The Connected Detour Edge Semi Toll Number of a Graph

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Abstract
A detour edge semi toll set of a connected graph G is called a connected detour edge semi toll set of G if the induced subgraph $G[S]$ is connected. The minimum cardinality of a connected detour semi toll set of G is the connected detour semi toll number and is denoted by $cdn_{est}(G)$. Any connected detour semi Euler set of cardinality $cdn_{est}(G)$ is called a cdn_{est}-set of G. Some general properties satisfied by this concept are studied . Some standard graphs are determined.It is shown that for every pair of a and b of integers with $2 \leq a < b$, there exists a connected graph G such that $dn_{est}(G) = a$ and $cdn_{est}(G) = b$.

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detour number,

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1. Introduction

By a graph $G = (V, E)$, we mean a finite , undirected connected graph without loops or multiple edges. The order and size of G are denoted by n and m respectively. For basic graph theoretic terminology , we refer to [2]. Two vertices u and v are said to be *adjacent* if uv is an edge of G. Two edges of G are said to be adjacent if they have a common vertex. A *walk* is defined as a finite length alternating sequence of vertices and edges. The total number of edges covered in a walk is called as length of the walk. It is a trail in which neither vertices nor edges are repeated i.e. if we traverse a graph such that we do not repeat a vertex and nor we repeat an edge.Any connected graph is called as an *Euler Graph* if and only if all its vertices are of even degree.If there exists a walk in the connected graph that starts and ends at the same vertex and visits every edge of the graph exactly once with or without repeating the vertices, then such a walk is called as an *Euler circuit*.

The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is thelength of a shortest u-v path in G. An $u-v$ path of length $d(u, v)$ is called an $u-v$ *geodesic*.The detour distance $D(u, v)$ between two vertices u and v in a connected graph G from u to v is defined as the length of a longest u-v path in G. An u-v path of length $D(u, v)$ is called an u-v *detour*. A vertex x is said to lie on an u-v detour P if x is a vertex of P including the vertices u and v. A detour set of G is a set $S \subseteq V(G)$ such that every vertex of G is contained in a detour joining some pair of vertices in S. The *detour numberdn*(G) of G is the minimum order of a detour set and any detour set of order $dn(G)$ is called *minimum detour set* of G or a *dn-set* of G. These conceptwere studied in [3-7,9].

A tolled walk T between u and v in G in a sequence at vertices of the form $T: u, w_1, w_2, \dots, v$ where $k \geq 1$ which enjoys the following three conditions.

- $w_i w_{i+1} \in E(G), \forall\ i$
- $uw_i \in E(G)$, iff $i = 1$.
- $vw_i \in E(G)$, iff $i = k$.

$T[u, v]$ = set of vertices lying in the uv tolled walk including u and v.

For $S \subseteq V(G)$, the tolled closure of G is $T[S] = \bigcup_{u,v \in S} T[u, v]$.A set $S \subseteq V(G)$ is called a tolled set if $T[S] = V[G]$. The minimum cardinality of a tolled set is called the *tolled number* of G and is denoted by $T(G)$. This concept were studied in [1,8].The following theorem are used in sequel.

In this paper, we define a new parameter.

Theorem 1.1[3] Each end vertex of a connected graph G belongs to every detour set of G is called the detour semi toll number of a graph.

Theorem 1.2[3] For the star$G = K_{1,n-1} (n \geq 2), d_n(G) = n - 1$.

2.The Connected detour edge semi toll number of a graph

Definition 2.1. A detour edge semi toll set of a connected graph G is called a connected detour edge semi toll set of G if the induced subgraph $G[S]$ is connected. The minimum cardinality of a connected detour edge semi toll set of G is the connected detour edge semi toll number and is denoted by $cdn_{est}(G)$. Any connected detour edge semi toll set of cardinality $cdn_{est}(G)$ is called a cdn_{est}-set of G.

Example 2.2. For the graph G given in Figure 2.1, $S_1 = \{v_3, v_4, v_5, v_6\}$,$S_2 = \{v_2, v_3, v_5, v_7\}$, $S_3 = \{v_3, v_4, v_6, v_7\}$and $S_4 = \{v_2, v_3, v_5, v_6\}$are four connected detour edge semi toll sets of G so that $cdn_{est}(G) = 4$.

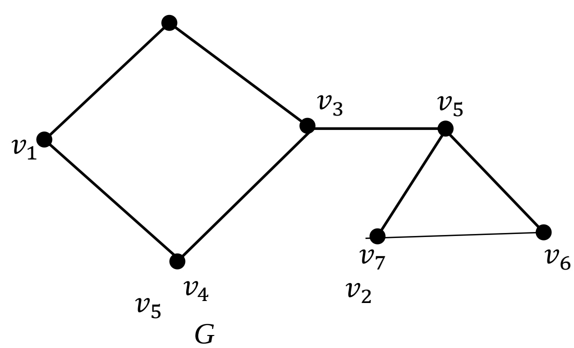

Figure 2.1

Observation 2.3 (i) Each end vertex of G belongs to every connected detour edge semi toll set of G.

