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Semi Centralizing Pair of Automorphisms of Prime Rings

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Abstract:

The concept of semi centralizing automorphism of rings is generalized as Semi - centralizing pair of automorphisms of rings and more general results are obtained.

1. Introduction:

Let T be an automorphism of ring. It is called commuting automorphism of R if x T(x)= T(x) x , ∀ x ∈ R and it is called a semi-commuting automorphism if either x T(x)= T(x) x (or) x T(x)= -T(x) x , for each x ∈ R . In[1] L.O. Chung and J.Luh have proved that a prime ring R of characteristic ≠ 2,3 possessing a non – trivial semi – commuting automorphism is necessarily a commutative integral domain. In[5] A.Kaya and C. Koc have defined semi – centralizing automorphism of a ring and proved that every semi – centralizing automorphism of a prime ring is commuting.

In this paper we generalize the concept of semi-centralizing automorphism of a ring as semi – centralizing pair of automorphism of a ring and more general results areproved.

2. Preliminary:

In this section we shall recollect some known definitions and results foreasy reference.

Definition 2.1

Let T be an automorphism of a ring R. T is called

i) a commuting automorphism if x T(x) = T(x) x , ∀ x ∈ R and
ii) an anti commuting automorphism if x T(x) = -T(x) x , ∀ x ∈ R .
iii) a semi commuting automorphism if either x T(x) = T(x) x (or) x T(x) = -T(x) x ∀ x ∈ R .
iv) a centralizing automorphism if x T(x) – T(x) x ∈ Z , ∀ x ∈ R .
v) an anti - centralizing automorphism if x T(x) + T(x) x ∈ Z , ∀ x ∈ R .
vi) a semi – centralizing automorphism if either x T(x) – T(x) x ∈ Z(or)x T(x) + T(x) x ∈ Z , ∀ x ∈ R .

Definition 2.2 [5]

Let T be an automorphism of R , we defineR + = { x ∈ R / x T(x) –T(x) x ∈ Z } and R - = { x ∈ R / x T(x) + T(x) x ∈ Z }**Lemma 2.3** [5]
Let R be any ring and T be a semi centralizing automorphism of R.
If x,y ∈ R + {resp in R - }
then x + y ∈ R + (resp in R -) if and only if x – y ∈ R +(resp in R -) .

Lemma 2.4 [5]
Let R be a prime ring and if $y^n = 0$, for all y ∉ R + where n>1 is a fixed integer, then $y^{n-1} = 0$.

Lemma 2.5 [4]
Let R be prime ring with non-trivial centralizing pair of automorphisms S and T such that S≠T. Then R is a commutative integral domain.

3. Main Results :

Definition 3.1
Let S and T be two non – trivial automorphisms of a ring. They are called ,a) a commuting pair of automorphism if S(x) T(x) = T(x) S(x) , ∀ x ∈ R.
b) an anti – commuting pair of automorphisms if S(x) T(x) = - T(x) S(x) , ∀ x ∈ R.c) a semi – commuting pair of automorphism if either S(x) T(x) = T(x) S(x) (or)
S(x) T(x) = - T(x) S(x) , ∀ x ∈ R.
d) a centralizing pair of automorphism if S(x) T(x) – T(x) S(x) ∈ Z, ∀ x ∈ R.
e) an anti – centralizing pair of automorphisms if S(x) T(x) + T(x) S(x) ∈ Z, ∀ x ∈ R.f) a semi – centralizing pair of automorphism if either S(x) T(x) – T(x) S(x) ∈ Z (or)
S(x) T(x) + T(x) S(x) ∈ Z for all x ∈ R .

Definition 3.2
Let R be any ring and S and T be two maps from R into R. We define,R + = { x ∈ R / S(x) T(x) – T(x) S(x) ∈ Z } and ,
R - = { x ∈ R / S(x) T(x) + T(x) S(x) ∈ Z }.

Remark 3.3
If S and T are semi centralizing automorphisms of R.Then R = R+ ∪ R - .

Lemma 3.4
Let R be any ring and S and T be non – trivial automorphisms of R.
If x,y ∈ R - { resp in R + } then x + y ∈ R - (resp in R +) if and only if x – y ∈ R -
(resp in R +) .

Proof :

Let x, y ∈ R - .

