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Two-Dimensional Landsberg Space with a Generalized (A, B)-Metric

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Abstract: - The present mathematical manipulation is designed to study a Finsler space with a generalized (α, β)-metric $F(\alpha, \beta) = \alpha + \beta + \frac{\beta^{t+1}}{\alpha^t}$ satisfying few conditions. We have obtained a condition for a Finsler space with a generalized (α, β)-metric to be a Berwald space. Further, we have proved that if a two-dimensional Finsler space with a generalized (α, β) metric F is a Landsberg space then it is a Berwald space.

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Key words: *Finsler space, Berwald space, (α, β)-metric, main scalar, Landsberg space, Cartan connections.*

1. Introduction

Many geometers have studied real Landsberg spaces, in particular real Berwald spaces, over the years. In 1926, L. Berwald [8] proposed a particular class of Finsler spaces, which was named after him in 1964. If the local coefficients of the Berwald connection depend only on position coordinates, a real Finsler space is known as a Berwald space. If the covariant derivative $C_{hij|k}$ of the C-tensor $C_{hij} = \dot{\partial}_h \dot{\partial}_i \dot{\partial}_j \frac{F^2}{4}$ satisfies $C_{hij|k}(x, y)y^k = 0$ in the Cartan connection $C\Gamma$, a Finsler space is termed as Landsberg space. $C_{hij|k} = 0$ characterizes a Berwald space. Berwald spaces are very interesting and essential since the connection is linear, and there are several examples of a Berwald space. However, no real example of a Landsberg space that isn't a Berwald space has yet been found. A Finsler space is essentially a Berwald space if it is a Landsberg space and satisfy some additional conditions [7]. A general Finsler space is a Landsberg space in the two-dimensional case if and only if its main scalar $I(x, y)$ satisfies $I_{|i}y^i = 0$ [20]. Recently, several authors [12-15, 21] are discussed the main scalar and Berwald spaces.

In 2003, Young Lee [22] obtained the two-dimensional Landsberg space with a special (α, β) metric and demonstrated that metric F^2 is Landsberg space and then it is a Berwald space. Shanker et al. [19] have proved the two-dimensional Finsler space with first approximate Matsumoto metric to be Landsberg space and then it is Berwald space. Shanker et al. [20], obtained the conditions for two-dimensional Finsler space F^2 with (α, β)-metric to be Landsberg space and then it is a Berwald space in two-dimension. Recently, Predeep Kumar et al. [11] have obatined the two-dimensional Landsberg and Berwald spaces of Finslerian space with special (α, β)-metric. Also, [1, 2, 3, 5, 16, 17] are investigated the Finsler space with (α, β)-metric.

The main aim of the present study is to find a two-dimensional Landsberg space with a generalized (α, β)-metric $F(\alpha, \beta) = \alpha + \beta + \frac{\beta^{t+1}}{\alpha^t}$ satisfying few conditions. Initially we have obtained the conditions for a Finsler space with a generalized (α, β)-metric to be berwald space. Further, with the aforementioned metric, we derive the difference vector and the main scalar of F^2. Finally, we have demonstrated the criteria for a two-dimensional Finsler space F^2 to be a Landsberg space and we prove that if F^2 is a Landsberg space with the given metric, then it is also a Berwald space. Firstly, we gave an introduction for Landsberg and Berwald space in section one. In section two, we go through some basic preliminaries of Berwald connections. In section three, we have derived the criteria for a Finsler space with a (α,β)-metric to be Berwald space. Finally, in section four we have discussed two dimensional Landsberg space with generalized (α,β)-metric.

2. Prerequisites

Let $F^n=(M^n,F(\alpha,\beta))$ be a n-dimensional Finsler space with (α,β)-metric and $R^n=(M^n,\alpha)$ the associated Riemannian space, where $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ be a Riemannian metric and $\beta=b_i(x)y^i$ be a differential 1-form defined on M. Since the metric tensor a_{ij} is invertible, we put $a^{ij}=\left(a_{ij}\right)^{-1}$.

The Riemannian metric α is not supposed to be positive-definite and we shall restrict our discussions to a domain of (x,y) where β does not vanish. The co-variant differentiation in the Levi-Civita connection $\left(\gamma^i_{jk}(x)\right)$ of R^n is denoted by the semi-colon. Let's look at the symbols for later use

i. $b^2=a^{rs}b_rb_s, b^i=a^{ir}b_r,$
ii. $2s_{ij}=b_{i;j}-b_{j;i}, 2r_{ij}=b_{i;j}+b_{j;i},$
iii. $s^i_j=a^{ir}s_{rj}, r^i_j=a^{ir}r_{rj}, r_i=b_rr^r_i, s_i=b_rs^r_i,$
iv. $L_\alpha=\frac{\partial L}{\partial\alpha}, L_{\alpha\alpha}=\frac{\partial^2L}{\partial\alpha^2}, L_\beta=\frac{\partial L}{\partial\beta}, L_{\beta\beta}=\frac{\partial^2L}{\partial\beta^2}.$

The Berwald connection $B\Gamma=\left(G^i_{jk},G^i_j,0\right)$ of F^n plays a significant role in the current paper. Denote by B^i_{jk} the difference tensor of Matsumoto [18] of G^i_{jk} from $\left(\gamma^i_{jk}\right)$

$$G^i_{jk}(x,y)=B^i_{jk}(x,y)+\gamma^i_{jk}(x,y) \tag{1}$$

With the subscript 0 and the contracting by y^i, we have

$$G^i_j=\gamma^i_{0j}+B^i_j,\ 2G^i=\gamma^i_{00}+2B^i \tag{2}$$

and then $B^i_j=\dot\partial_jB^i$ and $B^i_{jk}=\dot\partial_kB^i_j$. According to Matsumoto [18], the Berwald-connection $B\Gamma$ of a Finsler space with (α,β)-metric $F(\alpha,\beta)$ is given by (1) and (2), where B^i_{jk} are the components of a $(1,2)$-type Finsler tensor that is defined by

$$F_\alpha B^k_{ji}y^iy_k=\alpha F_\beta\left(b_{j;i}-B^k_{ji}b_k\right)y^j \tag{3}$$

According to Matsumoto [18], $B^i(x;y)$ is called the difference vector, if

$$\beta^2F_\alpha+\alpha\gamma^2F_{\alpha\alpha}\neq0$$

where $\gamma^2=b^2\alpha^2-\beta^2$. Hence, B^i can be written as

$$B^i=\frac{C}{\alpha}y^i+\frac{\alpha F_\beta}{F_\alpha}s^i_0-\frac{\alpha F_{\alpha\alpha}}{F\alpha}D\left(\frac{1}{\alpha}y^i-\frac{\alpha}{\beta}b^i\right) \tag{4}$$

were

$$C=\left(\frac{\beta F_\beta}{L}\right)D,\ D=\frac{\alpha\beta\left(r_{00}F_\alpha-2\alpha s_0F_\beta\right)}{2(\beta^2F_\alpha+\alpha\gamma^2F_{\alpha\alpha})}$$

Further, by means of Hashiguchi, Hojo and Matsumoto [10], we have

$$\begin{aligned}&\beta_{|i}y^i=r_{00}-2b_rB^r,\\&\alpha_{|i}=-\frac{F_\beta}{F_\alpha}\beta_{|i},\\&b^2_{|i}y^i=2(r_0+s_0),\\&\gamma^2_{|i}y^i=2(r_0+s_0)\alpha^2-2\left(\frac{F_\beta}{F_\alpha}b^2\alpha+\beta\right)(r_{00}-2b_rB^r).\end{aligned} \tag{5}$$

Lemma 2.1 [6]. *If* $\alpha^2 \equiv 0(mod\beta)$ *(i.e,* $a_{ij}y^iy^j$ *contains* $b_i(x)y^i$ *as a factor) then the dimension* n *is equal to 2 and* b^2 *vanishes. In this case we have 1 -form* $\delta = d_i(x)y^i$ *satisfying* $\alpha^2 = \beta\delta$ *and* $d_ib^i = 2$.

Lemma 2.2 [10]. *We consider the two-dimensional case*

i. *If* $b^2 = 0$ *then there exists* $\delta = d_i(x)y^i$ *such that* $\alpha^2 = \beta\delta$ *and* $d_ib^i = 2$,

ii. *If* $b^2 \neq 0$ *then there exist a sign* $\epsilon = \pm 1$ *and* $\delta = d_i(x)y^i$ *such that* $\alpha^2 = \frac{\beta^2}{b^2} + \epsilon\delta^2$ *and* $d_ib^i = 0$.

