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# Gaussian Product Anti Magic Labeling, Gaussian Product Odd, Even Anti Magic Labeling in Path Related Graphs

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**Abstract**

New labeling such as Gaussian Product Anti Magic Labeling, Gaussian Product Odd Anti Magic Labeling, and Gaussian Product even Anti Magic Labeling are introduced in this paper. We investigate if the above mentioned labelings exist for the path, Y-tree, comb, key, and star graphs.

**Keywords:** Graph labelings, antimagic graphs, Gaussian product antimagic graphs.

## Introduction

The concept of graph labeling was introduced by Rosa [3] in 1967. A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. The graph labeling problem that appears in graph theory has a fast development recently. This is not only due to its mathematical importance but also because of the wide range of the applications which includes X-rays, crystallography, coding theory, radar, astronomy, circuit design, and design of good Radar Type Codes, Missile Guidance Codes and Convolution Codes with optimal autocorrelation properties and communication design. An enormous body of literature has grown around the subject in the last five decades. They gave birth to families of graphs with attractive names such as graceful, harmonious, felicitous, elegant, cordial, *k*-sequentially additive, magic, bimagic, pair sum, anti-magic and prime graphs, etc. An useful survey to know about the numerous graph labeling methods is the one by J.A. Gallian recently [1].

## II Preliminaries

In this section, we give basic notions relevant to this paper.

**Definition 2.1.**A path P of length n in a graph G is a sequence $\{v_1, v_2, \dots, v_{(n-1)}, v_n\}$ of distinct vertices such that for $1 \le i \le (n-1)$, the vertices $v_i$ and $v_{i+1}$ are adjacent. We say that P is a $v_1 - v_n$ path. The vertices $v_1$ and $v_n$ are called the origin and terminus of P respectively. The vertices $v_2, \dots, v_{(n-1)}$ are called the internal vertices of P. A path on n vertices is denoted by $P_n$.

**Definition 2.2 :** By joining a pendant edge to the first vertex of degree two in a path graph the Y-tree graph is obtained.

**Definition 2.3 :** By adding pendent edge to each of the vertices of $P_n$ , we obtain a comb graph. It has 2n vertices and 2n-1 edges.

**Definition 2.4 :** By joining a single pendent edge to each vertex on $C_n$, the Huffman tree is a graph obtained.

**Definition 2.5 :** By connecting the edge between one vertex of $C_n$ and a vertex of degree two of Huffman tree**,** the Key graph is obtained.

**Definition 2.6.** With n vertices and n-1edges**,** the star graph $K_{1, n-1}$ is a complete bipartite graph.

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**Definition 2.7.** The vertex-weight of a vertex x in *G* under an edge labeling to be the sum of edge labels corresponding to all edges incident with x. Under a total labeling, vertex-weight of x is defined as the sum of the label of x and the edge labels corresponding to all the edges incident with x. If all vertices in *G* have the same weight *l* , we call the labeling vertex-magic edge labeling or vertex-magic total labeling, respectively and we call *l* a magic constant. If all vertices in *G* have different weights, then the labeling is called vertex-antimagic edge labeling or vertex-antimagic total labeling, respectively.

**Definition 2.8.** Gaussian antimagic labeling in a G(p,q) graph is a function $\psi : V \to \{\alpha+i\beta/a,b\in N\}$ $1 \leq \alpha \leq \beta \leq q$ such that the induced function $Ø^* : E \to N$ defined by $Ø^*(uv) = |Ø^*(u)|^2 + |Ø^*(v)|^2$ results all the edge labels are distinct. A graph which admits Gaussian antimagic labeling is called Gaussian antimagic graph.

**Definition 2.9.** Gaussian product antimagic labeling in a $G(p, q)$ graph is a function $\psi: V \to \{\alpha + i\beta \ / \ \alpha,\beta\in N\}$ such that the induced function $\tau^*: E \to N$ defined by $\tau^*(uv) = |\psi(u)|^2 \ |\psi(v)|^2$ results all the edge labels are distinct. A graph which admits Gaussian product antimagic labeling is called Gaussian product antimagic graph.

**Definition 2.10.** Gaussian product odd antimagic labeling in a $G(p, q)$ graph is a function $\psi: V \to \{\alpha + i\beta \ / \ \alpha,\beta \in N\}$ such that the induced function $\tau^*: E \to N$ defined by $\tau^*(uv) = |\psi(u)|^2 \ |\psi(v)|^2$ results all the edge labels are odd and distinct. A graph which admits Gaussian product odd anti magic labeling is called Gaussian product odd anti magic graph.

**Definition 2.11.**Gaussian product even antimagic labeling in a $G(p, q)$ graph is a function $\psi: V \to \{\alpha + i\beta \ / \ \alpha,\beta\in N\}$ such that the induced function $\tau^*: E \to N$ defined by $\tau^*(uv) = |\psi(u)|^2 \ |\psi(v)|^2$ results all the edge labels are even and distinct. A graph which admits Gaussian product even antimagic labeling is called Gaussian product even antimagic graph.

## III. Main Results

### GAUSSIAN PRODUCT ANTI MAGIC LABELING IN PATH RELATED GRAPHS

**Theorem 3. 1**: The path graph $P_n$, $n > 1$ admits Gaussian product antimagic labeling

**Proof**: Let $V= \{v_1, v_2, \ldots, v_k, v_{k+1}, v_{k+2}, \ldots, v_n \ / \text{ n is even, } k= \frac{n}{2}\}$ be the vertices and

$E = \{v_i v_{i+1} \ / \ 1 \leq i \leq 2k-1\}$be the edges of the path $P_n$.

Define a function $\psi: V \to \{ \alpha + i\beta \ / \ \alpha, \beta\in N \}$ such that

$\psi(v_e) = e + i(e + 1)$ ,e=1,2,3…,k

$\psi(v_l) = l + i(l + 2)$ ,$l$=k+1,k+2,k+3, ……..2k .

Define the induced function $\rho: E \to N$ such that $\rho(v_j v_{j+1}) = |\psi(v_j)|^2 \ |\psi(v_{j+1})|^2$

The edge labels are obtained as follows:

$\rho(v_j v_{j+1}) = |\psi(v_j)|^2 |\psi(v_{j+1})|^2, 1 \leq j \leq k-1$

$\rho(v_j v_{j+1}) = |\psi(v_j)|^2 |\psi(v_{j+1})|^2, k+1 \leq j \leq 2k-1$

$\rho(v_k v_{k+1}) = \frac{1}{4}(n^4+10n^3+38n^2+56n+40)$
$\rho(v_{n-1} v_n) = 4n^4+8n^3+12n^2+8n+8$

If $q \neq r$, $g \neq h$
$\rho(v_q v_{q+1}) = 4q^4+16q^3+24q^2+16q+5$
$\rho(v_r v_{r+1}) = 4r^4+16r^3+24r^2+16r+5$

$\rho(v_g v_{g+1}) = 4g^4+24g^3+60g^2+72g+40$
$\rho(v_h v_{h+1}) = 4h^4+24h^3+60h^2+72h+40$

This implies $\rho(v_q v_{q+1}) \neq \rho(v_r v_{r+1})$ , $\rho(v_g v_{g+1}) \neq \rho(v_h v_{h+1})$
$\rho(v_q v_{q+1}) \neq \rho(v_r v_{r+1}) \neq \rho(v_g v_{g+1}) \neq \rho(v_h v_{h+1})$
Thus $\rho(E) = \{65, 325, 1025, 2501, \dots, (\frac{1}{4})(n^4+10n^3+38n^2+56n+40), \dots (4n^4+8n^3+12n^2+8n+8)\}$ in which all the elements are distinct.

Therefore, the path graph $P_n$ is a Gaussian product antimagic graph.

Example3.1: The Gaussian antimagic labeling for Path with $n$(n is even)vertices is shown in figure -3. 1.

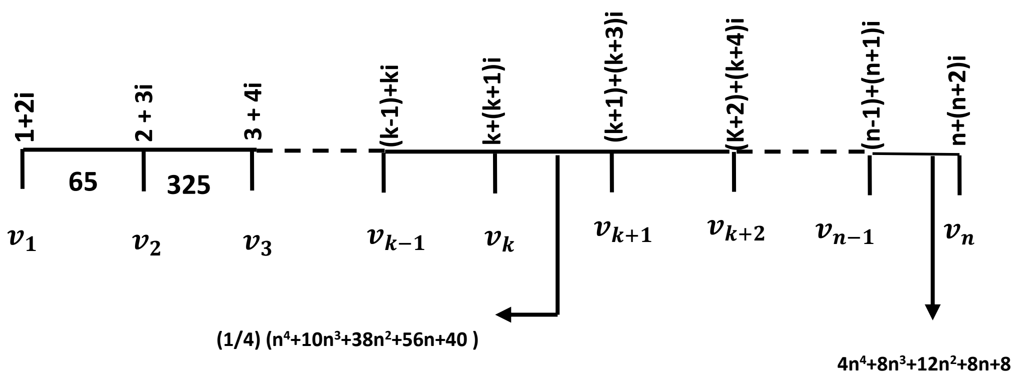


**Figure – 3.1.** The Gaussian antimagic labeling for Path with $n$ (n is even) vertices.

If n is Odd

Let $V=\{v_1, v_2, \dots, v_{\frac{n-1}{2}}, v_{\frac{n+1}{2}}, v_{\frac{n+3}{2}}, \dots, v_n\}$ be the vertices and $E = \{v_i v_{i+1} / 1 \le i \le n-1\}$ be the edges of the path $P_n$,

Define a function $\psi: V \to \{\alpha + i\beta / \alpha, \beta \in N\}$ such that

$\psi(v_e) = e + i(e+1)$ , $e=1,2,3\dots\dots\frac{(n-1)}{2}$

$\psi(v_l) = l + i(l+2)$ , $l = \frac{(n+1)}{2}, \frac{(n+3)}{2}, \frac{(n+5)}{2}, \dots, n$

Define the induced function $\rho: E \to N$ such that $\rho(v_j v_{j+1}) = |\psi(v_j)|^2 \ |\psi(v_{j+1})|^2$

The edge labels are obtained as follows:

$\rho(v_j v_{j+1}) = |\psi(v_j)|^2 |\psi(v_{j+1})|^2, 1 \le j \le \frac{(n-3)}{2}$

$\rho(v_j v_{j+1}) = |\psi(v_j)|^2 |\psi(v_{j+1})|^2, \frac{(n+1)}{2} \le j \le n-1$

$\rho(v_{\frac{n-1}{2}} v_{\frac{n+1}{2}}) = (\frac{1}{4})(n^4+6n^3+14n^2+6n+13)$ , $\rho(v_{n-1} v_n) = 4n^4+8n^3+12n^2+8n+8$
If $q \neq r$, $g \neq h$
$\rho(v_q v_{q+1}) = 4q^4+16q^3+24q^2+16q+5$ , $\rho(v_r v_{r+1}) = 4r^4+16r^3+24r^2+16r+5$
$\rho(v_g v_{g+1}) = 4g^4+24g^3+60g^2+72g+40$ , $\rho(v_h v_{h+1}) = 4h^4+24h^3+60h^2+72h+40$

This implies ρ ($v_q v_{q+1}$) ≠ ρ ($v_r v_{r+1}$), ρ ($v_g v_{g+1}$)≠ ρ ($v_h v_{h+1}$)
ρ ($v_q v_{q+1}$) ≠ ρ ($v_r v_{r+1}$)≠ ρ($v_g v_{g+1}$)≠ ρ ($v_h v_{h+1}$) ≠ ρ ($v_{n-1}v_n$)
Thus ρ ($E$) = {65, 325, 1025, 2501, … ,($\frac{1}{4}$) ($n^4+6n^3+14n^2+6n+13$ ), …($4n^4+8n^3+12n^2+8n+8$ ) } in which all the elements are distinct.

