elastic systems $\eta_1 = \eta_2 = 1/3$ et $\eta_1 = \eta_2 = 2/3$.

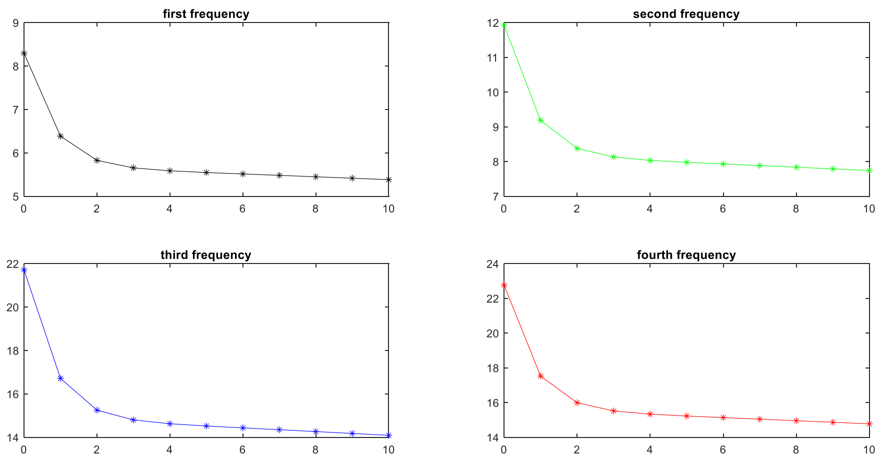

Fig.4. The variations of first four natural frequencies for different values of geometric factor

The four curves shown in Figure 4 show that the natural vibration frequencies for an FGM beam decrease when the geometric factor increases, i.e. when the beam becomes richer in metal.

5. Conclusion

In this study, we have dealt with a case that is much in demand in civil engineering and very rare in the literature, namely the free vibration within the framework of Euler-Bernoulli theory of a structure of three FGM beams coupled by a double spring-mass, we have examined this case by investigating their natural dimensionless vibration frequencies through the application of the section method and taking into account the properties of the materials with which the FGM beam is constructed, considering the case where its properties vary continuously along the thickness axis based on the power law, All these data are taken into consideration and included in the equation governing the FGM beam's vibratory dynamics, and taking into account the effect of coupling and their addition, especially in the conditions we have also studied the influence of several parameters on the adimensional natural frequency parameters, as well as the effect of boundary conditions and the tracing of vibration modes.

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