If there are two functions $f(x)$ and $g(x)$ satisfying $f\alpha^2 + g\beta^2 = 0$ then $f = g = 0$ is obvious, because $f \neq 0$ implies a contradiction $\alpha^2 = \frac{-g}{f}\beta^2$. Throughout the paper, we shall say "homogeneous polynomial (s) in (y^i) of degree r" as $hp(r)$ for brevity. Thus γ_{00}^i are $hp(2)$.

The (α,β)-metric is a class of Finsler metric in Finsler geometry. Let $F = \alpha\phi(b^2, s); s = \frac{\beta}{\alpha}$ is defined as a general (α,β)-metric. where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ be a Riemannian metric, $\beta = b_i(x)y^i$ be a differential one-form defined on manifold M and $\phi = \phi(s)$ is a C^∞ positive function on $(-b_0, b_0)$ satisfying

$$(b^2 - s^2)\phi''(s) - s\phi'(s) + \phi(s) > 0,\ |s| \leqslant b < b_0, \tag{6}$$

where $b := \| \beta(x) \|_\alpha$. It is known that $F = \alpha\phi(s)$ is a Finsler metric if and only if $\| \beta(x) \|_\alpha < b_0$ for any $x \in M$ [9].

In this study we have consider a generalized (α,β)-metric $F(\alpha,\beta) = \alpha + \beta + \frac{\beta^{t+1}}{\alpha^t}$ defined on manifold M, which is a generalized form of first approximate Matsumoto metric $F(\alpha,\beta) = \alpha + \beta + \frac{\beta^2}{\alpha}$. And the aforementioned metric can be written as $F = \alpha\phi(s)$ where $\phi(s) = 1 + s + s^{t+1}$. We know that $\phi(s) > 0$, then (6) becomes

$$1 + (t^2 + t)b^2s^{t-1} - (t^2 + 2t)s^{t+1} > 0,\ |s| \leqslant b < b_0$$

Let $s = b$, we get $b < \left(\frac{1}{t}\right)^{\left(\frac{1}{1+t}\right)}, t > 0, \forall b < b_0$.

If $b \to b_0$, then $b_0 < \left(\frac{1}{t}\right)^{\left(\frac{1}{1+t}\right)}$.

If $t = 1$, then $b_0 < 1$. So, $\| \beta(x) \|_\alpha < \left(\frac{1}{t}\right)^{\left(\frac{1}{1+t}\right)}$.

Therefore, $F(\alpha,\beta) = \alpha + \beta + \frac{\beta^{t+1}}{\alpha^t}$ is a Finsler metric.

3. Berwald Space

In this section, we have obtained the conditions for a Finsler space F^n with generalized (α,β)-metric to be a Berwald space. Let $F^n = (M^n, F(\alpha,\beta))$ be a n-dimensional Finsler space with a generalized (α,β)-metric given by

$$F(\alpha,\beta) = \alpha + \beta + \frac{\beta^{t+1}}{\alpha^t} \tag{7}$$

Let us assume $b^2 \neq 0$. Suppose $b^2 = 0$, then by lemma (2.2) we have $\alpha^t = \beta^{t+1}\delta$, so $F(\alpha,\beta) = \alpha + \beta + \delta$ which is Rander's metic. As a result, the presumption that $b^2 \neq 0$ is fair. Then from (7), we have

$$
\begin{aligned}
F_\alpha &= 1 - t\frac{\beta^{t+1}}{\alpha^{t+1}},\\
F_{\alpha\alpha} &= t(t+1)\frac{\beta^{t+1}}{\alpha^{t+2}},\\
F_\beta &= 1 + (t+1)\frac{\beta^{t}}{\alpha^{t}},\\
F_{\beta\beta} &= t(t+1)\frac{\beta^{t-1}}{\alpha^{t}}.
\end{aligned} \tag{8}
$$

Substituting (8) into (3), we get

$$(\alpha^{t+1} - t\beta^{t+1})B_{ji}^k y^i y_k + \alpha(-\alpha^{t+1} - (t+1)\alpha\beta^t)\big(b_{j|i} - B_{ji}^k b_k\big)y^j = 0. \tag{9}$$

Assume that the Finsler space with (α,β)-metric (7) be a Berwald space, i.e., $G_{jk}^i = G_{jk}^i(x)$. Then we have $B_{ji}^k = B_{ji}^k(x)$, so the left-hand side of equation (9) have the form

$$P_1(x,y) + \alpha P_2(x,y) = 0 \tag{10}$$

where P_1 and P_2 are polynomials in (y^i) while α is irrational in (y^i). Hence (9) shows $P_1 = P_2 = 0$.

By our assumption $b^2 \neq 0$ implies $(-\alpha^{t+1} - (t+1)\alpha\beta^t) \neq 0$. Hence, we have

$$B_{ji}^k a_{kh} y^i y^h = 0,\ \big(b_{j|i} - B_{ji}^k b_k\big)y^j = 0. \tag{11}$$

The former yields $B_{ji}^k a_{kh} + B_{hi}^k a_{kj} = 0$, so we have $B_{ji}^k = 0$. Then the latter leads to $b_{j|i} = 0$ directly.

Conversely, suppose $b_{j|i} = 0$, by well-known Okada's axioms, $\big(\gamma_{jk}^i, \gamma_{0j}^i, 0\big)$ becomes the Berwald connection of F^n. Hence F^n is a Berwald space. Thus, we have the following outcome

Theorem 3.1 *The Finsler space F^n with generalized (α,β)-metric $F(\alpha,\beta) = \alpha + \beta + \frac{\beta^{t+1}}{\alpha^t}$ satisfying $b^2 \neq 0$ is a Berwald space if and only if $b_{j|i} = 0$, and then the Berwald connection is essentially Riemannian $\big(\gamma_{jk}^i, \gamma_{0j}^i, 0\big)$.*

4. Two-dimensional Landsberg Space

Let the Finsler space $F^n = (M^n, F(\alpha,\beta))$ with an generalized (α,β)-metric given by equation (7) be a Landsberg Space. The difference vector B^i of the Finsler space has been first given in [18]. Here, by means of (4) and (8), we have

$$2B^i = \frac{A}{(\alpha^{t+1} - t\beta^{t+1})\Omega}\left(t(t+1)\alpha^2\beta^{t-1}b^i + \frac{By^i}{\alpha^t F}\right) + \frac{2\alpha^2(\alpha^t + (t+1)\beta^t)}{(\alpha^{t+1} - t\beta^{t+1})} s_0^i \tag{12}$$

where

$$
\begin{aligned}
A &= r_{00}(\alpha^{t+1} - t\beta^{t+1}) - 2\alpha^2 s_0(\alpha^t + (t+1)\beta^t),\\
B &= \alpha^{t+1} - (t^2 + 2t)\beta^{t+1} + t(t+1)b^2\alpha^2\beta^{t-1},\\
\Omega &= \alpha^{2t+1} - (t^2 + t + 1)\alpha^t\beta^{t+1} - (t^2 - 1)\alpha^{t+1}\beta^t - 2t(t+1)\beta^{2t+1}.
\end{aligned}
$$

It is trivial that $(\alpha^{t+1} - t\beta^{t+1}) \neq 0$ and $\Omega \neq 0$, because α is irrational in (y^i).

From (12) it follows that

$$r_{00} - 2B^r b_r = \frac{A\left[(\alpha^{t+1} - t\beta^{t+1})^2 + K\right]}{\alpha^t(\alpha^{t+1} - t\beta^{t+1})F\Omega} \tag{13}$$

where

$$K = -(1-t)\alpha^t\beta^{t+2} + t(t+1)b^2(-\alpha^3 + \alpha^{t+2}\beta t - 1)F.$$

Now we deal with the condition for a two-dimensional Finsler space F^2 with (7) to be a Landsberg space. It is known that in the two-dimensional case, a general Finsler space is a Landsberg space, if and only if it's main scalar $I(x,y)$ satisfies $I_{|i}y^i = 0$ ([4, 22]).