Thus, the path graph $P_n$ is a Gaussian product antimagic graph.

Example 3.2: The Gaussian anti magic labeling for Path with $n$ (‘n’ is Odd ) vertices is shown in figure – 3.2.

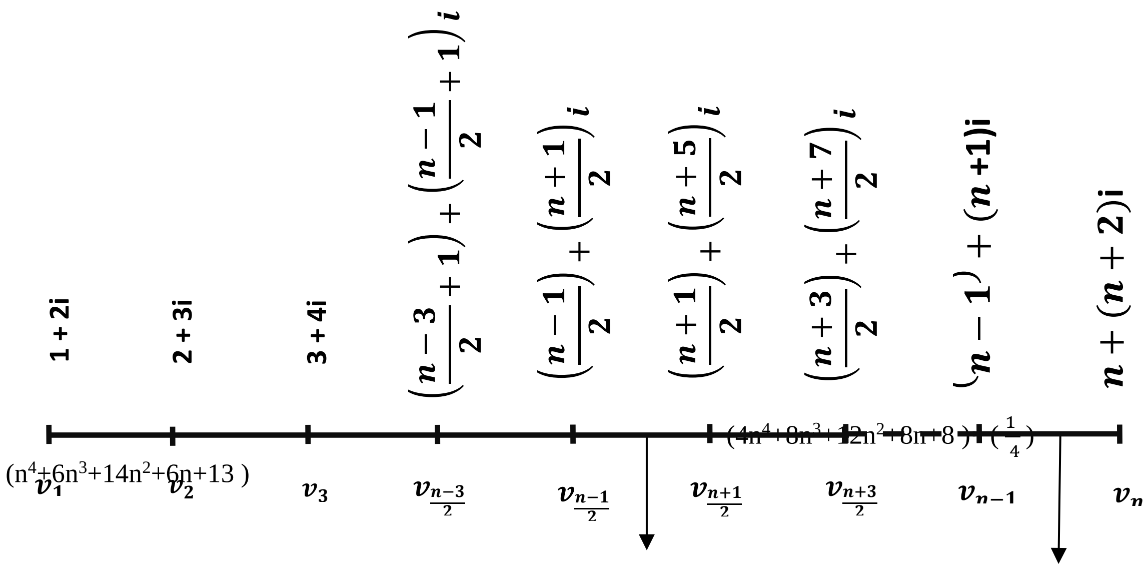


**Figure – 3.2** The Gaussian anti magic labeling for Path with $n$ (‘n’ is Odd ) vertices.

**Theorem 3. 2**:

The $Y_n -$ tree admits Gaussian product antimagic labeling.

**Proof:** Let {$v_1$, $v_2$, . . . ,$_k$, $v_{k+1}$, $v_{k+2}$, . . . , $v_n$ / n is even, k=$\frac{n}{2}$} be the vertices and

$E$ ={{$v_iv_{i+1}$ / 1 ≤$i$≤(2k−2)} ∪{ $v_{(n-2)} v_n$}} be the edges of the $Y_n$ tree.

Define a function ψ : V → {α + iβ / α, β ∈ N} such that

ψ ($v_e$) = e+i (e + 1) ,e =1,2,3……..k

ψ ($v_l$) = $l + i(l + 2)$ ,$l$ = (k+1),(k+2),……..(2k-1) ,

ψ ($v_n$) =n+i(n+2)

Define the induced function ρ:$E \to N$ such that ρ ($v_jv_{j+1}$) = | ψ ($v_j$)|$^2$ | ψ ($v_{j+1}$)|$^2$

The edge labels are obtained as follows:

ρ ($v_jv_{j+1}$) = |ψ ($v_j$)| $^2$ |ψ ($v_{j+1}$)| $^2$ ,1 ≤$j$≤k-**1**

ρ ($v_kv_{k+1}$) = ($\frac{1}{4}$) ($n^4+10n^3+38n^2+56n+40$ )

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$\rho(v_j v_{j+1}) = |\psi(v_j)|^2 |\psi(v_{j+1})|^2$, $k+1 \le j \le 2k-2$

$\rho(v_{n-2} v_n) = 4n^4+16$

If $q \neq r$, $g \neq h$
$\rho(v_q v_{q+1}) = 4q^4+16q^3+24q^2+16q+5$
$\rho(v_r v_{r+1}) = 4r^4+16r^3+24r^2+16r+5$
$\rho(v_g v_{g+1}) = 4g^4+24g^3+60g^2+72g+40$
$\rho(v_h v_{h+1}) = 4h^4+24h^3+60h^2+72h+40$

This implies $\rho(v_q v_{q+1}) \neq \rho(v_r v_{r+1})$ , $\rho(v_g v_{g+1}) \neq \rho(v_h v_{h+1})$

$\rho(v_q v_{q+1}) \neq \rho(v_r v_{r+1}) \neq \rho(v_g v_{g+1}) \neq \rho(v_h v_{h+1}) \neq \rho(v_{n-2} v_n)$

Thus $\rho(E) = \{\{65, 325, 1025, 2501, \ldots, (\frac{1}{4})(n^4+10n^3+38n^2+56n+40)$

$, \ldots (4n^4-8n^3+12n^2-8n+8), (4n^4+16)\}$ in which all the elements are distinct.

Therefore, the graph $Y_n$ tree is a Gaussian product antimagic graph.

Example 3.3:

The Gaussian product antimagic labeling for $Y_n$ – tree with n ('n' is even) vertices is shown in figure 3. 3.

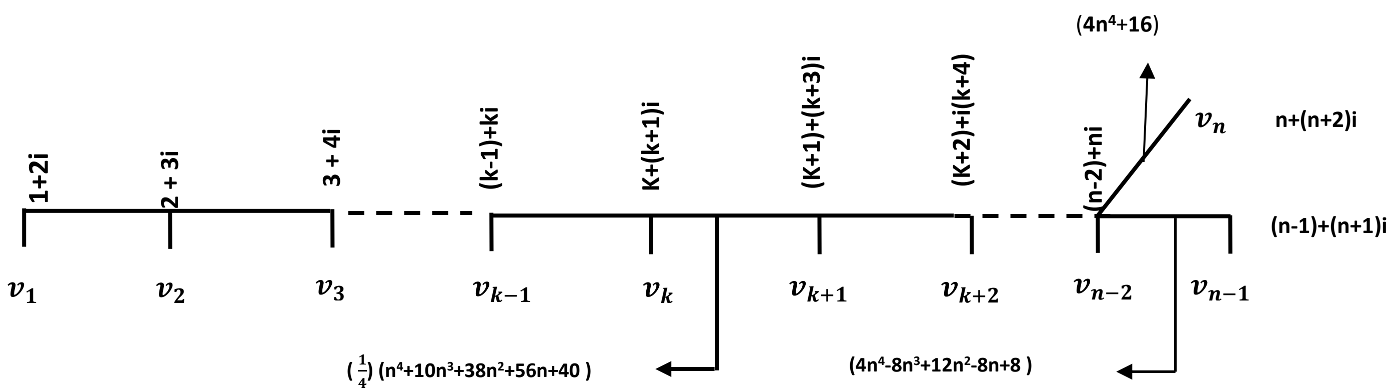


**Figure-3.3** The Gaussian product antimagic labeling for $Y_n$ – tree with n('n' is even) vertices.

If n is Odd

Let $\{v_1, v_2, \ldots, v_{\frac{n-1}{2}}, v_{\frac{n+1}{2}}, v_{\frac{n+3}{2}}, \ldots, v_n\}$ be the vertices and $E = \{\{v_i v_{i+1} / 1 \le i \le n-2\}$

$\cup \{v_{n-2} v_n\}\}$be the edges of the path $Y_n$.

Define a function $\psi: V \rightarrow \{\alpha + i\beta \,/\, \alpha, \beta \in N\}$ such that

$\psi(v_e) = e + i(e+1)$ ,$e = 1,2,3 \ldots \frac{(n-1)}{2}$

$\psi(v_z) = z + i(z+2)$ ,$z = \frac{(n+1)}{2}, \frac{(n+3)}{2}, \frac{(n+5)}{2} \ldots (n-1)$ ,

$\psi(v_n) = n+i(n+2)$

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Define the induced function $\rho: E \rightarrow N$ such that $\rho(v_j v_{j+1}) = |\psi(v_j)|^2 \ |\psi(v_{j+1})|^2$

The edge labels are obtained as follows:

$\rho(v_j v_{j+1}) = |\psi(v_j)|^2 |\psi(v_{j+1})|^2, 1 \le j \le \frac{(n-3)}{2}$

$\rho(v_j v_{j+1}) = |\psi(v_j)|^2 |\psi(v_{j+1})|^2, \frac{(n+1)}{2} \le j \le (n-2)$

$\rho(v_{\frac{(n-1)}{2}} \ v_{\frac{(n+1)}{2}}) = (\frac{1}{4})(n^4+6n^3+14n^2+6n+13)$
$\rho(v_{n-2} v_n) = 4n^4+16$

If $q \neq r, g \neq h$
$\rho(v_q v_{q+1}) = 4q^4+16q^3+24q^2+16q+5$
$\rho(v_r v_{r+1}) = 4r^4+16r^3+24r^2+16r+5$
$\rho(v_g v_{g+1}) = 4g^4+24g^3+60g^2+72g+40$
$\rho(v_h v_{h+1}) = 4h^4+24h^3+60h^2+72h+40$

This implies $\rho(v_q v_{q+1}) \neq \rho(v_r v_{r+1}), \rho(v_g v_{g+1}) \neq \rho(v_h v_{h+1})$

$\rho(v_q v_{q+1}) \neq \rho(v_r v_{r+1}) \neq \rho(v_g v_{g+1}) \neq \rho(v_h v_{h+1}) \neq \rho(v_{n-2} v_n)$

Thus $\rho(E) = \{65, 325, 1025, 2501, \ldots, (\frac{1}{4})(n^4+6n^3+14n^2+6n+13), \ldots (4n^4-8n^3+12n^2-8n+8), 4n^4+16\}$ in which all the elements are distinct.

Therefore, the path graph $P_n$ is a Gaussian product antimagic graph.

Example 3.4: The Gaussian product antimagic labeling for $Y_n$ – tree with n('n' is Odd ) vertices is shown in figure 3.4.