Then S(x) T(x) + T(x) S(x) ∈ Z ⟶ 1
S(y) T(y) + T(y) S(y) ∈ Z ⟶ 2
x + y ∈ R- iff S(x +y) T(x+y) + T(x+y) S(x+y) ∈ Z iff (S(x) +S(y)) (T(x)+T(y)) + (T(x)+T(y)) (S(x)+S(y)) ∈ Z

iff S(x) T(x) + S(x) T(y)+ S(y) T(x) + S(y)T(y)
T(x) S(x) + T(x) S(y)+ T(y) S(x) +T(y) S(y) ∈ Z
Using (1) and (2) , we get
x + y ∈ R- iff S(x) T(y) + S(y) T(x) + T(x) S(y) + T(y) S(x) ∈ Z .
iff -S(x) T(y) - S(y) T(x) - T(x) S(y) - T(y) S(x) ∈ Z .
iff S(x) T(x) – S(x) T(y) – S(y) T(x) + S(y) T(y) + T(x) S(x) – T(x) S(y)- T(y) S(x) +T(y) S(y) ∈ Z . iff S(x) (T(x) – T(y)) – S(y) (T(x) – T(y)) + T(x) (S(x) – S(y)) – T(y) (S(x) – S(y))∈Z
iff (S(x) – S(y)) (T(x) – T(y)) + (T(x) – T(y)) (S(x) -S(y)) ∈ Z , ∀ x, y ∈ R.
iff S(x - y) T(x -y) + T(x – y) S(x – y) ∈ Z , ∀ x, y ∈ R.iff x -y ∈ R - .
Repeating the above argument it can be proved that x + y ∈ R +
iff x – y ∈ R + .

Remark 3.5
Taking S = I ,the identity automorphism of R , we get lemma 1[5]

Lemma 3.6
Let R be a prime ring and S and T be semi centralizing pair ofautomorphisms of R.
If y ∉ R + , then $S(y^2)\ T(y^2) = 0$.

Proof :

<u>Case i)</u> Char R ≠ 2.
Since S and T are semi centralizing automorphisms of R , we haveR = R+ ∪ R - .
If y∉ R + , then y ∈ R - .
So, S(y) T(y) + T(y) S(y) ∈ Z ⟶ 1
Hence [S(y) T(y) + T(y) S(y) , S(y)] = 0
i.e , [S(y) T(y) , S(y)] + [T(y) S(y) , S(y)] = 0
S(y) [T(y) , S(y)] + [S(y), S(y)] T(y) + T(y) [S(y),S(y)] + [T(y),S(y)] S(y) = 0i.e , S(y) [T(y),S(y)] + [T(y), S(y)] S(y) = 0
S(y) (T(y) S(y) – S(y) T(y)) + (T(y) S(y) – S(y) T(y)) S(y) = 0
S(y) |(T(y) S(y) – $S(y^2)$ T(y)| + |T(y) $S(y^2)$ – S(y) T(y) S(y))| = 0$S(y^2)$ T(y) – T(y) $S(y^2)$ = 0
i.e , [$S(y^2)$, T(y)]=0 ,∀ y ∉ R + ⟶ 2
Also [S(y) T(y) + T(y) S(y), T(y)] = 0
i.e , [S(y) T(y) , T(y)] + [T(y) S(y) T(y)] = 0

$S(y)\,[T(y) , T(y)] + [S(y) , T(y)]\, T(y) + T(y)\, [S(y)\, T(y)] + [T(y) , T(y)]\, S(y) = 0$
$(S(y)\, T(y) – T(y)\, S(y))\, T(y) + T(y)\, (S(y)\, T(y) – T(y)\, S(y)) = 0.$
i.e , $S(y)\, T(y^2) – T(y)\, S(y)\, T(y) + T(y)\, S(y)\, T(y) – T(y^2)\, S(y) = 0$.i.e , $T(y^2)\, S(y) – S(y)\, T(y^2) = 0$.

i.e , $[T(y^2),S(y)] = 0 ,\forall\ y \notin R +$ ⟶ 3
Now, $[S(y^2 + y),T(y^2 + y)] = [S(y^2) + S(y),T(y^2) + T(y)]$
$= [S(y^2),T(y^2)] + [S(y^2),T(y)] + [S(y),T(y^2)] + [S(y), T(y)]$
$= S(y)[S(y),T(y^2)] + [S(y),T(y^2)]\, S(y) + [S(y),T(y)]$ $[S(y^2 + y),T(y^2+y)] = [S(y),T(y)]$ 4
Similarly, ⟶
$[S(y^2 - y),T(y^2 - y)] = [S(y),T(y)]$ ⟶ 5
Using (4) and (5), gives
$[S(y^2 + y),T(y^2+y)] = [S(y^2 - y),T(y^2 - y)]$
$= [S(y),T(y)] ,\forall\ y \notin R +$ ⟶ 6
Now $y \notin R + \Rightarrow [S(y),T(y)] \notin Z$.