The main scalar of F^2 is obtained as follows

$$\epsilon I^2 = \frac{\gamma^2 M^2}{4\alpha^t F\Omega^3} \tag{14}$$

where

$$M = 3\alpha^{2t+1} - a_1\alpha^t\beta^{t+1} - a_2\beta^{2t+1} + a_3 b^2\alpha^{t+2}\beta^{t-1} + a_4 b^2\alpha^2\beta^{2t+1} - a_5\alpha^{t+1}\beta^t + a_6 b^2\alpha^{t+3}\beta^{t-2},$$
$$\Omega = \alpha^{t+1} - (t^2+2t)\beta^{t+1} + t(t+1)b^2\alpha^2\beta^{t-1}.$$

The co-variant differentiation of (14) leads to

$$4\alpha^{2t}F\Omega^4\epsilon I^2_{|i} = M\left(\alpha^t\Omega M\gamma^2_{|i} + 2\alpha^t\Omega\gamma^2 M_{|i} - t\alpha^{t-1}\Omega M\gamma^2\alpha_{|i} - 3\alpha^t M\gamma^2\Omega_{|i}\right) \tag{15}$$

Trasvecting (15) by y^i, we get

$$4\alpha^{2t}F\Omega^4\epsilon I^2_{|i}y^i = M\left(P\gamma^2_{|i}y^i + XM_{|i}y^i - Y\alpha_{|i}y^i - Z\Omega_{|i}y^i\right) \tag{16}$$

where

$$P = 3\alpha^{4t+2} - a_1\alpha^{3t+1}\beta^{t+1} - a_2\alpha^{2t+1}\beta^{2t+1} + a_3 b^2\alpha^{3t+3}\beta^{t-1} + a_4\alpha^{2t+3}\beta^{2t+1} - a_5\alpha^{3t+2}\beta^t + a_6 b^2\alpha^{4t+4}\beta^{t-2} - 3a_7\alpha^{2t+1}\beta^{t+1} + a_8\alpha^{2t}\beta^{2t+2} + a_9\alpha^t\beta^{3t+2} - a_{10}b^2\alpha^{2t+2}\beta^{2t} - a_{11}b^2\alpha^{t+2}\beta^{3t+2} + a_{12}\alpha^{2t+1}\beta^{2t+1} - a_{13}b^2\alpha^{2t+3}\beta^{2t-1} + a_{14}b^2\alpha^{3t+3}\beta^{t-1} - a_{15}b^2\alpha^{2t+2}\beta^{2t} - a_{16}b^2\alpha^{t+2}\beta^{3t} + a_{17}b^2\alpha^{2t+4}\beta^{2t-2} + a_{18}b^4\alpha^{t+4}\beta^{3t} - a_{19}b^2\alpha^{2t+3}\beta^{2t-1} + a_{20}b^2\alpha^{2t+5}\beta^{2t-3},$$

$$X = 2b^2\alpha^{2t+3} - 2\alpha^{2t+1}\beta^2 - 2b^2(t^2+2t)\alpha^{t+2}\beta^{t+1} + 2(t^2+2t)\alpha^t\beta^{t+3} + 2t(t+1)b^4\alpha^{t+4}\beta^{t-1} - 2t(t+1)b^2\alpha^{t+2}\beta^{t+1},$$

$$Y = 3t(b^2\alpha^{4t+3} - \alpha^{4t+1}\beta^2) + ta_1(\alpha^{3t}\beta^{t+3} - b^2\alpha^{3t+2}\beta^{t+1}) + ta_2(\alpha^{2t}\beta^{2t+3} - b^2\alpha^{2t+2}\beta^{2t+1}) + ta_3(b^4\alpha^{3t+4}\beta^{t-1} - b^2\alpha^{3t+2}\beta^{t+1}) + ta_4(b^4\alpha^{3t+4}\beta^{2t+1} - b^2\alpha^{3t+2}\beta^{2t+3}) + ta_5(\alpha^{3t+1}\beta^{t+2} - b^2\alpha^{3t+3}\beta^t) + ta_6(b^4\alpha^{3t+5}\beta^{t-2} - b^2\alpha^{3t+3}\beta^t) + 3ta_7(\alpha^{3t}\beta^{t+3} - b^2\alpha^{3t+2}\beta^{t+1}) + ta_8(b^2\alpha^{2t+1}\beta^{2t+2} - \alpha^{2t-1}\beta^{2t+4}) + ta_9(b^2\alpha^{3t+2}\beta^{t+1} - \alpha^{t-1}\beta^{3t+4}) + ta_{10}(b^2\alpha^{2t+1}\beta^{2t+2} - b^4\alpha^{2t+3}\beta^{2t}) + ta_{11}(b^2\alpha^{t+1}\beta^{3t+4} - b^4\alpha^{t+3}\beta^{3t+2}) + ta_{12}(b^2\alpha^{2t+2}\beta^{2t+1} - \alpha^{2t}\beta^{2t+3}) + ta_{13}(b^2\alpha^{2t+2}\beta^{2t+1} - b^4\alpha^{2t+4}\beta^{2t-1}) + ta_{14}(b^4\alpha^{3t+4}\beta^{t-1} - b^2\alpha^{3t+2}\beta^{t+1}) + ta_{15}(b^2\alpha^{2t+1}\beta^{2t+2} - b^4\alpha^{2t+3}\beta^{2t}) + ta_{16}(b^2\alpha^{t+1}\beta^{3t+2} - b^4\alpha^{t+3}\beta^{3t}) + ta_{17}(b^4\alpha^{2t+5}\beta^{2t-2} - b^2\alpha^{2t+3}\beta^{2t}) + ta_{18}(b^6\alpha^{t+5}\beta^{3t} - b^4\alpha^{t+3}\beta^{3t+2}) + ta_{19}(b^2\alpha^{3t+2}\beta^{2t+1} - b^4\alpha^{3t+4}\beta^{2t-1}) + ta_{20}(b^6\alpha^{2t+6}\beta^{2t-3} - b^4\alpha^{2t+4}\beta^{2t-1}),$$

$$Z = 9(b^2\alpha^{3t+3} - \alpha^{3t+1}\beta^2) + 3a_1(\alpha^{2t}\beta^{t+3} - b^2\alpha^{2t+2}\beta^{t+1}) + 3a_2(\alpha^t\beta^{2t+3} - b^2\alpha^{t+2}\beta^{2t+1}) + 3a_3(b^4\alpha^{2t+4}\beta^{t-1} - b^2\alpha^{2t+2}\beta^{t+1}) + 3a_4(b^4\alpha^{t+4}\beta^{2t+1} - b^2\alpha^{t+2}\beta^{2t+3}) + 3a_5(\alpha^{2t+1}\beta^{t+2} - b^2\alpha^{2t+3}\beta^t) + 3a_6(b^4\alpha^{2t+5}\beta^{t-2} - b^2\alpha^{2t+3}\beta^t).$$

Thus, the equation (16) is rewritten in the form

$$4\alpha^{2t}F\Omega^4\epsilon I^2_{|i}y^i = M\left(P\gamma^2_{|i}y^i + Q\alpha_{|i}y^i + Sb^2_{|i}y^i + R\beta_{|i}y^i\right), \tag{17}$$