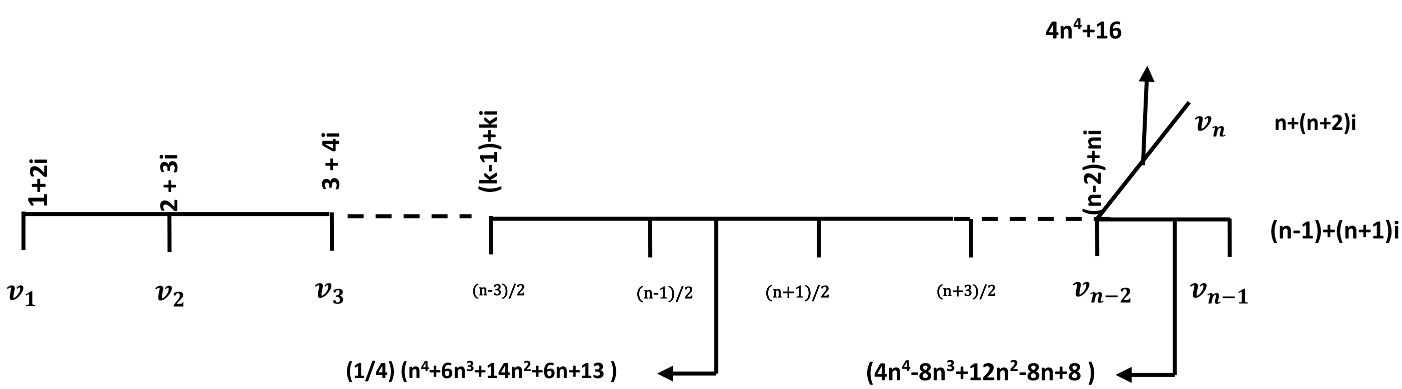


**Figure 3.4.** The Gaussian product antimagic labeling for $Y_n$ – tree with n ('n' is Odd ) vertices.

**Theorem 3.3:** The comb graph of odd length admits Gaussian product antimagic labeling.

**Proof**: Let $\{v_1, v_2, \ldots, v_n, w_1, w_2, w_3, \ldots w_n\}$ be the vertices of comb graph.

Let $E = \{\{v_i v_{i+1} / i = 1,3,5,\ldots(n-1)\} \cup \{w_i w_{i+1} / i = 1,3,5,\ldots(n-1)\} \cup \{v_i v_{i+2} / i = 2,4,6,\ldots n\} \cup$

$\{w_i w_{i+2} / i = 2,4,6,\ldots n\}\}$ be the edges of the comb graph. where n is even

Define a function ψ: V → { α + iβ / α, β ∈ N }such that

$\psi(v_e) = e + i(e+1)$ ,$1 \le e \le n$

$\mu(w_z) = z + i(z+2)$ ,$1 \le z \le n$

Define the induced function $\rho : E \to N$ such that $\rho(v_j v_{j+1}) = |\psi(v_j)|^2 \; |\psi(v_{j+1})|^2$. The edge labels are obtained as follows:

$\rho(v_j v_{j+1}) = |\psi(v_j)|^2 \; |\psi(v_{j+1})|^2$ , j = 1,3,5,…$(n-1)$

$\rho(w_j w_{j+1}) = |\mu(w_j)|^2 \; |\mu(w_{j+1})|^2$ , j= 1,3,5,…$(n-1)$

$\rho(v_j v_{j+2}) = |\psi(v_j)|^2 \; |\psi(v_{j+2})|^2$ , j = 2,4,6,…$n$

$\rho(w_j w_{j+2}) = |\mu(w_j)|^2 \; |\mu(w_{j+2})|^2$ , j= 2,4,6,…$n$

$\rho(v_{k-1} v_k) = 4k^4+1$, k=2,4,6…n

$\rho(v_{n-1} v_n) = 4n^4+1$

$\rho(w_{k-1} w_k) = 4k^4+8k^3+12k^2+8k+8$ , k=2,4,6…n

$\rho(v_k v_{k+2}) = 4k^4+24k^3+48k^2+36k+13$ , k=2,4,6…(n-2)

$\rho(v_n w_2) = 40n^2+40n+20$

$\rho(w_k w_{k+2}) = 4k^4+32k^3+96k^2+128k+80$ , k=2,4,6…(n-2)

$\rho(w_{n-2} w_n) = (4n^4+16)$

If $q \ne r$, $g \ne h$, $l \ne m$ , $o \ne p$
$\rho(v_q v_{q+1}) = 4q^4+16q^3+24q^2+16q+5$
$\rho(v_r v_{r+1}) = 4r^4+16r^3+24r^2+16r+5$
$\rho(w_g w_{g+1}) = 4g^4+24g^3+60g^2+72g+40$
$\rho(w_h w_{h+1}) = 4h^4+24h^3+60h^2+72h+40$
$\rho(v_l v_{l+1}) = 64l^4+192l^3+192l^2+72l+13$
$\rho(v_m v_{m+1}) = 64m^4+192m^3+192m^2+72m+13$
$\rho(w_o w_{o+1}) = 64o^4+256o^3+384o^2+256o+80$
$\rho(w_p w_{p+1}) = 64p^4+256p^3+384p^2+256p+80$

This implies $\rho(v_q v_{q+1}) \ne \rho(v_r v_{r+1})$ ,
$\rho(w_g w_{g+1}) \ne \rho(w_h w_{h+1})$
$\rho(v_l v_{l+1}) \ne \rho(v_m v_{m+1})$
$\rho(w_o w_{o+1}) \ne \rho(w_p w_{p+1})$,

$\rho(v_q v_{q+1}) \ne \rho(v_r v_{r+1}) \ne \rho(w_g w_{g+1}) \ne \rho(w_h w_{h+1}) \ne \rho(v_l v_{l+1}) \ne \rho(v_m v_{m+1}) \ne \rho(w_o w_{o+1}) \ne \rho(w_p w_{p+1}) \ne \rho(w_{n-2} w_n)$
Thus $\rho(E)$ ={65,1025,5185, …,$(4n^4+1)$,200,1768,7400,…,$(4n^4+8n^3+12n^2+8n+8)$,533,3485,12325,…,$(40n^2+40n+20)$,1040,5200,16400,…,$(4n^4+16)$}

in which all the elements are distinct.

Therefore, the comb graph is a Gaussian product antimagic graph.

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Example.3.5: The Gaussian product even antimagic labeling for comb graph with $2n$ vertices is shown in figure 3.5.

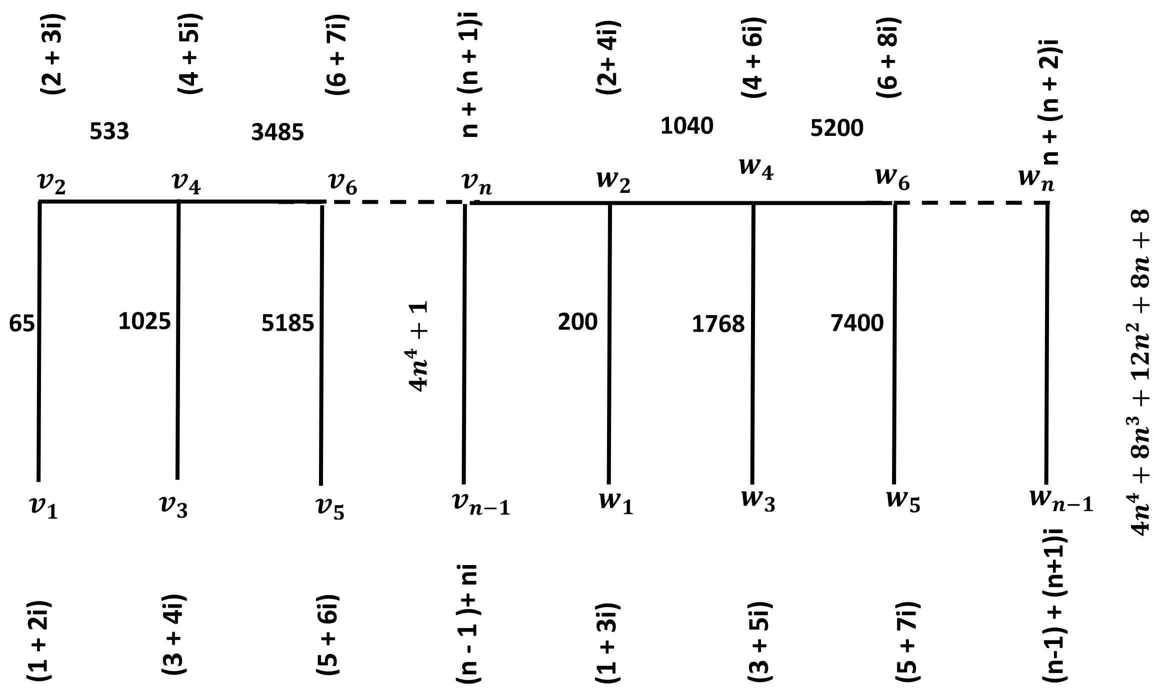



**Figure- 3.5.** The Gaussian product even antimagic labeling for comb graph with $2n$ vertices.

**Theorem 3.4**: The key graph admits Gaussian product antimagic labeling.

**Proof**: Let $V = \{v_1, v_2, \ldots, v_{\frac{3n}{2}}, w_1, w_2, w_3, \ldots w_{\frac{3n}{2}}\}$ be the vertices and

$E = \{\{v_h v_{h+1} \,/\, 1 \le h \le \frac{3n}{2}-1\} \cup \{w_h w_{h+1} \,/\, 1 \le h \le \frac{n}{2}-1\} \cup \{w_{n/2}\, v_{n/2}\} \cup \{v_{(n/2)+h} w_{(n/2)+h} / \, 1 \le h \le n\}$ be the edges of the key graph. Where n is even.

Define a function $\psi: V \to \{\alpha + i\beta \,/\, \alpha, \beta \in N\}$ such that

$\psi(v_e) = e + i(e + 1)$ ,$1 \le e \le \frac{3n}{2}$

$\mu(w_z) = z + i(z + 2)$ ,$1 \le z \le \frac{3n}{2}$

Define the induced function $\rho: E \to N$ such that

$\rho(v_j v_{j+1}) = |\psi(v_j)|^2 \;|\, \psi(v_{j+1})|^2$.

The edge labels are obtained as follows:

$\rho(v_h v_{h+1}) = 4h^4+1$, $1 \le h \le \frac{3n}{2}-1$

$\rho(w_h w_{h+1}) = 4h^4+24h^3+60h^2+72h+40$ , $1 \le h \le \frac{n}{2}-1$

ρ ( $w_{\frac{n}{2}}\ v_{\frac{n}{2}}$ ) $=\frac{1}{4}$($n^4$ +6$n^3$+18$n^2$+24n+16)

ρ ( $v_{\frac{n}{2}+h}\ \ w_{\frac{n}{2}+h}$ ) = $4h^4+12h^3+18h^2+12h+4,\ 1 \le h \le n$

ρ ($v_{\frac{3n}{2}}\ w_{\frac{3n}{2}}$)= $\frac{1}{4}$($81n^4+162n^3+162n^2+72n+16$)