(ii) Each cut vertex of G belongs to every connected detour edge semi toll set of G.

(iii) $2 \leq dn_{est}(G) \leq cdn_{est}(G) \leq n$, where $n \geq 2$.

Theorem 2.4. For the path $G = P_n(n \geq 3)$, $cdn_{est}(G) = n$.

Proof. From Observation 2.3 (i) and (ii) each end vertices and cut vertices are belongs to every connected detour edge semi toll set of G so that $cdn_{est}(G) = n$. ■

Theorem 2.5. For the complete graph $G = K_n(n \geq 3)$, $cdn_{est}(G) = 2$.

Proof. Let u and v be any two adjacent vertices of G. Then $S = \{u, v\}$ is a connected detour edge semi toll set of

G so that $cdn_{est}(G) = 2$. ■

Theorem 2.6. For the cycle $G = C_n (n \geq 3)$, $cdn_{est}(G) = 2$.

Proof. Let u, v be any two adjacent vertices of G. Then $S = \{u, v\}$ is a connected detour edge semi toll set of G so that $cdn_{est}(G) = 2$. ■

Theorem 2.7. For the fan graph $G = K_1 + P_{n-1}$ $(n \geq 4)$, $cdn_{est}(G) = 2$.

Proof. Let $V(K_1) = \{x\}$and $V(P_{n-1}) = \{v_1, v_2, \dots, v_{n-1}\}$. Then $S = \{x, v_1\}$ is a connected detour edge semi toll set of G so that $cdn_{est}(G) = 2$. ■

Theorem 2.8. For the wheel graph $G = K_1 + C_{n-1}$ $(n \geq 4)$, $cdn_{est}(G) = 2$.

Proof. Let $V(K_1) = \{x\}$and $V(C_{n-1}) = \{v_1, v_2, \dots, v_{n-1}\}$. Then $S = \{x, v_1\}$ is a connected detour edge semi toll set of G so that $cdn_{est}(G) = 2$. ■

Theorem 2.9. For the star graph$G = K_{1,n-1}(n \geq 3)$, $cdn_{est}(G) = n$.

Proof. Let S be the end vertices and cut vertex of G. Then by Observation 2.3 (i) and (ii), S is a subset of every connected detour edge semi toll set of G and that $cdn_{est}(G) \geq |S|$. Since S is a connected detour edge semi toll set of G so that $cdn_{est}(G) = n$. ■

Theorem 2.10. For the Helm graph $G = H_n, cdn_{est}(G) = 2n$, where n is the number of end vertices .
Proof. Let$S$ be the set of end vertices and cut vertices of G. Then by Observation 2.3 (i) and (ii) , $cdn_{est}(G) = 2n$. ■

Theorem 2.11. For the complete bipartite graph $G = K_{m,n}, m \geq 2, n \geq 2$, $cdn_{est}(G) = 2$.

Proof. Let u, v be any two vertices of G. Then $S = \{u, v\}$ is a connected detour edge semi toll set of G so that $cdn_{est}(G) = 2$. ■

Definition 2.12. A vertex v is said to be detour edge toll vertex if v is not an internal vertex of any x-y detour edge semi toll path of G.

Example 2.13. For the graph G given in Figure 2.2, v_9 is not an internal edge of any x-y detour edge semi toll path of G, so that v_9 is the detour edge semi toll vertex of G.

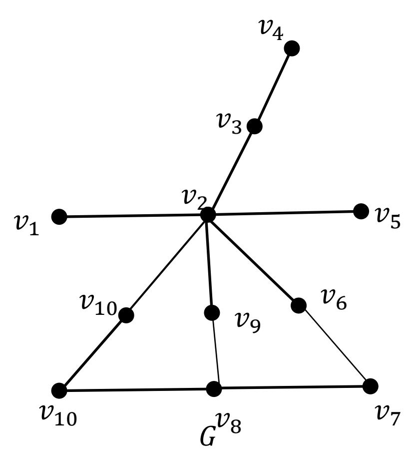

Figure 2.2

Remark 2.14. Every end vertex of G is the detour edge semi toll vertex of G but the converse need not be true.

Observation 2.15. Every detour edge semi toll vertex set belong to any connected detour edge semi toll se of G.

Theorem 2.16. Let G be the connected graph of order $n \geq 2$. If $cdn_{est}(G) = 2$, then every vertex of G lies on a u-v detour edge semi toll diametral walk of G.