where

$$Q=\{Q_1=3t(\alpha^{4t+1}\beta^2-b^2\alpha^{4t+3})-ta_1\alpha^{3t}\beta^{t+3}-ta_2\alpha^{2t}\beta^{2t+3}+ta_3b^2\alpha^{3t+2}\beta^{t+1}+ta_4b^2\alpha^{3t+2}\beta^{2t+3}-ta_5\alpha^{3t+1}\beta^{t+2}+ta_6b^2\alpha^{3t+3}\beta^{t}-3ta_7\alpha^{3t}\beta^{t+3}+ta_8\alpha^{2t-1}\beta^{2t+4}+ta_9\alpha^{t-1}\beta^{3t+4}-ta_{10}b^2\alpha^{2t+1}\beta^{2t+2}-ta_{11}b^2\alpha^{t+1}\beta^{3t+4}+ta_{12}\alpha^{2t}\beta^{2t+3}-ta_{13}b^2\alpha^{2t+2}\beta^{2t+1}+ta_{14}b^2\alpha^{3t+2}\beta^{t+1}-ta_{15}b^2\alpha^{2t+1}\beta^{2t+2}-ta_{16}b^2\alpha^{t+1}\beta^{3t+2}+ta_{16}b^4\alpha^{t+3}\beta^{3t}+ta_{17}b^2\alpha^{2t+3}\beta^{2t}-ta_{18}b^6\alpha^{t+5}\beta^{3t}+ta_{19}b^4\alpha^{3t+4}\beta^{2t-1}+ta_{20}b^4\alpha^{2t+4}\beta^{2t-1}+a_{21}b^2\alpha^{4t+3}+a_{22}\alpha^{3t}\beta^{t+3}+a_{23}b^4\alpha^{3t+4}\beta^{t-1}-2ta_1b^2\alpha^{3t+2}-a_{24}\alpha^{2t-1}\beta^{t+3}-a_{25}b^4\alpha^{2t+3}\beta^{t-1}+a_{26}b^4\alpha^{3t+4}\beta^{t-1}-a_{27}b^4\alpha^{2t+3}\beta^{2t}+a_{28}b^6\alpha^{2t+5}\beta^{2t-2}-a_{29}b^2\alpha^{2t+3}\beta^{2t+3}-a_{30}b^2\alpha^{t+3}\beta^{3t+2}+a_{31}b^2\alpha^{t+5}\beta^{3t}-a_{32}b^2\alpha^{3t+3}\beta^{t}-a_{33}\alpha^{2t}\beta^{2t+3}-a_{34}b^4\alpha^{2t+4}\beta^{2t-1}+a_{35}b^4\alpha^{3t+5}\beta^{t-2}-a_{36}b^4\alpha^{2t+4}\beta^{2t-1}+a_{37}b^6\alpha^{2t+6}\beta^{2t-3}-9(t+1)b^2\alpha^{4t+3}-a_{38}\alpha^{3t}\beta^{t+4}+a_{39}\alpha^{t+1}\beta^{3t+4}-a_{40}b^4\alpha^{3t+4}\beta^{t-1}+a_{41}b^2\alpha^{2t+2}\beta^{2t+3}+a_{42}b^2\alpha^{3t+3}\beta^{t}-a_{43}b^4\alpha^{3t+5}\beta^{t-2}+a_{44}b^4\alpha^{4t+4}\beta^{t-1}-a_{45}b^4\alpha^{3t+3}\beta^{2t}+a_{46}b^4\alpha^{t+3}\beta^{3t}+a_{47}b^4\alpha^{2t+3}\beta^{2t}-a_{48}b^6\alpha^{t+5}\beta^{3t}+a_{48}b^4\alpha^{t+3}\beta^{3t+2}+a_{49}b^4\alpha^{2t+4}\beta^{2t-1}+a_{50}(b^4\alpha^{2t+4}\beta^{2t-1}-b^2\alpha^{2t+6}\beta^{2t-3})\}+\alpha\{Q_2=ta_1b^2\alpha^{3t+1}\beta^{t+1}+ta_2b^2\alpha^{2t+1}\beta^{2t+1}-ta_3b^4\alpha^{3t+1}\beta^{t-1}-ta_4b^4\alpha^{3t+1}\beta^{2t+1}+ta_5b^2\alpha^{3t+2}\beta^{t}-ta_6b^4\alpha^{3t+4}\beta^{t-2}+3ta_7b^2\alpha^{3t+1}\beta^{t+1}-ta_8b^2\alpha^{2t}\beta^{2t+2}-ta_9b^2\alpha^{3t+1}\beta^{t+1}+ta_{10}b^4\alpha^{2t+2}\beta^{2t}+ta_{11}b^4\alpha^{t+2}\beta^{3t+2}-ta_{12}b^2\alpha^{2t+1}\beta^{2t+1}+ta_{13}b^4\alpha^{2t+3}\beta^{2t-1}-ta_{14}b^4\alpha^{3t+3}\beta^{t-1}+ta_{15}b^4\alpha^{2t+2}\beta^{2t}-ta_{17}b^4\alpha^{2t+4}\beta 7^{2t-2}+ta_{18}b^4\alpha^{t+2}\beta^{3t+2}-ta_{19}b^2\alpha^{3t+1}\beta^{2t+1}-ta_{20}b^6\alpha^{2t+5}\beta^{2t-3}-a_{21}\alpha^{4t}\beta^2-a_{22}b^2\alpha^{3t+1}\beta^{t+1}-a_{23}b^2\alpha^{3t+1}\beta^{t+1}+2ta_1\alpha^{3t}\beta^2+a_{24}b^2\alpha^{2t}\beta^{t+1}+a_{25}b^2\alpha^{2t}\beta^{t+1}-a_{26}b^2\alpha^{3t+1}\beta^{t+1}+a_{27}b^2\alpha^{2t}\beta^{2t+2}-a_{28}b^4\alpha^{2t+2}\beta^{2t}+a_{29}b^4\alpha^{2t+3}\beta^{2t+1}+a_{30}\alpha^{t}\beta^{3t+4}-a_{31}b^4\alpha^{t+2}\beta^{3t+2}+a_{32}\alpha^{3t}\beta^{t+2}+a_{33}b^2\alpha^{2t+1}\beta^{2t+1}+a_{34}b^2\alpha^{2t+1}\beta^{2t+1}-a_{35}b^2\alpha^{3t+2}\beta^{t}+a_{36}b^2\alpha^{3t+1}\beta^{2t+1}-a_{37}b^4\alpha^{2t+3}\beta^{2t-1}+9(t+1)\alpha^{4t}\beta^2+a_{38}b^2\alpha^{3t+1}\beta^{t+1}-a_{39}b^2\alpha^{2t+1}\beta^{2t+3}+a_{40}b^2\alpha^{3t+1}\beta^{t+1}-a_{41}b^4\alpha^{2t+3}\beta^{2t+1}-a_{42}\alpha^{3t}\beta^{t+2}+a_{43}b^2\alpha^{3t+2}\beta^{t}-a_{44}b^2\alpha^{4t+1}\beta^{t+1}+a_{45}b^2\alpha^{3t}\beta^{2t+4}-a_{46}b^2\alpha^{t}\beta^{3t+2}-a_{47}b^6\alpha^{2t+4}\beta^{2t-2}-a_{49}b^2\alpha^{2t+2}\beta^{2t+1}\}$$

$$R=a_{51}(\alpha^{3t+1}\beta^{t+2}-b^2\alpha^{3t+3}\beta^{t})+a_{52}(b^2\alpha^{3t+2}\beta^{2t+1}-\alpha^{2t}\beta^{2t+3})+a_{53}(b^2\alpha^{2t+2}\beta^{2t+1}-b^4\alpha^{5t+4}\beta^{2t-1})+a_{54}(\alpha^{2t+1}\beta^{2t+2}-b^2\alpha^{2t+3}\beta^{2t})+a_{55}(b^2\alpha^{t+2}\beta^{3t+1}-\alpha^{t}\beta^{3t+3})+a_{56}(b^2\alpha^{t+2}\beta^{3t+1}-b^4\alpha^{t+4}\beta^{3t-1})+a_{57}(b^4\alpha^{3t+5}\beta^{t-2}-b^2\alpha^{3t+3}\beta^{t})+a_{58}(b^2\alpha^{2t+2}\beta^{2t+1}-b^4\alpha^{2t+4}\beta^{2t-1})+a_{59}(b^4\alpha^{2t+6}\beta^{2t-2}-b^2\alpha^{2t+4}\beta^{2t})+a_{60}(b^4\alpha^{2t+5}\beta^{2t}-b^2\alpha^{2t+3}\beta^{2t+2})+a_{61}(b^2\alpha^{t+2}\beta^{3t+3}-b^4\alpha^{t+4}\beta^{3t+1})+a_{62}(b^6\alpha^{t+6}\beta^{3t-1}-b^4\alpha^{t+4}\beta^{3t+1})+a_{63}(\alpha^{3t+2}\beta^{3}-b^2\alpha^{3t+4}\beta)+a_{64}(b^2\alpha^{2t+3}\beta^{t+2}-\alpha^{2t+1}\beta^{t+4})+a_{65}(b^2\alpha^{2t+3}\beta^{t+2}-b^4\alpha^{2t+5}\beta^{t})+a_{66}(b^4\alpha^{3t+6}\beta^{t-3}-b^2\alpha^{3t+4}\beta^{t-1})+a_{67}(b^2\alpha^{2t+3}\beta^{2t}-b^4\alpha^{2t+5}\beta^{t-2})+a_{68}(b^2\alpha^{2t+7}\beta^{2t-4}-b^4\alpha^{2t+5}\beta^{2t-2})+a_{69}(b^2\alpha^{3t+2}\beta^{t}-\alpha^{3t+1}\beta^{t+2})+a_{70}(\alpha^{2t}\beta^{2t+3}-b^2\alpha^{2t+2}\beta^{2t+1})+a_{71}(\alpha^{t}\beta^{3t+3}-b^2\alpha^{t+2}\beta^{3t+1})+a_{72}(b^4\alpha^{2t+4}\beta^{2t-1}-b^2\alpha^{2t+2}\beta^{2t+1})+a_{73}(b^4\alpha^{t+4}\beta^{3t+1}-b^2\alpha^{t+2}\beta^{3t+3})+a_{74}(\alpha^{2t+1}\beta^{2t+2}-b^2\alpha^{2t+3}\beta^{2t})+a_{75}(b^4\alpha^{2t+5}\beta^{2t-2}-b^2\alpha^{2t+3}\beta^{2t})+a_{76}(b^4\alpha^{4t+5}\beta^{t-2}-b^2\alpha^{4t+3}\beta^{t})+a_{77}(b^2\alpha^{3t+2}\beta^{2t+3}-b^4\alpha^{3t+4}\beta^{2t-1})+a_{78}(b^4\alpha^{t+4}\beta^{3t-1}-b^2\alpha^{t+2}\beta^{3t+1})+a_{79}(b^4\alpha^{2t+4}\beta^{2t-1}-b^6\alpha^{2t+6}\beta^{2t-3})+a_{80}(b^4\alpha^{t+4}\beta^{3t+2}-b^6\alpha^{t+6}\beta^{3t-1})+a_{81}(b^4\alpha^{2t+5}\beta^{2t-2}-b^2\alpha^{2t+4}\beta^{2t})+a_{82}(b^4\alpha^{2t+5}\beta^{2t-2}-b^6\alpha^{2t+7}\beta^{2t-4}).$$