Hence, $[S(y^2 + y), T(y^2 + y)] = [S(y^2 – y),T(y^2 – 2)] \notin Z$.
So neither $y^2 + y$ nor $y^2 – y$ belongs to $R +$ .So $y^2 + y \in R –$ and $y^2 - y \in R –$.
Since $y \in R\_$, it is clear that $ny \in R –$ for all possible integer n.Now, $(y^2 + y) + (y^2 – y) = 2y \in R –$,
So by lemma 3.4
$(y^2 + y) + (y^2 – y) = 2\, y^2 \in R –$.
Hence $y^2 \in R –$.
Hence $S(y^2)\, T(y^2) + T(y^2)\, S(y^2) \in Z$ ⟶ 7
Now, $S(y^2)\, T(y^2) = S(y)\, (S(y)\, T(y^2))$
$= S(y)\, (T(y^2)\, S(y))$ (Using (3))
$= (S(y)\, (T(y^2))\, S(y)$
$= (T(y^2)\, S(y))\, S(y)$ (Using (3))
$S(y^2)\, T(y^2) = T(y^2)\, S(y^2)$ ⟶ 8
From (7) and (8) we get,
$2\, S(y^2)\, T(y^2) = 2\, T(y^2)\, S(y^2) \in Z$
Hence $S(y^2)\, T(y^2) = T(y^2)\, S(y^2) \in Z$, $\forall\ y \notin R +$ ⟶ 9
Since $y^2 + y \in R+$ using(2) we get,$[S(y^2 + y)^2 ,T(y^2 + y)] = 0$
$[S\, (y^4 + 2y^3 + y^2),T(y^2 + y)] = 0$
$[S(y^4),T(y^2)] + [S(y^4),T(y)] + 2\, [S(y^3),T(y^2)] + 2\, [S(y^3),T(y)] + [S(y^2),T(y^2)]$
$+ [S(y^2),T(y)] = 0$.
Since $y^2 \notin R+$,by (2) $[S(y^4),T(y^2)] = 0$.
Now, $[S(y^4),T(y)] = S(y^2)\, [S(y^2),T(y)] + [S(y^2), T(y)]\, S(y^2) = 0$.
Also ,$[S(y^2),T(y^2)] = S(y^2)\, [S(y),T(y^2)] + [S(y^2),T(y^2)]\, S(y)$.
$= \{S(y)[S(y),T(y2)] + [S(y),T(y^2)]\, S(y)\}\, S(y)$ (using (3))
$= 0$.
Also,$[S(y^2),T(y^2)] = S(y)\, [S(y),T(y^2)] + [S(y),T(y)]S(y)$
$= 0$
Hence, $2\, [S(y^3),T(y)] = 0$.

Since, Char R ≠ 2 , we get$[S(y^3),T(y)] = 0$
i.e, $S(y^2)[S(y),T(y)] + [S(y^2),T(y)] S(y) = 0$i.e, $S(y^2) [S(y),T(y)] = 0$, ∀ y ∉ R+ .
∴ $T(y^2) S(y^2)[S(y),T(y)] = 0$, ∀ y ∉R+ .
By (9), $T(y^2) S(y^2) \in Z$, Since R is prime and $[S(y),T(y)] \neq 0$, we have $T(y^2) S(y^2) = 0$.i.e, $S(y^2) T(y^2) = T(y^2) S(y^2) = 0$, ∀ y ∉ R+ .
Case (ii) :
Char (R) = 2 . Then x = -x ∀ x∈ R.
So every semi centralizing pair of automorphisms are centralizing pair of automorphisms. So byTheorem 2.5 [4] they are commuting pair of automorphisms .
So, $[S(y^2),T(y^2)] = 0$
i.e, $S(y^2) T(y^2) – T(y^2) S(y^2) = 0$
i.e, $S(y^2) T(y^2) = T(y^2) S(y^2) = - T(y^2) S(y^2)$
Now, $2 S(y^2) T(y^2) = S(y^2) T(y^2) + S(y^2) T(y^2)$.
$= S(y^2) T(y^2) – T(y^2) S(y^2)$.
= 0.
Since Char R = 2, we get $S(y^2) T(y^2) = 0$,∀ y ∉ R+ .
Hence the proof.