If $q \neq r$, $g \neq h, l \neq m$

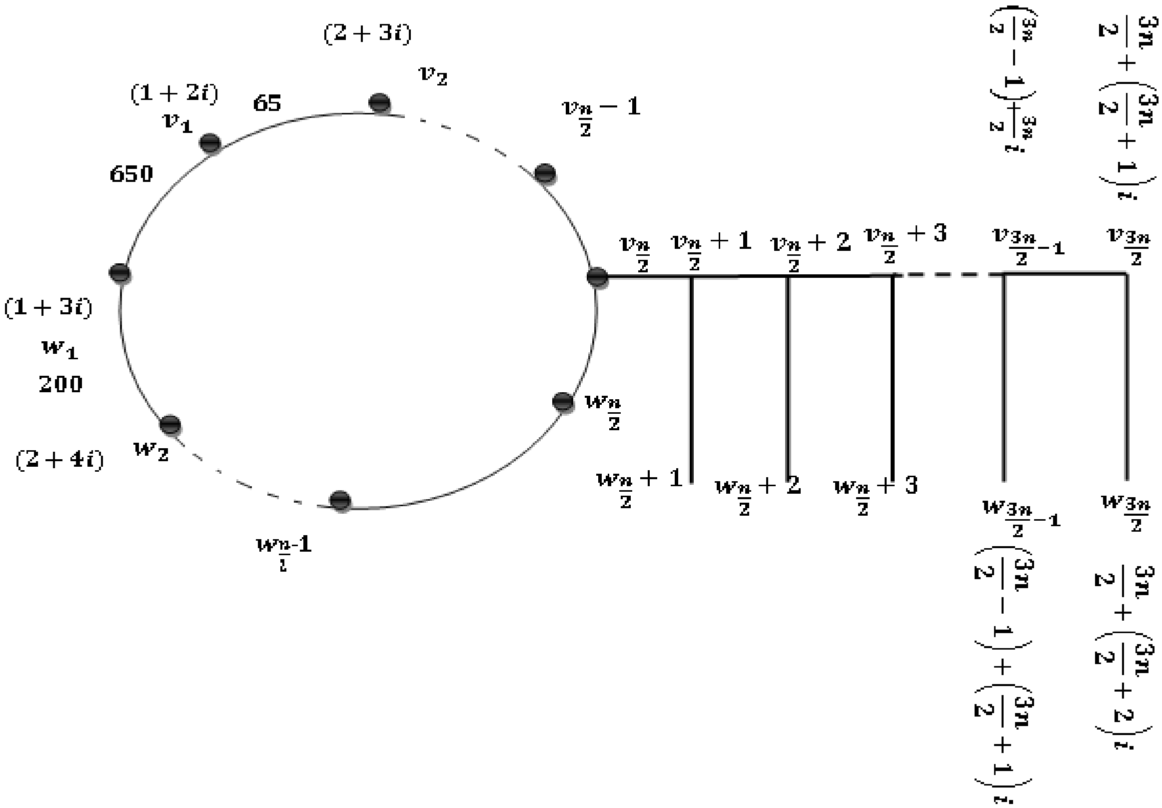
ρ ($v_q\ v_{q+1}$ ) = $4q^4+16q^3+24q^2+16q+5$

ρ ($v_r\ v_{r+1}$ ) = $4r^4+16r^3+24r^2+16r+5$

ρ ($w_g\ w_{g+1}$) = $4g^4+24g^3+60g^2+72g+40$
ρ ($w_h\ w_{h+1}$) = $4h^4+24h^3+60h^2+72h+40$
ρ ($w_l\ v_l$) = $4l^4+12l^3+18l^2+12l+4$
ρ ($w_m\ v_m$) = $4m^4+12m^3+18m^2+12m+4$

This implies ρ ($v_q\ v_{q+1}$ ) ≠ ρ ($v_r\ v_{r+1}$) ,
ρ ($w_g\ w_{g+1}$) ≠ ρ ($w_h\ w_{h+1}$)
ρ ($w_l\ v_l$) ≠ ρ ($w_m\ v_m$)

ρ ($v_q\ v_{q+1}$ ) ≠ ρ ($v_r\ v_{r+1}$) ≠ ρ ($w_g\ w_{g+1}$ ) ≠ ρ ($w_h\ w_{h+1}$) ≠ ρ ($w_l\ v_l$) ≠ ρ ($w_m\ v_m$)
Thus ρ (E) = {65,325,1025,…,$\frac{1}{4}$($81n^4+4$), 200, 680, 1768,…, $\frac{1}{4}$($n^4$ +6$n^3$+18$n^2$+24n+16), … , $\frac{1}{4}$($81n^4+162n^3+162n^2+72n+16$) } in which all the elements are distinct. Therefore, The Key graph admits Gaussian product anti magic labeling.



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Example 3.6: The Gaussian even antimagic labeling for Key graph with 3n vertices is given in figure -3.6

**Figure -3.6.** The Gaussian even antimagic labeling for Key graph with 3n vertices.

If n is odd

Let $V = \{v_1, v_2, \ldots, v_{\frac{3n+1}{2}}, w_1, w_2, w_3, \ldots, w_{\frac{3n-1}{2}}\}$ be the vertices and

$E = \{\{v_h v_{h+1} \,/\, 1 \le h \le \frac{(3n-3)}{2}\} \cup \{w_h w_{h+1} \,/\, 1 \le h \le \frac{(n-3)}{2}\} \cup \{w_{\frac{n-1}{2}}, v_{\frac{n+1}{2}}\}$

$\cup\{ v_{\frac{n+1}{2}+h}\, w_{\frac{n-1}{2}+h} \,/\, 1 \le h \le n\}$ be the edges of the key graph.

Define a function $\psi : V \to \{ \alpha + i\beta \,/\, \alpha, \beta \in N \}$ such that

$\psi(v_e) = e + i(e + 1)$ $,1 \le e \le \frac{(3n+1)}{2}$

$\mu(w_z) = z + i(z + 2)$ $,1 \le z \le \frac{(3n-1)}{2}$

Define the induced function $\rho: E \to N$ such that

$\rho(v_j v_{j+1}) = |\psi(v_j)|^2 \; | \psi(v_{j+1})|^2$.

The edge labels are obtained as follows:

$\rho(v_h v_{h+1}) = 4h^4+1,\ 1 \le h \le \frac{(3n-1)}{2}$

$\rho(w_h w_{h+1}) = 4h^4+24h^3+60h^2+36h+40,\ 1 \le h \le \frac{(n-3)}{2}$

$\rho(w_{\frac{n-1}{2}} v_{\frac{n+1}{2}}) = \frac{1}{4}(n^4+6n^3+18n^2+30n+25)$

$\rho(v_{h+1} w_h) = 4h^4+4h^3+6h^2+4h+4,\ \frac{(n+1)}{2} \le h \le \frac{(3n-1)}{2}$

$\rho(v_{\frac{3n+1}{2}}\ v_{\frac{3n-1}{2}}) = \frac{1}{4}(81n^4+162n^3+162n^2+90n+25)$

If $q \neq r$, $g \neq h$, $l \neq m$

$\rho(v_q v_{q+1}) = 4q^4+16q^3+24q^2+16q+5$

$\rho(v_r v_{r+1}) = 4r^4+16r^3+24r^2+16r+5$

$\rho(w_g w_{g+1}) = 4g^4+24g^3+60g^2+72g+40$
$\rho(w_h w_{h+1}) = 4h^4+24h^3+60h^2+72h+40$
$\rho(v_l w_{(l-1)}) = 4l^4+4l^3+6l^2+4l+2$
$\rho(v_m w_{(m-1)}) = 4m^4+4m^3+6m^2+4m+2$

This implies $\rho(v_q v_{q+1}) \neq \rho(v_r v_{r+1})$,
$\rho(w_g w_{g+1}) \neq \rho(w_h w_{h+1})$
$\rho(v_l w_{(l-1)}) \neq \rho(v_m w_{(m-1)})$
$\rho(v_q v_{q+1}) \neq \rho(v_r v_{r+1}) \neq \rho(w_g w_{g+1}) \neq \rho(w_h w_{h+1}) \neq \rho(v_l w_{(l-1)}) \neq \rho(v_m w_{(m-1)})$

Thus ρ (E) = {65,325,1025,…, $\frac{1}{4}$ $(81n^4 +108n^3+54n^2+12n+5)$ , 200, 680, 1768,……..,$\frac{1}{4}(n^4+6n^2+25)$,
…,$\frac{1}{4}(81n^4+162n^3+162n^2+90n+25)$ }

in which all the elements are distinct.

Therefore, The Key graph admits Gaussian product antimagic labeling.

Example 3.7: The Gaussian antimagic labeling for Key graph with 3n (n is odd) vertices is given in figure -3.7

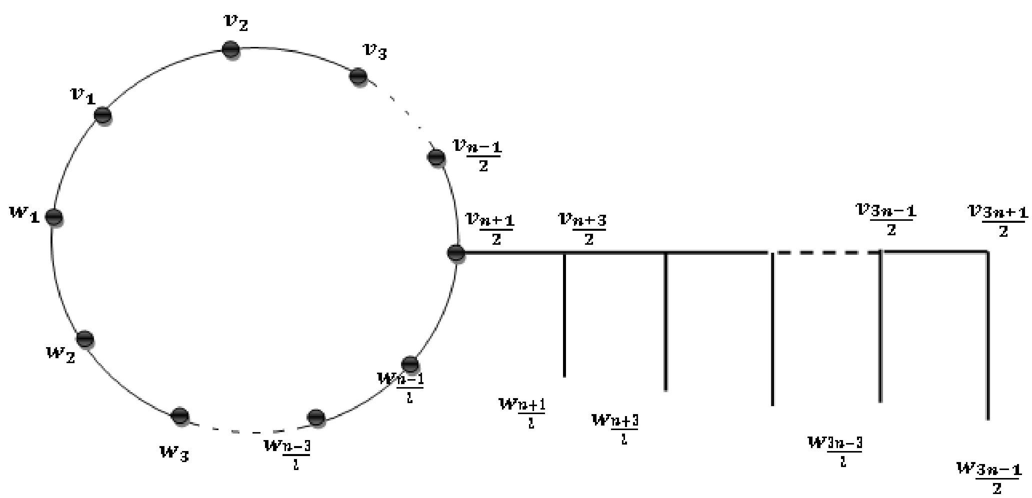


**Figure 3.7.** The Gaussian antimagic labeling for Key graph with 3n (n is odd) vertices.

**Theorem 3.5**:

The star graph admits Gaussian product antimagic labeling.

**Proof**: Let $\{v_1, v_2, \dots, v_k, v_{k+1}, v_{k+2}, \dots, v_n$ / n is even,$k=\frac{n}{2}\}$ be the vertices and

$E = \{v_1v_{h+1} / 1 \le h \le n-1\}$ be the edges of the star graph.

Define a function ψ : V → { α + iβ / α, β ∈ N }and such that

$\psi(v_e) = e + i(e + 1)$ ,$1 \le e \le k$ ,

$\psi(v_l) = l + i(l + 2)$ ,$(k+1) \le l \le n$

Define the induced function ρ : E ⟶ N such that

$\rho(v_jv_{j+1}) = |\psi(v_j)|^2 \ | \psi(v_{j+1})|^2$

The edge labels are obtained as follows:

$\rho (v_1, v_h) = 10h^2+10h+5$ ,$2 \le h \le k$,where $v_1=1+2i$

$\rho (v_1, v_h) = 10h^2+20h+20$ ,$k+1 \le h \le n$,where $v_1=1+2i$,

$\rho(v_1, v_n) = 10n^2+20n+20$

Thus $\rho(E) = \{65, 125, 205, \ldots, (5/2)(n^2+8n+20), \ldots, (10n^2+20n+20)\}$ in which all the elements aredistinct. Therefore, The Star Graph admits Gaussian product antimagic labeling.