$$S=a_{83}(b^2\alpha^{3t+5}\beta^{t-1}-b^2\alpha^{3t+3}\beta^{t+1})+a_{84}(\alpha^{2t+2}\beta^{2t+2}-b^2\alpha^{2t+4}\beta^{2t})+a_{85}(b^4\alpha^{2t+6}\beta^{2t-2}-b^2\alpha^{2t+4}\beta^{2t})+a_{86}(b^2\alpha^{2t+5}\beta^{2t+1}-\alpha^{2t+4}\beta^{2t+3})+a_{87}(\alpha^{t+2}\beta^{3t+4}-b^2\alpha^{t+4}\beta^{3t+2})+a_{88}(b^4\alpha^{t+6}\beta^{3t}-b^2\alpha^{t+4}\beta^{3t+2})+a_{89}(b^2\alpha^{3t+6}\beta^{t-2}-\alpha^{3t+4}\beta^{t})+a_{90}(\alpha^{2t+3}\beta^{2t+1}-b^2\alpha^{2t+5}\beta^{2t-1})+a_{91}(b^4\alpha^{2t+7}\beta^{2t-3}-b^2\alpha^{t+4}\beta^{3t})+a_{92}(b^2\alpha^{4t+5}\beta^{t-1}-\alpha^{4t+3}\beta^{t+1})+a_{93}(\alpha^{3t+2}\beta^{2t+4}-b^2\alpha^{3t+4}\beta^{2t})+a_{94}(b^2\alpha^{t+4}\beta^{3t}-\alpha^{t+2}\beta^{3t+2})+a_{95}(b^2\alpha^{2t+4}\beta^{2t}-b^4\alpha^{2t+6}\beta^{2t-2})+a_{96}(b^2\alpha^{t+4}\beta^{3t+2}-b^4\alpha^{t+6}\beta^{3t})+a_{97}(b^2\alpha^{2t+5}\beta^{2t-1}-\alpha^{2t+4}\beta^{2t+1})+a_{98}(b^2\alpha^{2t+5}\beta^{2t-1}-b^4\alpha^{2t+7}\beta^{2t-3}),$$

in which

$a_1 = t^3 + 6t^2 + 8t,$	$a_2 = 4t^3 + 12t^2 + 8t,$	$a_3 = t^3 + 3t^2 + 2t,$
$a_4 = 4t^3 + 6t^2 + 2t,$	$a_5 = t^3 + 3t^2 - 4,$	$a_6 = t^3 - t,$
$a_7 = t^2 + 2t,$	$a_8 = t^5 + 8t^4 + 12t^3 + 26t^2$	$a_9 = 4t^5 + 20^4 + 32t^3 + 16$
$a_{10} = t^5 + 5t^4 + 8t^3 + 4t^2,$	$a_{11} = 4t^5 + 14t^4 + 14t^3 + 4$	$a_{13} = t^5 + 2t^4 - t^3 - 2t^2,$
$a_{12} = t^5 + 5t^4 + 6t^3 - 4t^2 - 8$	$a_{14} = 3t^2 + 3t,$	$a_{15} = t^5 + 7t^4 + 14t^3 + 8t^2$
$a_{16} = 4t^5 + 16t^4 + 20t^3 + 8t^2$	$a_{17} = t^5 + 4t^4 + 5t^3 + 2t^2,$	$a_{18} = 4t^5 + 10t^4 + 8t^3 + 2t$
$a_{19} = 3t^5 + 6t^4 + 3t^3 - 4t^2 -$	$a_{20} = t^5 + t^4 - t^3 - t^2,$	$a_{21} = 6(2t + 1),$
$a_{22} = 6(2t + 1)(t^2 + 2t),$	$a_{23} = 6t(t + 1)(2t + 1),$	$a_{24} = 2t(t^2 + 2t)a_1,$
$a_{25} = 2t^2(t + 1)a_1,$	$a_{26} = 2(t + 2)a_3,$	$a_{27} = 2(t + 2)(t^2 + 2t)a_3,$
$a_{28} = 2t(t + 1)(t + 2)a_3,$	$a_{29} = 4a_4,$	$a_{30} = 4(t^2 + 2t)a_4,$
$a_{31} = 4t(t + 1)a_4,$	$a_{32} = 2(t + 1)a_5,$	$a_{33} = 2(t + 1)(t^{+}2t)a_5,$
$a_{34} = 2t(t + 1)^2 a_5,$	$a_{35} = 2(t + 3)a_6,$	$a_{36} = 2(t + 3)(t^{+}2t)a_6,$
$a_{37} = 2t(t + 1)(t + 3)a_6,$	$a_{38} = 3(t + 1)a_1,$	$a_{39} = 3(2t + 1)a_2,$
$a_{40} = 3(t + 1)a_3,$	$a_{41} = 3(t + 1)a_4,$	$a_{42} = 3(t + 1)a_5,$
$a_{43} = 3(t + 1)a_6,$	$a_{44} = 18t(t + 1),$	$a_{45} = 6t(t + 1)a_1,$
$a_{46} = 6t(t + 1)a_2,$	$a_{47} = 6t(t + 1)a_3,$	$a_{48} = 6t(t + 1)a_4,$
$a_{49} = 6t(t + 1)a_5,$	$a_{50} = 6t(t + 1)a_6,$	$a_{51} = 2(t + 1)a_1,$
$a_{52} = 2(t + 1)(t^2 + 2t)a_1,$	$a_{53} = 2t(t + 1)^2 a_1,$	$a_{54} = 2(2t + 1)a_2,$
$a_{55} = 2(2t + 1)(t^2 + 2t)a_2,$	$a_{56} = 4t(2t + 1)(t + 1)a_2,$	$a_{57} = 2(t - 1)a_3,$
$a_{58} = 2(t - 1)(t^2 + 2t)a_3,$	$a_{59} = 2t(t - 1)(t + 1)a_3,$	$a_{60} = 2(2t + 1)a_4,$
$a_{61} = 2(2t + 1)(t^2 + 2t)a_4,$	$a_{62} = 2t(2t + 1)(t + 1)a_4,$	$a_{63} = 2ta_5,$
$a_{64} = 2t(t^2 + 2t)a_5,$	$a_{65} = 2t^2(t + 1)a_5,$	$a_{66} = 2(t - 2)a_6,$
$a_{67} = 2(t - 2)(t^2 + 2t)a_6,$	$a_{68} = 2t(t - 2)(t + 1)a_6,$	$a_{69} = 9(t + 1)(t^2 + 2t),$
$a_{70} = 3(t + 1)(t^2 + 2t)a_1,$	$a_{71} = 3(t + 1)(t^2 + 2t)a_2,$	$a_{72} = 3(t + 1)(t^2 + 2t)a_3,$
$a_{73} = 3(t + 1)(t^2 + 2t)a_4,$	$a_{74} = 3(t + 1)(t^2 + 2t)a_5,$	$a_{75} = 3(t + 1)(t^2 + 2t)a_6,$