Proof. Let $cdn_{est}(G) = 2$. Let $S = \{u, v\}$ be a cdn_{est}-set of G. Then every vertex of G lies on u-v detour edge semi toll diametral walk of G. On the contrary, suppose that P is not a u-v detour edge semi toll diametral walk of G.Then there exists at least one vertex, say $x \in V(G) - V(P)$ such that x is not an internal vertex of u-v detour edge semi toll walk of G, which is a contradiction. Therefore every vertex of G lies on a u-v detour edge semi toll diametral walk of G. ■

Remark 2.17. The converse of Theorem 2.16 need not be true. For $G = P_n (n \geq 3)$ with $V(G) = \{v_1, v_2, \ldots, v_n\}$, every vertex of G lies on the v_1-v_n detour edge semi toll walk of G. However by Theorem 2.3, $cdn_{est}(G) = n\ (n \geq 3)$.

Theorem 2.18. Let G be the connected graph of order $n \geq 3$. Then $cdn_{est}(G) = n$ if and only if every vertex of G is either a cut vertex or a detour edge semi toll vertex of G.

Proof. If every vertex of G is either a cut vertex or a detour edge semi toll vertex of G, then the result follows from Observation 2.3 (ii) and 2.15.

Conversely, let $cdn_{est}(G) = n$. We prove that every vertex of G is either a cut vertex or a detour edge semi toll vertex of G. On the contrary, suppose that there exists a vertex x such that x is neither a cut vertex nor an detour edge semi toll vertex of G.Let$S = V(G) - \{x\}$. Since x is not a pendant vertex of G, x ∈ $dn_{est}(G)$. Since x is not a cut vertex of $G, G[S]$ is connected. Then S is a connected detour edge semi toll set of G and so $cdn_{est}(G) \leq n - 1$, which is a contradiction. Therefore every vertex of G is either a cut vertex of G or a detour edge semi toll vertex of G. ■

Theorem 2.19. For every pair of a and b of integers with $2 \leq a < b$, there exists a connected graph G such that $dn_{est}(G) = a$and $cdn_{est}(G) = b$.

Proof. For $a = b$ let $G = K_{1,a}$. Then by Theorems 1.2 and 2.9, $dn_{est}(G) = cdn_{est}(G) = a$. So let $a < b$. Let $P_{b-a+1} :\ u_0, u_1, u_2, \ldots, u_{b-a}$be a path of order $b - a + 1$. Let P_i: $x_i, y_i (1 \leq i \leq b - a)$ be a copy of pathon two vertices. Let G be the graph obtained from $P_{b-a+1}$and P_i $(1 \leq i \leq b - a - 1)$by joining x_i and y_i $(1 \leq i \leq b - a - 1)$ with u_i $(1 \leq i \leq b - a - 1)$. The graph G is shown in Figure 2.3.

First we prove that $dn_{est}(G) = a$. Let $Z = \{u_0, z_1, z_2, \ldots, z_{a-1}\}$be the set of all end vertices of G. Then by Observation 2.3(ii),Z is a subset of every detour edge semi toll set of $G$and so $dn_{est}(G) \geq a$. Since Z is a detour edge semi toll set of G, we have $dn_{est}(G) = a$.

Next we prove that $cdn_{est}(G) = b$. Let$Z_1 = Z \cup \{u_1, u_2, \ldots, u_{b-a}\}$ be the set of all end vertices and cut vertices of G.By Observation 2.3 (i) and (ii), Z_1 is a subset of every connected detour edge semi toll set of G and so $cdn_{est}(G) \geq b$. Since Z_1 is a connected detour edge semi toll set of $G, cdn_{est}(G) = b$.

■

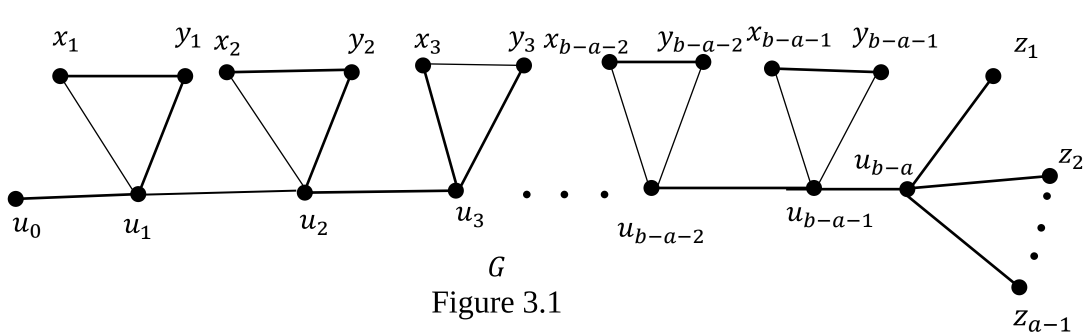

Figure 3.1

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