Example 3.8: The Gaussian even antimagic labeling for Star graph with $n$+1 (n is even)vertices is given in figure – 3.8

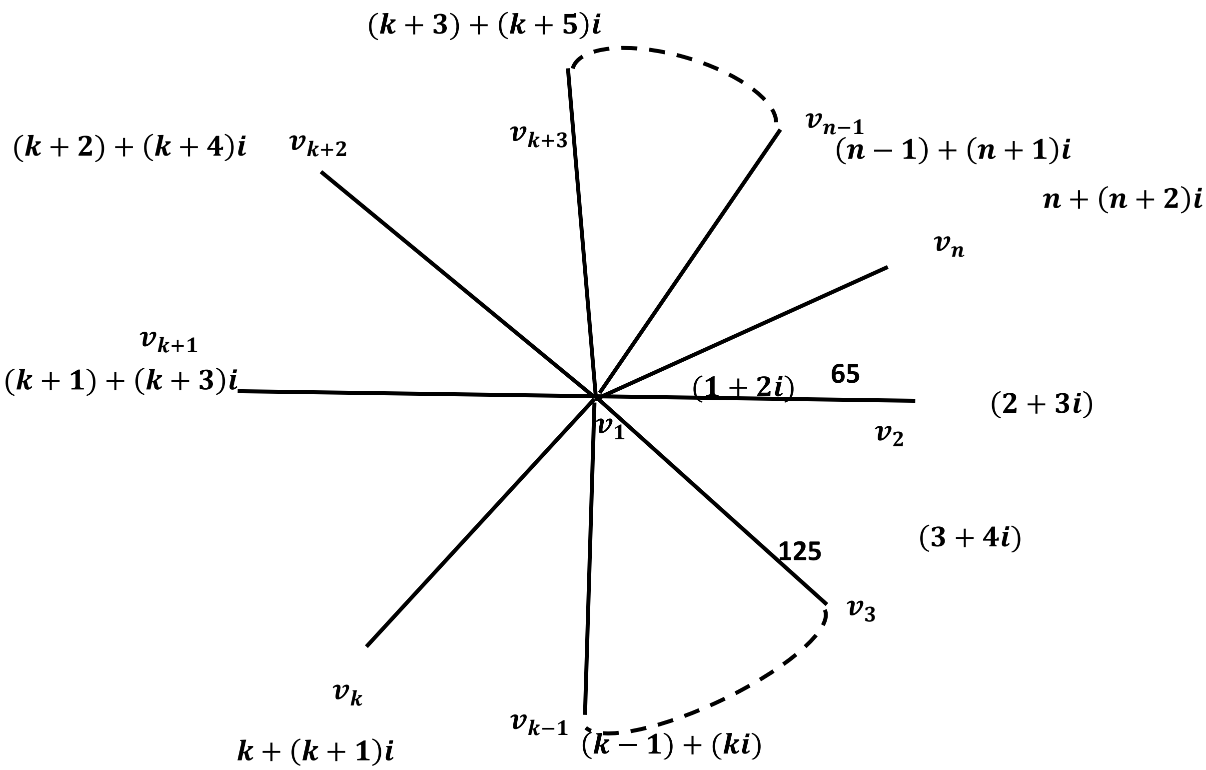


**Figure – 3.8.** The Gaussian even antimagic labeling for Star graph with $n$+1 (n is even) vertices.

If n is odd Let V= $\{v_1, v_2, \ldots, v_{\frac{n-1}{2}}, v_{\frac{n+1}{2}}, v_{\frac{n+3}{2}}, \ldots, v_n\}$ be the vertices and and E = $\{v_1v_{h+1} / 1 \le h \le n-1\}$ be the edges of the star graph.

Define a function $\psi: V \to \{\alpha + i\beta / \alpha, \beta \in N\}$and such that

$\psi(v_e) = e + i(e+1)$ ,e=1,2,3,…,$\frac{n-1}{2}$ , $\psi(v_l) = l + i(l+2)$ , $l = \frac{n+1}{2}, \frac{n+3}{2}, \ldots, n$

Define the induced function $\rho : E \to N$ such that $\rho(v_jv_{j+1}) = |\psi(v_j)|^2 \; |\psi(v_{j+1})|^2$

The edge labels are obtained as follows:

$\rho(v_1, v_{h+1}) = 10h^2+10h+5$ ,$1 \le h \le \frac{n-3}{2}$,where $v_1$=1+2i

$\rho(v_1, v_h) = 10h^2+20h+20$ , $\frac{n+1}{2} \le h \le n$,where $v_1$=1+2i , $\rho(v_1, v_n) = 10n^2+20n+20$

Thus ρ (E) ={65,125,205,…,$\frac{5}{2}$ ($n^2$+1), $\frac{5}{2}$ ($n^2$+6n+13),…,($10n^2$+20n+20) } in which all the elements are distinct.

Thus, the Star Graph admits Gaussian product antimagic labeling.

Example 3.9.: The Gaussian even antimagic labeling for Star graph with $n$+1 (n is odd ) vertices is given in figure – 3.9.

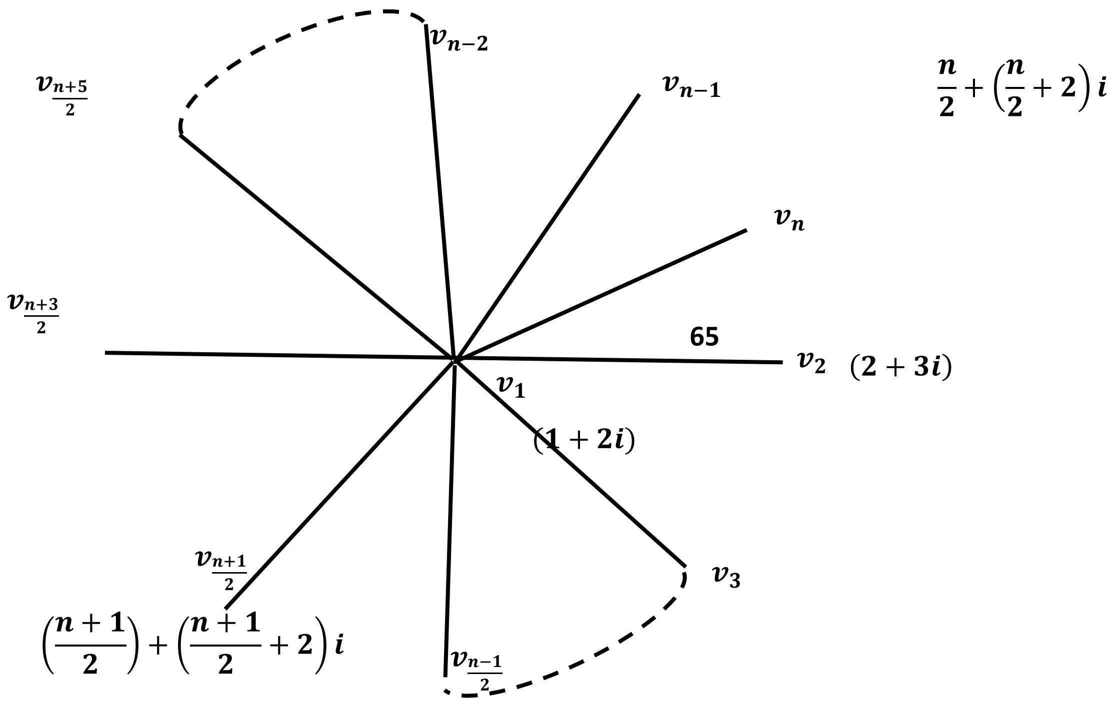


**Figure 3.9.** The Gaussian even antimagic labeling for Star graph with $n$+1 (n is odd ) vertices.

**Theorem 3.6**: The path graph $P_n$, $n$> 1 admits Gaussian product odd antimagic labeling.

$$\frac{n-1}{2}+\left(\frac{n-1}{2}+1\right)i$$

**Proof**: Let {$v_1$, $v_2$, . . . , $v_n$} be the vertices and $E$ = {$v_i v_{i+1}$ / 1 ≤$i$≤$n$−1} be the edges of the path $P_n$. Define a function ψ: V → { α + iβ / α, β ∈ N } such that

ψ($v_e$) = e + $i$(e + 1) , 1≤ e≤ n

Define the induced function ρ:$E$ → $N$ such that ρ($v_j v_{j+1}$) = $|\psi(v_j)|^2$ | $\psi(v_{j+1})|^2$

The edge labels are obtained as follows:

ρ ($v_j v_{j+1}$) = $|\psi(v_j)|^2$ $|\psi(v_{j+1})|^2$ ,1 ≤$j$≤$n$−1

ρ ($v_k v_{k+1}$) = $4k^4$+$16k^3$+$24k^2$ +16k+5,   1≤ k ≤ n-3

ρ ($v_{n-2}$ $v_{n-1}$) = $4n^4$-$16n^3$+$24n^2$-16n+5

ρ ($v_{n-1} v_n$) = $4n^4$+1

If  k≠ $q$,

$\rho\,(v_q\, v_{q+1}) = 4q^4+16q^3+24q^2+16q+5$
$\rho\,(v_k\, v_{k+1}) = 4k^4+16k^3+24k^2+16k+5$

This implies
$\rho\,(v_k v_{k+1}) \neq \rho\,(v_q\, v_{q+1}) \neq \rho\,(v_{n-2}\, v_{n-1}) \neq \rho\,(v_{n-1} v_n)$
Thus $\rho\,(E) = \{65, 325, 1025, 2501, \ldots, 4n^4-16n^3+24n^2-16n+5, 4n^4+1\}$ in which all the elements are odd and distinct.

Therefore, the path graph $P_n$, is a Gaussian product odd antimagic graph.

3. 10: The Gaussian product odd antimagic labeling for $P_n$ with $n$ vertices is shown in figure – 3.10.

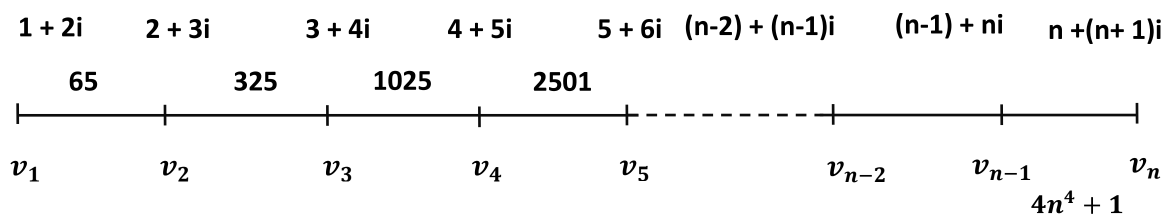


**Figure 3.10.** The Gaussian product odd antimagic labeling for $P_n$ with $n$ vertices .

**Theorem 3.7**: The $Y_n -$ tree admits Gaussian product odd antimagic labeling.

**Proof**: Let V=$\{v_1, v_2, \ldots, v_n\}$ be the vertices and $E = \{\{ v_i v_{i+1} / 1 \leq i \leq n-2\} \cup \{v_{n-2}\, v_n\}\}$ be the edges of the $Y_n$ tree.