$a_{76} = 9t(t^2 - 1),$	$a_{77} = 6t(t^2 - 1)a_1,$	$a_{78} = 6t(t^2 - 1)a_2,$
$a_{79} = 6t(t^2 - 1)a_3,$	$a_{80} = 6t(t^2 - 1)a_4,$	$a_{81} = 6t(t^2 - 1)a_5,$
$a_{82} = 6t(t^2 - 1)a_6,$	$a_{83} = 3a_3,$	$a_{84} = 2(t^2 + 2t)a_3,$
$a_{85} = 2t(t + 1)a_3,$	$a_{86} = 2a_4,$	$a_{87} = 2(t^2 + 2t)a_4,$
$a_{88} = 2t(t + 1)a_4,$	$a_{89} = 2a_6,$	$a_{90} = 2(t^2 + 2t)a_6,$
$a_{91} = 2t(t + 1)a_6,$	$a_{92} = 9t(t + 1),$	$a_{93} = 3t(t + 1)a_1,$
$a_{94} = 3t(t + 1)a_2,$	$a_{95} = 3t(t + 1)a_3,$	$a_{96} = 3t(t + 1)a_4,$
$a_{97} = 3t(t + 1)a_5,$	$a_{98} = 3t(t + 1)a_6.$	

Consequently, the two dimensional Finsler space F^2 with (7) is a Landsberg space, if and only if

$$P\gamma^2_{|i}y^i + Q\alpha_{|i}y^i + Sb^2_{|i}y^i + R\beta_{|i}y^i = 0 \tag{18}$$

when $M \neq 0$. If $M = 0$, then $b^2 = 0$, namely, it is a contradiction.

By means of equation (5), the above equation is written as

$$\begin{aligned} &2(\alpha^{t+1} - t\beta^{t+1})(\alpha^2 P + S)(r_0 + s_0) + \{(\alpha^{t+1} - t\beta^{t+1})R - Q\alpha(\alpha^t + (t+1)\beta^t) \\ &\quad -[\alpha^2 b^2(\alpha^t + (t+1)\beta^t) + \beta(\alpha^{t+1} - t\beta^{t+1})]P\}(r_{00} - 2b_r B^r) = 0. \end{aligned} \tag{19}$$

Substituting the values of P, Q, R, S and $(r_{00} - 2b_r B^r)$ in (19) and separating the rational and irrational terms with respect to (y^i), we get

$$\{\alpha^{2t+2}E_1 + \alpha^{2t}\beta F_1 r_{00} + 2\alpha^{2t+2}G_1 s_0\} + \alpha\{\alpha^{2t+2}E_2 + F_2 r_{00} + 2\alpha^{2t}\beta G_2 s_0\} = 0 \tag{20}$$

where

$E_1 = 2\{\alpha^{2t+2} + t(t+1)b^2(\alpha^{t+3}\beta^{t-1} + \alpha^{t+2}\beta^t) + (t-t^3)b^2\alpha^2\beta^{2t} - t\beta^{t+2} - t^2(t+1)b^2(\beta^{3t+1}\alpha + \alpha^{1-t}\beta^{3t})\}(\alpha^{t+1} - t\beta^{t+1})\{3\alpha^{3t+3} + a_3b^2\alpha^{2t+4}\beta^{t-1} - a_5\alpha^{2t+3}\beta^t + a_6b^2\alpha^{3t+5}\beta^{t-2} + a_9\alpha\beta^{3t+2} - a_{10}b^2\alpha^{t+3}\beta^{2t} - a_{13}b^2\alpha^{t+4}\beta^{2t-1} + a_{14}b^2\alpha^{2t+4}\beta^{t-1} - a_{15}b^2\alpha^{t+3}\beta^{2t} - a_{16}b^2\alpha^3\beta^{3t} + a_{17}b^2\alpha^{t+5}\beta^{2t-2} + a_{18}b^4\alpha^5\beta^{3t} - a_{19}b^2\alpha^{t+4}\beta^{2t-1} + a_{20}b^2\alpha^{t+6}\beta^{2t-3} + a_{83}\alpha^{t+1}\beta^{2t+3} - a_{84}b^2\alpha^{t+3}\beta^{2t} + a_{85}(b^4\alpha^{t+5}\beta^{2t-2} - b^2\alpha^{t+3}\beta^{2t}) + a_{88}b^4\alpha^5\beta^{3t} + a_{89}(b^2\alpha^{2t+5}\beta^{t-2} - \alpha^{2t+3}\beta^t) - a_{90}b^2\alpha^{t+4}\beta^{2t-1} + a_{91}(b^4\alpha^{t+6}\beta^{2t-3} - b^2\alpha^3\beta^{3t}) + a_{92}b^2\alpha^{3t+4}\beta^{t-1} - a_{93}b^2\alpha^{2t+3}\beta^{2t} + a_{94}b^2\alpha^3\beta^{3t} + a_{95}(b^2\alpha^{t+3}\beta^{2t} - b^4\alpha^{t+5}\beta^{2t-2}) - a_{96}b^4\alpha^5\beta^{3t} - a_{97}b^2\alpha^{t+4}\beta^{2t-1} + a_{98}(b^2\alpha^{t+4}\beta^{2t-1} - b^4\alpha^{t+6}\beta^{2t-3})\}$,

$E_2 = 2\{(1-t^2-3t)\alpha^{2t+1}\beta + \alpha^{3t} - (t^2+2t)\alpha^{2t}\beta^{t+1} + (t^3+t^2-2t)b^2\alpha^t\beta^{2t+1} + t(t^2+2t)(\alpha^{t-1}\beta^{2t+2} + \alpha^{-1}\beta^{3t+1}) - tb^2\alpha^{2t}\beta^{2t} +\}(\alpha^{t+1} - t\beta^{t+1})\{-a_1\alpha^{2t+2}\beta^{t+1} - a_2\alpha^{t+2}\beta^{2t+1} + a_4\alpha^{t+4}\beta^{2t+1} - 3a_7\alpha^{t+2}\beta^{t+1} + a_8\alpha^{t-1}\beta^{2t+2} - a_{11}b^2\alpha^3\beta^{3t+2} + a_{12}\alpha^{t+2}\beta^{2t+1} - a_{83}b^2\alpha^{2t+2}\beta^{t+1} + a_{84}\alpha^{t+1}\beta^{2t+2} + a_{86}(b^2\alpha^{t+4}\beta^{2t+1} - \alpha^{t+3}\beta^{2t+3}) + a_{87}(\alpha\beta^{3t+4} - b^2\alpha^3\beta^{3t+2}) - a_{88}b^2\alpha^3\beta^{3t+2} + a_{90}\alpha^{t+2}\beta^{2t+1} - a_{92}\alpha^{3t+2}\beta^{t+1} + a_{93}\alpha^{2t+1}\beta^{2t+4} - a_{94}\alpha\beta^{3t+2} + a_{96}b^2\alpha^3\beta^{3t+2} - a_{97}\alpha^{t+3}\beta^{2t+1} - a_{98}b^4\alpha^{t+6}\beta^{2t-3}\}$,