Remark 3.8

Taking S = R, the identity automorphism of R, we get lemma 2[5].

Theorem 3.9

Let R be a prime ring and S and T be a semi centralizing pair ofautomorphisms of R. Then S and T are commuting pair of automorphisms.

Proof :
Let x ∈ R and y ∈ R+ . Consider the element $x y^2+y$,since S and T are semi centralizing pair of automorphisms of R. We have
$c = S(x y^2+ y^2) +T (x y^2+ y^2) \pm T(x y^2+y2) S(x y^2+ y^2) \in Z$.
i.e, $c = S(x y^2)T(x y^2) + S(x y^2)T(y^2) + S(y^2)T(x y^2) + S(y^2) T(y^2) \pm T(x y^2)S(x y^2) +T(x y^2)S(y^2)+T(y^2)S(x y^2)+T(y^2)S(y^2) \in Z$.
Using lemma 3.6 we get
$c = S(x y^2)T(x y^2) + S(x)S(y^2)T(y^2)+S(y^2)T(x)T(y^2) \pm T(x y^2)S(x y^2)+T(x)T(y^2)S(y^2) + T(y^2)S(x)S(y^2) \in Z$.
Again using lemma 3.6, we get
$c = S(x)S(y^2)T(x)T(y^2) + S(y^2)T(x)T(y^2) \pm T(x)T(y^2)S(x)S(y^2) + T(y^2)S(x) S(y^2) \in Z$.
$c T(y^2) = S(x)S(y^2)T(x)T(y^2)T(y^2) + S(y^2)T(x)T(y^2)T(y^2) \pm T(x)T(y^2)S(x)S(y^2)T(y^2) + T(y^2)S(x)S(y^2)T(y^2) \in Z$.
i.e, $c T(y^2) = S(x)S(y^2)T(x)T(y^4) + S(y^2)T(x)T(y^4) \in Z$. ⟶ 1
Similarly considering the element $x y^2 – y^2$, we get

$d\,T(y^2) = S(x)S(y^2)T(x)T(y^4) - S(y^2)T(x)T(y^4) \in Z$, for some $d \in Z$. ⟶ 2

(1) – (2) gives,

$(c - d)\ T(y^2) = 2\ S(y^2)\ T(x)\ T(y^4)$.

$\therefore T(y^2)\ (c - d)\ T(y^2) = 2\ T(y^2)\ S(y^2)\ T(x)\ T(y^4) \in Z$. ⟶ 3

Using lemma 3.6, we get

$(c - d)\ T(y^4) = 0$ (using $c - d \in Z$)

If $c \neq d$, $T(y^4) = 0$ and so $y^4 = 0$ ⟶ 4

If c = d, then from (4) and (2), we get

$S(x)S(y^2)T(x)T(y^4) + S(y^2)T(x)T(y^4) = S(x)S(y^2)T(x)T(y^4) - S(y^2)T(x)T(y^4)$ .i.e, $2\ S(y^2)T(x)T(y^4) = 0$.

Since R is prime either $S(y^2) = 0$ (or) $T(y^4) = 0$.In either case we get $y^4 = 0$.

Thus if $y \notin R+$, then $y^4 = 0$ ⟶ 5

Then using lemma 2.4 repeatedly, we get y = 0, Hence R = R+ and so S and T are commutingpair of automorphism of R.

Corollary 3.10

A prime ring R possessing a non – trivial semi centralizing pair of automorphism is a commutative integral domain .

Corollary 3.11

A prime ring R possessing a non – trivial semi commuting pairof automorphism is a commutative integral domain .

Proof :

Since every semi – commuting pair of automorphisms is asemi centralizing pair, we get the result.

Remark 3.12

Taking S = R , the identity automorphism of R, we get Theorem [5].

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