Define a function $\psi : V \to \{ \alpha + i\beta \,/\, \alpha, \beta \in N \}$ such that $\psi(v_e) = e + i(e+1)$ , $1 \leq e \leq n$

Define the induced function $\rho: E \to N$ such that $\rho(v_j v_{j+1}) = |\psi(v_j)|^2 \;\; |\psi(v_{j+1})|^2$

The edge labels are obtained as follows:

$\rho\,(v_j v_{j+1}) = |f(v_j)|^2 \;\; |f(v_{j+1})|^2$ , $1 \leq j \leq n-1$

$\rho\,(v_k v_{k+1}) = 4k^4+16k^3+24k^2+16k+5$, $1 \leq k \leq n-2$

$\rho\,(v_{n-2} v_n) = 4n^4-8n^3+4n+5$

If $k \neq q$,

$\rho\,(v_q v_{q+1}) = 4q^4+16q^3+24q^2+16q+5$

$\rho\,(v_k v_{k+1}) = 4k^4+16k^3+24k^2+16k+5$

This implies

$\rho\,(v_k\, v_{k+1}) \neq \rho\,(v_q\, v_{q+1}) \neq \rho\,(v_{n-2}\, v_n)$

Thus $\rho\,(E) = \{65, 325, 1025, \ldots, 2n^4-6n^3+7n^2-6n+5, 4n^4-8n^3+4n+5\}$ in which all the elements are odd and distinct.

Thus, the $Y_n$ – tree is a Gaussian product odd antimagic graph.
Example 3.11: The Gaussian product odd antimagic labeling for $Y_n-$ tree with $n$ vertices is shown in figure – 3.11

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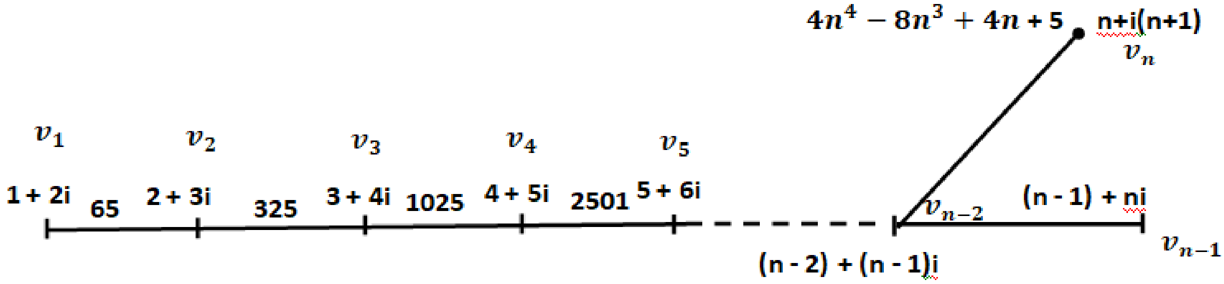


**Figure – 3.11.** The Gaussian product odd antimagic labeling for $Y_n$− tree with $n$ vertices.

**Theorem 3.8:**

The comb graph of odd length admits Gaussian product odd antimagic labeling.

**Proof**: Let $\{v_1, v_2, . . . ,_{2n}\}$ be the vertices of comb graph.

Let $E = \{\{v_i v_{i+1} / 1 \leq i \leq n-1\} \cup \{v_i v_{n+i} / 1 \leq i \leq n\}\}$ be the edges of the comb graph of odd length.

Define a function ψ: V → { α + iβ / α, β ∈ N } such that

$\psi(v_e) = e + i(e+1)$

Define the induced function $\rho: E \rightarrow N$ such that $\rho(v_j v_{j+1}) = |\psi(v_j)|^2 \mid \psi(v_{j+1})|^2$.

The edge labels are obtained as follows:

$\rho\,(v_j v_{j+1}) = |\psi(v_j)|^2\ |\psi(v_{j+1})|^2 ,\ 1 \leq j \leq n-2$

$\rho\,(v_k v_{k+1}) = 4k^4+16k^3+24k^2+16k+5, \quad 1 \leq k \leq n-3$

$\rho\,(v_{n-2}\, v_{n-1}) = 4n^4-16n^3+24n^2-16n+5$ , $\rho\,(v_{n-1} v_n) = 4n^4+1$

$\rho\,(v_1\, v_{n+1}) = 10n^2+30n+25$ , $\rho\,(v_n\, v_{2n}) = 16n^4+24n^3+18n^2+6n+1$

$\rho\,(v_k v_{n+k}) = (2k^2+2k+1)(\ 2n^2+2n+2k^2+2k+4nk+1), \quad 1 \leq k \leq n-1$

If $k \neq q$,
$\rho\,(v_q\, v_{q+1}) = 4q^4+16q^3+24q^2+16q+5$ , $\rho\,(v_k v_{k+1}) = 4k^4+16k^3+24k^2+16k+5$
$\rho\,(v_m\, v_{n+m}) = (2m^2+2m+1)(\ 2n^2+2n+2m^2+2m+4nm+1)$
This implies $\rho\,(v_k v_{k+1}) \neq \rho\,(v_q v_{q+1})$, $\rho\,(v_k v_{n+k}) \neq \rho\,(v_m\, v_{n+m})$
$\rho\,(v_k\, v_{k+1}) \neq \rho\,(v_q\, v_{q+1}) \neq \rho\,(v_k v_{n+k}) \neq \rho\,(v_m\, v_{n+m}) \neq \rho\,(v_{n-1} v_n) \neq \rho\,(v_n, v_{2n})$
Thus $\rho\,(E)$ = {65, 325,1025, … , $(4n^4-16n^3+24n^2-16n+5)$,$(4n^4+1)$,$(10n^2+30n+25)$…,$(16n^4+24n^3+18n^2+6n+1)$ } in which all the elements are odd and distinct.

Therefore, the comb graph is a Gaussian product odd antimagic graph.

Example.3.12: The Gaussian product odd antimagic labeling for comb graph with 2n vertices is shown in figure-3.12

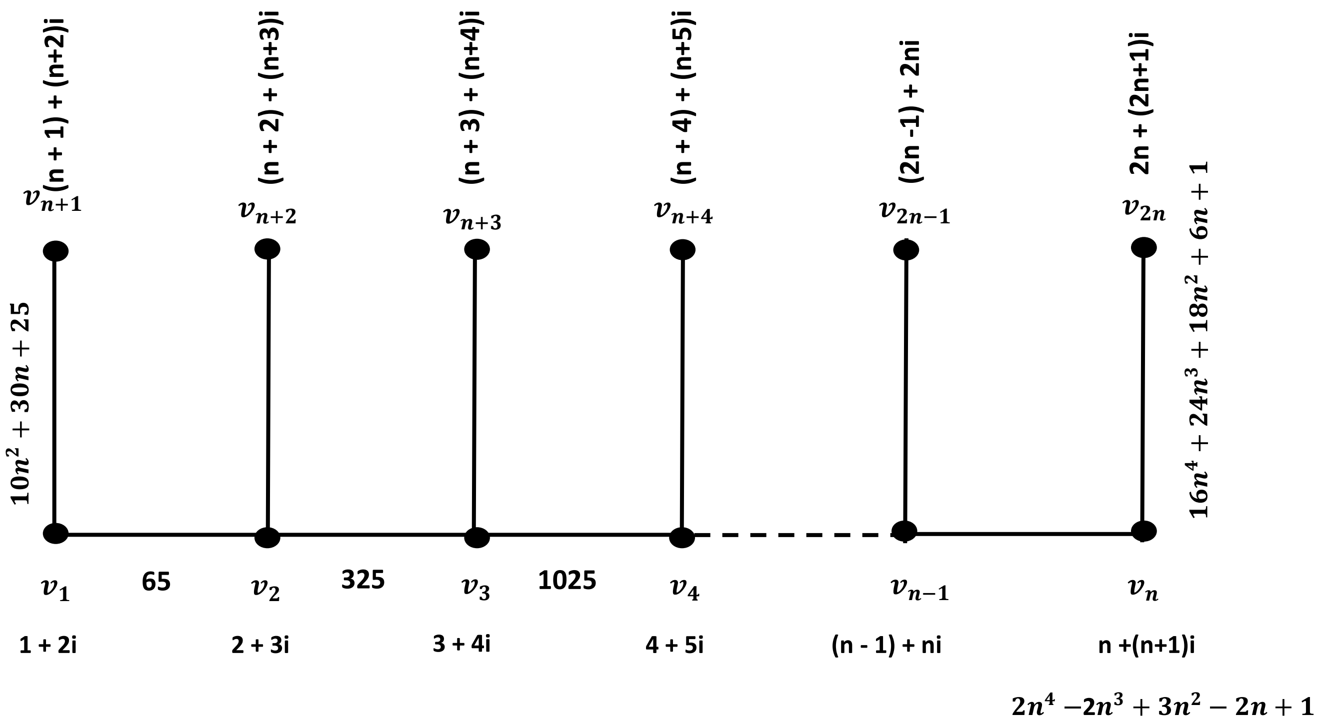


**Figure 3.12.** The Gaussian product odd antimagic labeling for comb graph with 2n vertices

**Theorem 3.9**: The key graph admits Gaussian product odd antimagic labeling.

**Proof**: Let $V = \{v_1, v_2, \ldots, v_{3n}\}$ be the vertices and be the edges

$E = \{\{v_k v_{k+1} \ / \ 1 \le k \le n-1\} \cup \{v_k v_{n+k} \ / \ 1 \le k \le n\} \cup \{v_{2n+k} \, v_{2n+k+1} \ / \ 1 \le k \le n-1\} \cup \{ v_{3n} \, v_{2n+1}\} \cup \{ v_{2n+1} \, v_1\}\}$ of the key graph.

Define a function $\psi : V \to \{ \alpha + i\beta \ / \ \alpha, \beta \in N \}$

such that $\psi(v_e) = e + i(e+ 1), 1 \le e \le 3n$.

Define the induced function $\rho$: $E \to N$ such that $\rho(v_j v_{j+1}) = |\psi(v_j)|^2 \ | \psi(v_{j+1})|^2$.

The edge labels are obtained as follows:

$\rho (v_k \, v_{k+1}) = 4k^4+16k^3+24k^2 +16k+5, \quad 1 \le k \le n-1$

$\rho (v_k \, v_{n+k}) = (2k^2+2k+1)(2n^2+2n+2k^2+2k+4nk+1) , 1 \le k \le n, n \in N$

$\rho (v_{2n+h} \, v_{2n+h+1} ) = 4(2n+h+1)^4+1 \ ,1 \le h \le n - 1, n \in N$

$\rho (v_{3n} v_{2n+1} ) = 144n^4+264n^3+170n^2+42n+5, n \in N$

$\rho (v_{2n+1} \, v_1 ) = 40n^2+60n+25, n \in N$

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$\rho\ (v_n, v_{2n}) = 16n^4 + 24n^3 + 18n^2 + 6n + 1.$

$\rho\ (v_{n-1}v_n) = 4n^4+1$

If $k \neq q$, $g \neq h$
$\rho\ (v_q v_{q+1}) = 4q^4+16q^3+24q^2+16q+5$
$\rho\ (v_k v_{k+1}) = 4k^4+16k^3+24k^2+16k+5$
$\rho\ (v_q v_{n+q}) = (2q^2+2q+1)(2n^2+2n+2q^2+2q+4nq+1)$
$\rho\ (v_{2n+g} v_{2n+g+1}) = 4(2n+g+1)^4+1$

This implies
$\rho\ (v_k v_{k+1}) \neq \rho\ (v_q v_{q+1}) \neq \rho\ (v_{n-1}v_n) \neq \rho\ (v_q v_{n+q}) \neq \rho\ (v_k v_{n+k}) \neq \rho\ (v_{2n+g} v_{2n+g+1}) \neq$

$\rho\ (v_{2n+h} v_{2n+h+1}) \neq \rho\ (v_n, v_{2n})$
Thus $\rho$ (E) = {(65, 325, 1025,...,$4n^4+1$

, $(10n^2+30n+25)$,$(26n^2+130n+169)$…, $(16n^4 + 24n^3 + 18n^2 + 6n + 1)$,…,

$(144n^4+264n^3+170n^2+42n+5)$,…,$(40n^2+60n+25)$}

in which all the elements are odd and distinct.