$F_2 = \{t^2\beta^{2t+2} + (1-t)\alpha^t\beta^{t+3} + \alpha^{3t+1}(t^2+t)b^2(\alpha^{t+2}\beta^{t-1} - \alpha^3 - \alpha^2\beta^{t+1})\}(\alpha^{t+1} - t\beta^{t+1})\{(\alpha^{t+1} - t\beta^{t+1})(a_{51}\alpha^{3t+1}\beta^{t+2} - a_{52}\alpha^{2t}\beta^{2t+3} + a_{53}b^2\alpha^{2t+2}\beta^{2t+1} - a_{54}b^2\alpha^{2t+3}\beta^{2t} - a_{55}\alpha^t\beta^{3t+3} - a_{56}b^4\alpha^{t+4}\beta^{3t-1} + a_{57}b^4\alpha^{3t+5}\beta^{t_2} + a_{58}b^2\alpha^{2t+2}\beta^{2t+1} - a_{59}b^2\alpha^{2t+4}\beta^{2t} - a_{60}b^2\alpha^{2t+3}\beta^{2t+2} + a_{61}(b^2\alpha^{t+2}\beta^{3t+3} - b^4\alpha^{t+4}\beta^{3t+1}) + a_{62}b^6\alpha^{t+6}\beta^{3t-1} - a_{63}b^2\alpha^{3t+4}\beta + a_{64}(b^2\alpha^{2t+3}\beta^{t+2} - \alpha^{2t+1}\beta^{t+4}) + a_{65}b^2\alpha^{2t+3}\beta^{t+2} + a_{67}b^2\alpha^{2t+3}\beta^{2t} + a_{68}(b^2\alpha^{2t+7}\beta^{2t-4} - b^4\alpha^{2t+5}\beta^{2t-2}) - a_{69}\alpha^{3t+1}\beta^{t+2} - a_{70}b^2\alpha^{2t+2}\beta^{2t+1} + a_{71}(\alpha^t\beta^{3t+3} - b^2\alpha^{t+2}\beta^{3t+1}) - a_{72}b^2\alpha^{2t+2}\beta^{2t+1} + a_{73}(b^4\alpha^{t+4}\beta^{3t+1} - b^2\alpha^{t+2}\beta^{3t+3}) + a_{74}\alpha^{2t+1}\beta^{2t+2} - a_{75}b^2\alpha^{2t+3}\beta^{2t} + a_{76}b^2\alpha^{4t+3}\beta^t + a_{77}b^2\alpha^{3t+2}\beta^{2t+3} + a_{78}b^4\alpha^{t+4}\beta^{3t-1} + a_{80}b^4\alpha^{t+4}\beta^{3t+2} - a_{81}b^2\alpha^{2t+4}\beta^{2t} + a_{82}b^4\alpha^{2t+5}\beta^{2t-2}) - (Q)(\alpha^{t+1} + (2t+1)\alpha\beta^t) - \left(b^2\alpha^2(\alpha^t + (t+1)\beta) + \beta(\alpha^{t+1} - t\beta^{t+1})\right)(-a_1\alpha^{2t+2}\beta^{t+1} - a_2\alpha^{t+2}\beta^{2t+1} + a_4\alpha^{t+4}\beta^{2t+1} - 3a_7\alpha^{t+2}\beta^{t+1} + a_8\alpha^{t-1}\beta^{2t+2} - a_{11}b^2\alpha^3\beta^{3t+2} + a_{12}\alpha^{t+2}\beta^{2t+1} + a_{17}b^2\alpha^{2t+4}\beta^{2t-2} + a_{18}b^4\alpha^{t+4}\beta^{3t} - a_{19}b^2\alpha^{2t+3}\beta^{2t-1})\}$,

$G_1 = (\alpha^t + (t+1)\beta^t)\{2t\alpha^2\beta^{t+1} + (t-1)\alpha\beta^{t+2} + t(1+t)(b^2\alpha^2 - b^2\alpha^2\beta^t + \alpha^{4-t}) + \alpha^{t+2}\}\{(-a_{51}b^2\alpha^{3t+3}\beta^t + a_{52}b^2\alpha^{3t+2}\beta^{2t+1} - a_{53}b^4\alpha^{5t+4}\beta^{2t-1} + a_{54}\alpha^{2t+1}\beta^{2t+2} + a_{55}b^2\alpha^{t+2}\beta^{3t+1} - a_{56}b^2\alpha^{t+2}\beta^{3t+1} + a_{57}b^4\alpha^{3t+5}\beta^{t_2} + a_{59}b^4\alpha^{2t+6}\beta^{2t-2} + a_{60}b^4\alpha^{2t+5}\beta^{2t} + a_{62}b^6\alpha^{t+6}\beta^{3t-1} + a_{63}(\alpha^{3t+2}\beta^3 - b^2\alpha^{3t+4}\beta) - a_{65}b^4\alpha^{2t+5}\beta^t + a_{66}(b^4\alpha^{3t+6}\beta^{t-3} - b^2\alpha^{3t+4}\beta^{t-1}) - a_{67}b^4\alpha^{2t+5}\beta^{t-2} + a_{68}(b^2\alpha^{2t+7}\beta^{2t-4} - b^4\alpha^{2t+5}\beta^{2t-2}) + a_{69}b^2\alpha^{3t+2}\beta^t + a_{70}\alpha^{2t}\beta^{2t+3} + a_{72}b^4\alpha^{2t+4}\beta^{2t-1} + a_{74}\alpha^{2t+1}\beta^{2t+2} + a_{75}b^4\alpha^{2t+5}\beta^{2t-2} + a_{76}(b^4\alpha^{4t+5}\beta^{t-2} - b^2\alpha^{4t+3}\beta^t) - a_{77}b^4\alpha^{3t+4}\beta^{2t-1} + a_{78}b^4\alpha^{t+4}\beta^{3t-1} + a_{79}(b^4\alpha^{2t+4}\beta^{2t-1} - b^6\alpha^{2t+6}\beta^{2t-3}) - a_{80}b^4\alpha^{t+6}\beta^{3t-1} + a_{81}(b^4\alpha^{2t+5}\beta^{2t-2} - b^2\alpha^{2t+4}\beta^{2t}) + a_{82}(b^4\alpha^{2t+5}\beta^{2t-2} - b^6\alpha^{2t+7}\beta^{2t-4})\big) - (Q)(\alpha^{t+1} + (2t+1)\alpha\beta^t)\big\} - (b^2\alpha^2(\alpha^t + (t+1)\beta) + \beta(\alpha^{t+1} - t\beta^{t+1}))(3\alpha^{4t+2} + a_3b^2\alpha^{3t+3}\beta^{t-1} - a_5\alpha^{3t+2}\beta^t + a_6b^2\alpha^{4t+4}\beta^{t-2} + a_9\alpha^t\beta^{3t+2} - a_{10}b^2\alpha^{2t+4}\beta^{2t} - a_{13}b^2\alpha^{2t+3}\beta^{2t-1} + a_{14}b^2\alpha^{3t+3}\beta^{t-1} - a_{15}b^2\alpha^{2t+2}\beta^{2t} - a_{16}b^2\alpha^{t+2}\beta^{3t} + a_{20}b^2\alpha^{2t+4}\beta^{2t-3})$.

$G_2 = \{t^2\alpha^{t-1}\beta^{2t+2} + (1-t)\alpha^t\beta^{t+3} + \alpha^{3t+1}(t^2+t)b^2(\alpha^{t+2}\beta^{t-1} - \alpha^3 - \alpha^2\beta^{t+1})\}(\alpha^{t+1} - t\beta^{t+1})\{(\alpha^{t+1} - t\beta^{t+1})(a_{51}\alpha^{3t+1}\beta^{t+2} - a_{52}\alpha^{2t}\beta^{2t+3} + a_{53}b^2\alpha^{2t+2}\beta^{2t+1} - a_{54}b^2\alpha^{2t+3}\beta^{2t} - a_{55}\alpha^t\beta^{3t+3} - a_{56}b^4\alpha^{t+4}\beta^{3t-1} + a_{57}b^4\alpha^{3t+5}\beta^t + a_{58}b^2\alpha^{2t+2}\beta^{2t+1} - a_{59}b^2\alpha^{2t+4}\beta^{2t} - a_{60}b^2\alpha^{2t+3}\beta^{2t+2} + a_{61}(b^2\alpha^{t+2}\beta^{3t+3} - b^4\alpha^{t+4}\beta^{3t+1}) + a_{62}b^6\alpha^{t+6}\beta^{3t-1} - a_{63}b^2\alpha^{3t+4}\beta + a_{64}(b^2\alpha^{2t+3}\beta^{t+2} - \alpha^{2t+1}\beta^{t+4}) + a_{65}b^2\alpha^{2t+3}\beta^{t+2} + a_{67}b^2\alpha^{2t+3}\beta^{2t} + a_{68}(b^2\alpha^{2t+7}\beta^{2t-4} - b^4\alpha^{2t+5}\beta^{2t-2}) - a_{69}\alpha^{3t+1}\beta^{t+2} - a_{70}b^2\alpha^{2t+2}\beta^{2t+1} + a_{71}(\alpha^t\beta^{3t+3} - b^2\alpha^{t+2}\beta^{3t+1}) - a_{72}b^2\alpha^{2t+2}\beta^{2t+1} + a_{73}(b^4\alpha^{t+4}\beta^{3t+1} - b^2\alpha^{t+2}\beta^{3t+3}) + a_{74}\alpha^{2t+1}\beta^{2t+2} - a_{75}b^2\alpha^{2t+3}\beta^{2t} + a_{76}b^2\alpha^{4t+3}\beta^t + a_{77}b^2\alpha^{3t+2}\beta^{2t+3} + a_{78}b^4\alpha^{t+4}\beta^{3t-1} + a_{80}b^4\alpha^{t+4}\beta^{3t+2} - a_{81}b^2\alpha^{2t+4}\beta^{2t} + a_{82}b^4\alpha^{2t+5}\beta^{2t-2}) - (Q)(\alpha^{t+1} + (2t+1)\alpha\beta^t) - \left(b^2\alpha^2(\alpha^t + (t+1)\beta) + \beta(\alpha^{t+1} - t\beta^{t+1})\right)(-a_1\alpha^{2t+2}\beta^{t+1} - a_2\alpha^{t+2}\beta^{2t+1} + a_4\alpha^{t+4}\beta^{2t+1} - 3a_7\alpha^{t+2}\beta^{t+1} + a_8\alpha^{t-1}\beta^{2t+2} - a_{11}b^2\alpha^3\beta^{3t+2} + a_{12}\alpha^{t+2}\beta^{2t+1} + a_{17}b^2\alpha^{2t+4}\beta^{2t-2} + a_{18}b^4\alpha^{t+4}\beta^{3t} - a_{19}b^2\alpha^{2t+3}\beta^{2t-1})\}$,