Thus, the Key graph admits Gaussian product odd antimagic labeling.

Example 3.13: The Gaussian product odd antimagic labeling for Key graph with 3n vertices is given in figure -3.1

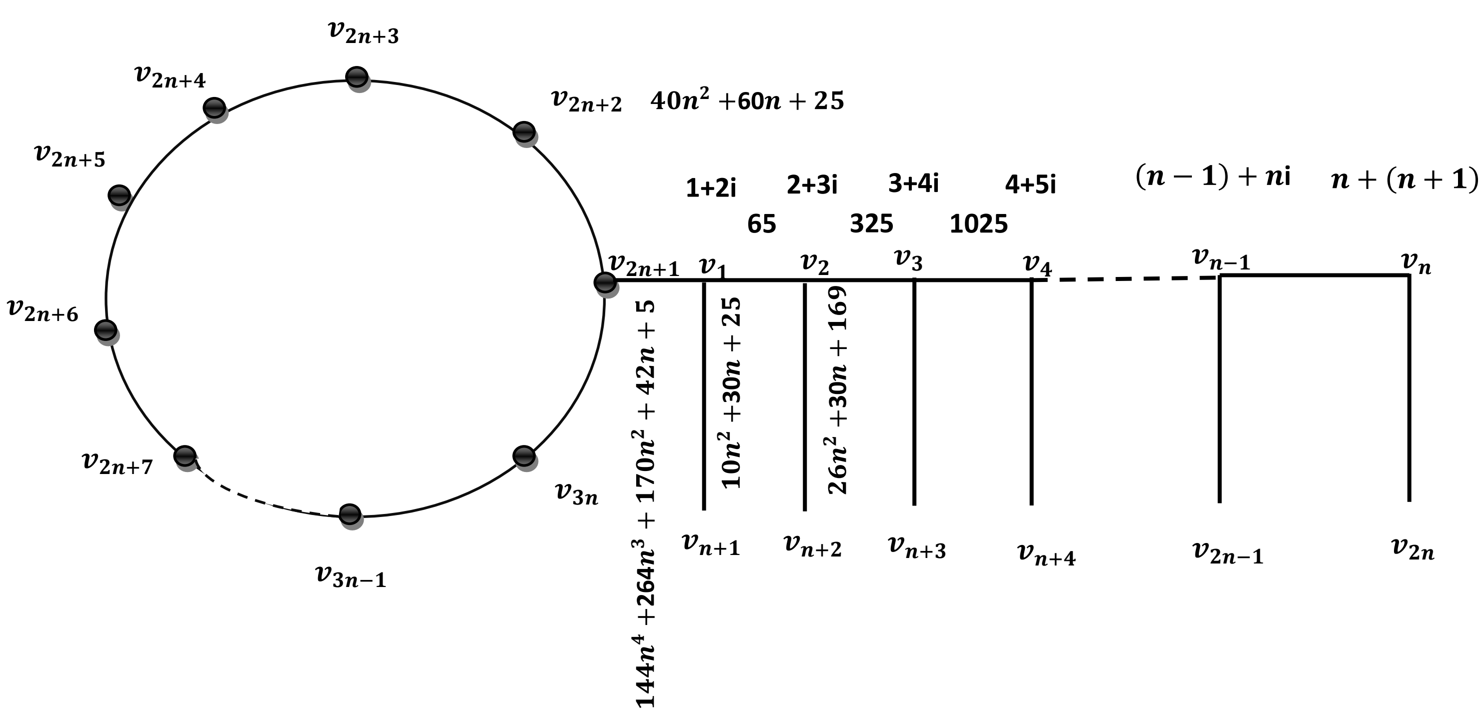


**Figure -3.13** .The Gaussian product odd antimagic labeling for Key graph with 3n vertices.

**Theorem 3.10**: The star graph admits Gaussian product odd antimagic labeling.

**Proof**: Let V={ $v_1, v_2, \ldots, v_n$ } be the vertices and

$E = \{ {}_n v_k / 1 \le k \le n-1 \}$ be the edges of the star graph.

Define a function ψ: V → { α + iβ / α, β ∈ N } and such that

$\psi(v_e) = e + i(e+1)$ , $1 \le e \le n$

Define the induced function ρ : E → N such that $\rho(v_j v_{j+1}) = |\psi(v_j)|^2 \; | \psi(v_{j+1})|^2$.

The edge labels are obtained as follows:

$\rho(v_k, v_n) = (2k^2+2k+1)(2n^2+2n+1)$ , $1 \le k \le n-2$

$\rho(v_{n-1}, v_n) = 4n^4+1$

Thus ρ (E) ={$(10n^2+10n+5), (26n^2+26n+13), \ldots, (4n^4+1)$} in which all the elements are odd and distinct. Therefore, The Star Graph admits Gaussian antimagic labeling.

**Example 3.14:**

The Gaussian productodd antimagic labeling for Star graph with $n$ vertices is

given in figure – 3.14

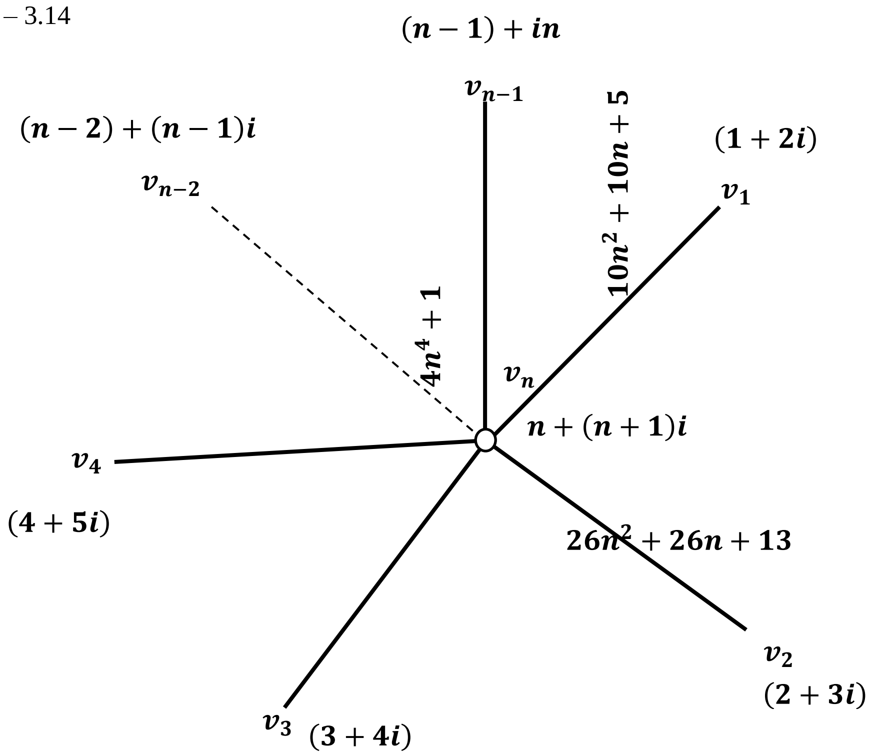


**Figure 3.14.** The Gaussian product odd antimagic labeling for Star graph with $n$ vertices.

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**GAUSSIAN PRODUCT EVEN ANTI MAGIC LABELING IN PATH RELATED GRAPHS**

**Theorem 3.11**: The path graph $P_n$, $> 1$ admits Gaussian product even antimagiclabelling

. **Proof**: Let V= $\{v_1, v_2, \ldots, v_n\}$ be the vertices and $E = \{v_i v_{i+1} / 1 \leq i \leq n-1\}$ be the edges of the path $P_n$.

Define a function ψ: V → { α + iβ / α, β ∈ N } such that

$\psi(v_e) = e + i(e+2)$

Define the induced function

$\rho: E \rightarrow N$ such that $\rho(v_j v_{j+1}) = |\psi(v_j)|^2 \; | \psi(v_{j+1})|^2$

The edge labels are obtained as follows:

$\rho\,(v_j v_{j+1}) = |\psi(v_j)|^2 \, |\psi(v_{j+1})|^2, 1 \leq j \leq n-1$

$\rho\,(v_k v_{k+1}) = 4k^4+24k^3+60k^2+72k+40, \quad 1 \leq k \leq n-1$

$\rho\,(v_{n-1} v_n) = 4n^4+8n^3+12n^2+8n+8$

If $q \neq k$, $\rho\,(v_q v_{q+1}) = 4q^4+24q^3+60q^2+72q+40$
This implies $\rho\,(v_k v_{k+1}) \neq \rho\,(v_q v_{q+1})$
$\rho\,(v_k v_{k+1}) \neq \rho\,(v_q v_{q+1}) \neq \rho\,(v_{n-1} v_n)$

Thus $\rho\,(E) = \{200, 680, 1768, 3848, 7400, \ldots, (4n^4+8n^3+12n^2+8n+8)\}$ in which all the elements are even and distinct.

Therefore, the path graph $P_n$, is a Gaussian product even antimagic graph.

Example 3.15: The Gaussian even antimagiclabeling for $P_n$ with $n$ vertices is shown in figure – 3.15.

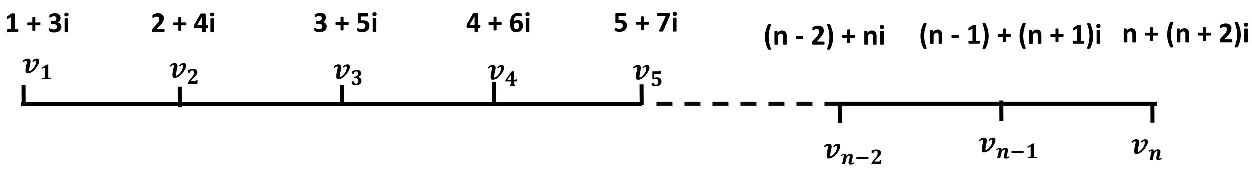


**Figure – 3.15.** The Gaussian even antimagiclabeling for $P_n$ with $n$ vertices.

**Theorem 3.12:**

The $Y_n$ − tree admits Gaussian product even antimagic labeling.

**Proof:** Let V=$\{v_1, v_2, \ldots, v_n\}$ be the vertices and $E = \{\{v_i v_{i+1} / 1 \leq i \leq n-2\} \cup \{v_{n-2} v_n\}\}$ be the edges of the $Y_n$ tree.

Define a function ψ : V → { α + iβ / α, β ∈ N } such that $\psi(v_e) = e + i(e+2)$, $1 \leq e \leq n$

Define the induced function

$\rho: E \rightarrow N$ such that $\rho(v_j v_{j+1}) = |\psi(v_j)|^2 \; | \psi(v_{j+1})|^2$

The edge labels are obtained as follows:

$\rho\,(v_j v_{j+1}) = |\psi(v_j)|^2\ |\psi(v_{j+1})|^2$, $1 \le j \le n-2$

$\rho\,(v_k v_{k+1}) = 4k^4+24k^3+60k^2+72k+40$, $1 \le k \le n-2$

$\rho\,(v_{n-2} v_n) = 4n^4+16$,

If $q \neq k$, $\rho\,(v_q v_{q+1}) = 4q^4+24q^3+60q^2+72q+40$
This implies $\rho\,(v_k v_{k+1}) \neq \rho\,(v_q v_{q+1})$
$\rho\,(v_k v_{k+1}) \neq \rho\,(v_q v_{q+1}) \neq \rho\,(v_{n-1} v_n) \neq \rho\,(v_{n-2} v_n)$

Thus $\rho\,(E) = \{200, 680, 1768, \ldots, (4n^4+16)\}$ in which all the elements are even and distinct. Therefore, the $Y_n$ – tree is a Gaussian product even antimagic graph.