$$F_1 = \{(t^2+t)(b^2\alpha^2\beta^{t-1} - \alpha^{4-t} + b^2\alpha^{3-t}\beta^{2t-1}) + (1-t)(\alpha\beta^{t+1} + \alpha^{1-t}\beta^{2t+2}) - 2t\alpha^2\beta^t\}(\alpha^{t+1} - t\beta^{t+1})\{(\alpha^{t+1} - t\beta^{t+1})(-a_{51}b^2\alpha^{3t+3}\beta^t + a_{52}b^2\alpha^{3t+2}\beta^{2t+1} - a_{53}b^4\alpha^{5t+4}\beta^{2t-1} + a_{54}\alpha^{2t+1}\beta^{2t+2} + a_{55}b^2\alpha^{t+2}\beta^{3t+1} - a_{56}b^2\alpha^{t+2}\beta^{3t+1} + a_{57}b^4\alpha^{3t+5}\beta^t + a_{59}b^4\alpha^{2t+6}\beta^{2t-2} + a_{60}b^4\alpha^{2t+5}\beta^{2t} + a_{62}b^6\alpha^{t+6}\beta^{3t-1} + a_{63}(\alpha^{3t+2}\beta^3 - b^2\alpha^{3t+4}\beta) - a_{65} - b^4\alpha^{2t+5}\beta^t + a_{66}(b^4\alpha^{3t+6}\beta^{t-3} - b^2\alpha^{3t+4}\beta^{t-1}) - a_{67}b^4\alpha^{2t+5}\beta^{t-2} + a_{68}(b^2\alpha^{2t+7}\beta^{2t-4} - b^4\alpha^{2t+5}\beta^{2t-2}) + a_{69}b^2\alpha^{3t+2}\beta^t + a_{70}\alpha^{2t}\beta^{2t+3} + a_{72}b^4\alpha^{2t+4}\beta^{2t-1} + a_{74}\alpha^{2t+1}\beta^{2t+2} + a_{75}b^4\alpha^{2t+5}\beta^{2t-2} + a_{76}(b^4\alpha^{4t+5}\beta^{t-2} - b^2\alpha^{4t+3}\beta^t) - a_{77}b^4\alpha^{3t+4}\beta^{2t-1} + a_{78}b^4\alpha^{t+4}\beta^{3t-1} + a_{79}(b^4\alpha^{2t+4}\beta^{2t-1} - b^6\alpha^{2t+6}\beta^{2t-3}) - a_{80}b^4\alpha^{t+6}\beta^{3t-1} + a_{81}(b^4\alpha^{2t+5}\beta^{2t-2} - b^2\alpha^{2t+4}\beta^{2t}) + a_{82}(b^4\alpha^{2t+5}\beta^{2t-2} - b^6\alpha^{2t+7}\beta^{2t-4})\big) - (Q)(\alpha^{t+1} + (2t+1)\alpha\beta^t) - \big(b^2\alpha^2(\alpha^t + (t+1)\beta) + \beta(\alpha^{t+1} - t\beta^{t+1})\big)(3\alpha^{4t+2} + a_3b^2\alpha^{3t+3}\beta^{t-1} - a_5\alpha^{3t+2}\beta^t + a_6b^2\alpha^{4t+4}\beta^{t-2} + a_9\alpha^t\beta^{3t+2} - a_{10}b^2\alpha^{2t+4}\beta^{2t} - a_{13}b^2\alpha^{2t+3}\beta^{2t-1} + a_{14}b^2\alpha^{3t+3}\beta^{t-1} - a_{15}b^2\alpha^{2t+2}\beta^{2t} - a_{16}b^2\alpha^{t+2}\beta^{3t} + a_{20}b^2\alpha^{2t+4}\beta^{2t-3})\},$$

The equation (20) yields two equations as follows

$$\alpha^2E_1(r_0+s_0) + \beta F_1 r_{00} + 2\alpha^2 G_1 s_0 = 0, \tag{21}$$

$$\alpha^{2t+2}\beta E_2(r_0+s_0) + F_2 r_{00} + 2\alpha^2 G_2 s_0 = 0. \tag{22}$$

From (22), we get

$$-a_{39}t^2\beta^{6t+6}r_{00} \equiv 0(\mathrm{mod}\,\alpha^2). \tag{23}$$

Therefore, there exists a function $f(x)$ such that $r_{00} = \alpha^2 f(x)$. Thus, we obtain

$$r_{ij} = a_{ij}f(x). \tag{24}$$

Transvection above equation by b^iy^i leads to

$$r_0 = \beta f(x),\ r_j = b_j f(x). \tag{25}$$

Eliminating (r_0+s_0) from (21) and (22) using (23), we get

$$\alpha^2 f(x)(\alpha^{2t}\beta E_2F_1 - E_1F_2) + 2\alpha^2\beta s_0(\alpha^{2t}E_2G_1 - E_1G_2) = 0 \tag{26}$$

From $\alpha^2 \neq 0(\mathrm{mod}\beta)$ it follows that there exists a function $g(x)$ satisfy $s_0 = g(x)\beta$.

Thus (26) reduces to

$$\alpha^{2t}\beta^2(f(x)E_2F_1 + 2g(x)E_1G_1) - (f(x)E_1F_2 + 2\beta^2 g(x)E_1G_2) = 0. \tag{27}$$

Since only the term $-a_{39}a_{83}t^2(f(x)+4g(x))\beta^{12t+10}$ of $(f(x)E_1F_2 + 2\beta^2 g(x)E_1G_2)$ apparently does not contain α^2, we must have $hp(12t+8)V_{12t+8}$ such that $\beta^{12t+10} = \alpha^2 V_{12t+8}$. But it is a contradiction because of $\alpha^2 \neq 0(\mathrm{mod}\beta)$, that is, $(f(x)E_1F_2 + 2\beta^2 g(x)E_1G_2)$ does not contain α^2 as a factor. Hence $(f(x)E_1F_2 + 2\beta^2 g(x)E_1G_2)$ must be zero, which implies $f(x) = g(x) = 0$, this leads to $s_0 = 0$ and $s_i = 0$. From (24), we get $r_{ij} = 0$.

Recapping up, we obtain $r_{ij} = 0$ and $s_i = 0$, that is

$$b_{i|j} + b_{j|i} = 0,\ b^r b_{r|i} = 0. \tag{28}$$

Therefore $b_i(x)$ is the so-called Killing vector field with a constant length.

According to [8], the condition (28) is equivalent to $b_{i|j} = 0$. So, we have

Theorem 4.1 Let two-dimensional Finsler space F^2 with a generalized (α,β)-metric $F(\alpha,\beta) = \alpha + \beta + \frac{\beta^{t+1}}{\alpha^t}$ satisfying $b^2 \neq 0$. If F^2 is a Landsberg space, then F^2 is a Berwald space.

5. Conclusion

The purpose of this research is to find a Landsberg space in a two-dimensional Finsler space F^2 with a generalised (α,β)-metric $F(\alpha,\beta)=\alpha+\beta+\frac{\beta^{t+1}}{\alpha^t}$ that meets certain conditions. First, we have determined the criterion for a Finsler space with a generalized (α,β)-metric to be a Berwald space. Further, we have calculated the difference vector and the main scalar of F^2 using the aforementioned metric. Finally, we have demonstrated that if the Finsler space F^2 with the (α,β)-metric is a Landsberg space under certain conditions, it becomes a Berwald space.

6. Competing Interests

The authors declare that they have no competing interests.

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