Example 3.16: The Gaussian even antimagic labeling for $Y_n$− tree with $n$ vertices is shown in figure – 3.16

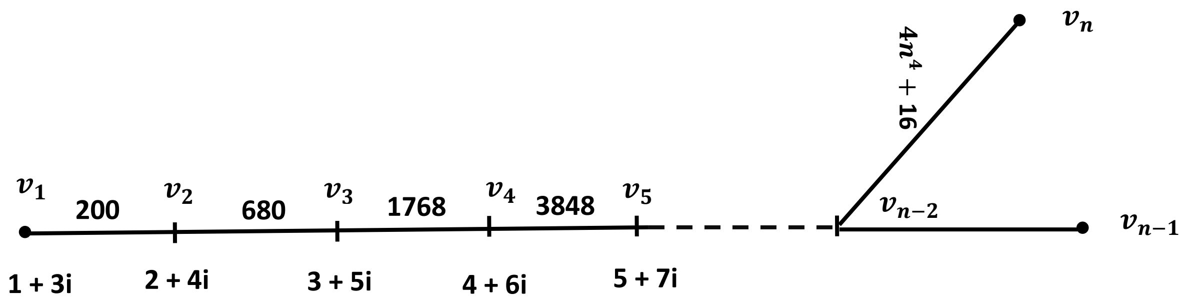


**Figure – 3.16.** The Gaussian even antimagic labeling for $Y_n$− tree with $n$ vertices.

**Theorem 3.13:** The comb graph of odd length admits Gaussian product even antimagic labeling.

**Proof:** Let $\{v_1, v_2, \ldots, v_{2n}\}$ be the vertices of comb graph.

Let $E = \{\{v_i v_{i+1} / 1 \le i \le n-1\} \cup \{v_i v_{n+i} / 1 \le i \le n\}\}$ be the edges of the comb graph of odd length.

Define a function $\psi : V \rightarrow \{\alpha + i\beta \,/\, \alpha, \beta \in N\}$ such that $\psi(v_e) = e + i(e+2)$

Define the induced function $\rho : E \rightarrow N$ such that $\rho(v_j v_{j+1}) = |\psi(v_j)|^2\ |\psi(v_{j+1})|^2$.

The edge labels are obtained as follows:

$\rho\,(v_j v_{j+1}) = |\psi(v_j)|^2\ |\psi(v_{j+1})|^2$, $1 \le j \le n-1$

$\rho\,(v_k v_{k+1}) = 4k^4+24k^3+60k^2+72k+40$, $1 \le k \le n-1$

$\rho\,(v_{n-1} v_n) = 4n^4+8n^3+12n^2+8n+8$, $\rho\,(v_{n+k} v_k) = 4(k^2+2k+2)(k^2+2k+2+n^2+2n+2nk)$

$\rho\,(v_{2n} v_n) = 16n^4+48n^3+72n^2+48n+16$

If $q \neq k \neq g$
$\rho\,(v_q v_{q+1}) = 4q^4+24q^3+60q^2+72q+40$
$\rho\,(v_{n+g} v_g) = 4(g^2+2g+2)(g^2+2g+2+g^2+2g+2ng)$

This implies $\rho\,(v_k v_{k+1}) \neq \rho\,(v_q v_{q+1})$, $\rho\,(v_{n+k} v_k) \neq \rho\,(v_{n+g} v_g)$
$\rho\,(v_k v_{k+1}) \neq \rho\,(v_q v_{q+1}) \neq \rho\,(v_{n+k} v_k) \neq \rho\,(v_{n+g} v_g) \neq \rho\,(v_{2n} v_n)$

Thus ρ $(E)$ ={200, 680,1768, …,$(4n^4+8n^3+12n^2+8n+8)$,$(20n^2+80n+100)$,$(40n^2+240n+400)$,$(68n^2+544n+1156)$,…, $(16n^4+48n^3+72n^2+48n+16)$}

in which all the elements are even and distinct.

Therefore, the comb graph is a Gaussian product even antimagic graph.

Example 3.17: The Gaussian product even antimagic labeling for comb graph with $2n$ vertices is shown in figure-3.17

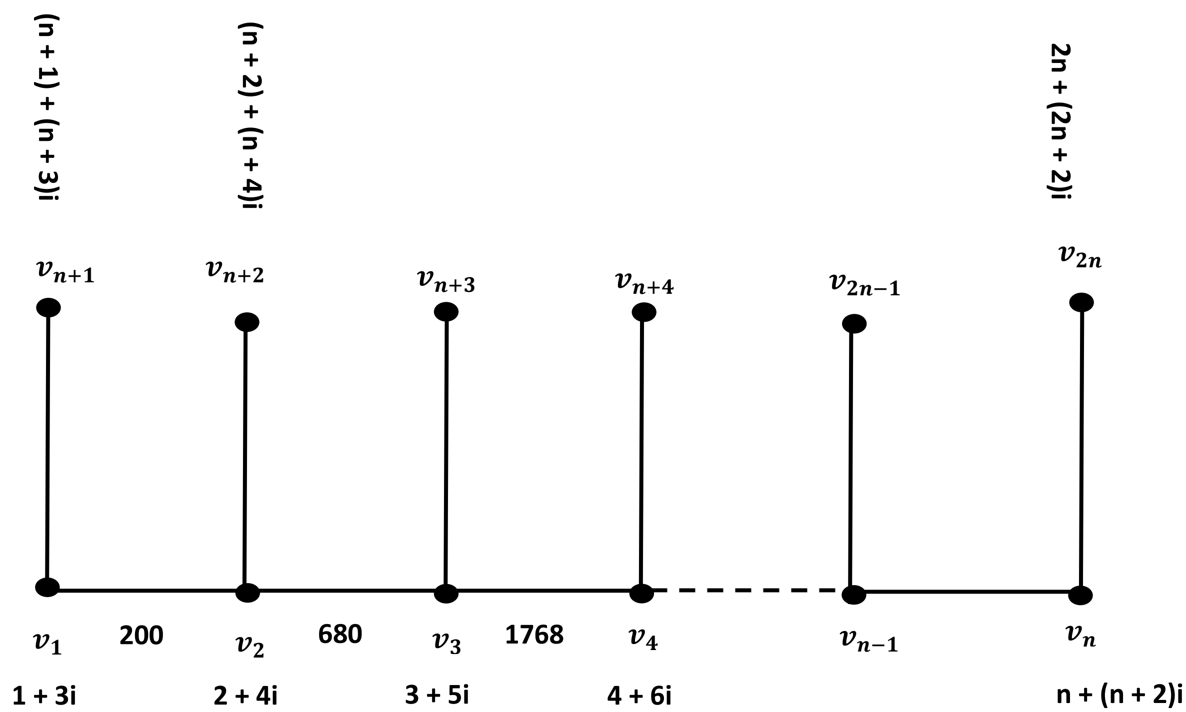


**Figure 3.17.** The Gaussian product even antimagic labeling for comb graph with $2n$ vertices.

**Theorem 3.14:** The key graph admits Gaussian product even antimagic labeling.

Proof: Let $V = \{v_1, v_2, \ldots, v_{3n}\}$ be the vertices and be the edges

E = {$\{v_h v_{h+1} \ / \ 1 \le h \le n-1\} \cup \{v_h v_{n+h} \ / \ 1 \le h \le n\} \cup \{v_{2n+h}\ v_{2n+h+1} \ / \ 1 \le h \le n-1\} \cup \{v_{3n}\ v_{2n+1}\} \cup \{v_{2n+1}\ v_1\}$} of the key graph.

Define a function ψ : V → { α + iβ / α, β ∈ N } such that $\psi(v_e)$ = e+i(e+2), 1 ≤ e ≤3n.

Define the induced function ρ: E ⟶ N such that $\rho(v_j v_{j+1}) = |\psi(v_j)|^2 \ |\psi(v_{j+1})|^2$.

The edge labels are obtained as follows:

ρ $(v_k v_{k+1})$ =$4k^4+24k^3+60k^2+72k+40$, $1 \le k \le n-1$,

ρ $(v_{n-1}\ v_n)$ =$4n^4+8n^3+12n^2+8n+8$

ρ $(v_{n+k\ \ k})$ = $4(k^2+2k+2)(k^2+2k+2+n^2+2n+2nk)$

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ρ ($v_{2n+k}$ $v_{2n+k+1}$ ) =$(8n^2+8n+2k^2+4k+8nk+4)^2+(8n^2+8n+2k^2+4k+8nk+4)(8n+4k+6)$ ,$1 \leq h \leq (n-1)$,

ρ ($v_{3n}$ $v_{2n+1}$ ) = $144n^4+384n^3+404n^2+184n+40$, n∈ N

ρ ($v_{2n+1}$ $v_1$ ) =$80n^2+160n+100$, n∈ N

ρ ($v_{2n}$ $v_n$)= $16n^4+48n^3+72n^2+48n+16$

If $k\neq q$, $g\neq k$ $k\neq h$
ρ ($v_q$ $v_{q+1}$ ) = $4q^4+24q^3+60q^2+72q+40$
ρ ($v_{n+g}v_g$) = $4(g^2+2g+2)(g^2+2g+2+n^2+2n+2ng)$
ρ ($v_{2n+h}$ $v_{2n+h+1}$ ) =$(8n^2+8n+2h^2+4h+8nh+4)^2+(8n^2+8n+2h^2+4h+8nh+4)(8n+4h+6)$
This implies
ρ ($v_k$ $v_{k+1}$ ) ≠ $\tau$($v_q$ $v_{q+1}$ ), ρ ($v_{n+k}$ $_k$) ≠ $\tau$ ($v_{n+g}v_g$) , ρ ($v_{2n+k}$ $v_{2n+k+1}$ )≠ ρ ($v_{2n+h}$ $v_{2n+h+1}$ )

ρ ($v_k$ $v_{k+1}$ ) ≠ ρ ($v_q$ $v_{q+1}$ )≠ ρ ($v_{n+k}$ $_k$) ≠ ρ ($v_{n+g}v_g$) ≠ ρ ($v_{2n+k}$ $v_{2n+k+1}$ )≠ ρ ($v_{2n+h}$ $v_{2n+h+1}$ )≠
ρ ($v_{3n}$ $v_{2n+1}$ )≠ ρ ($v_{2n+1}$ $v_1$ ) ≠ ρ ($v_{2n}$ $v_n$)
Thus ρ (E) = {(200, 680, 1768,…, $(4n^4+8n^3+12n^2+8n+8)$,$(20n^2 + 80n + 100)$,$(40n^2+240n+400)$,… , $(16n^4+48n^3+72n^2+48n+16)$,… ,$(144n^4+384n^3+404n^2+184n+40)$,}

in which all the elements are even and distinct.

Thus, the Key graph admits Gaussian producteven antimagic labeling.

Example 3.18: The Gaussianevenantimagic labeling for Key graph with 3n vertices is given in figure -3.18

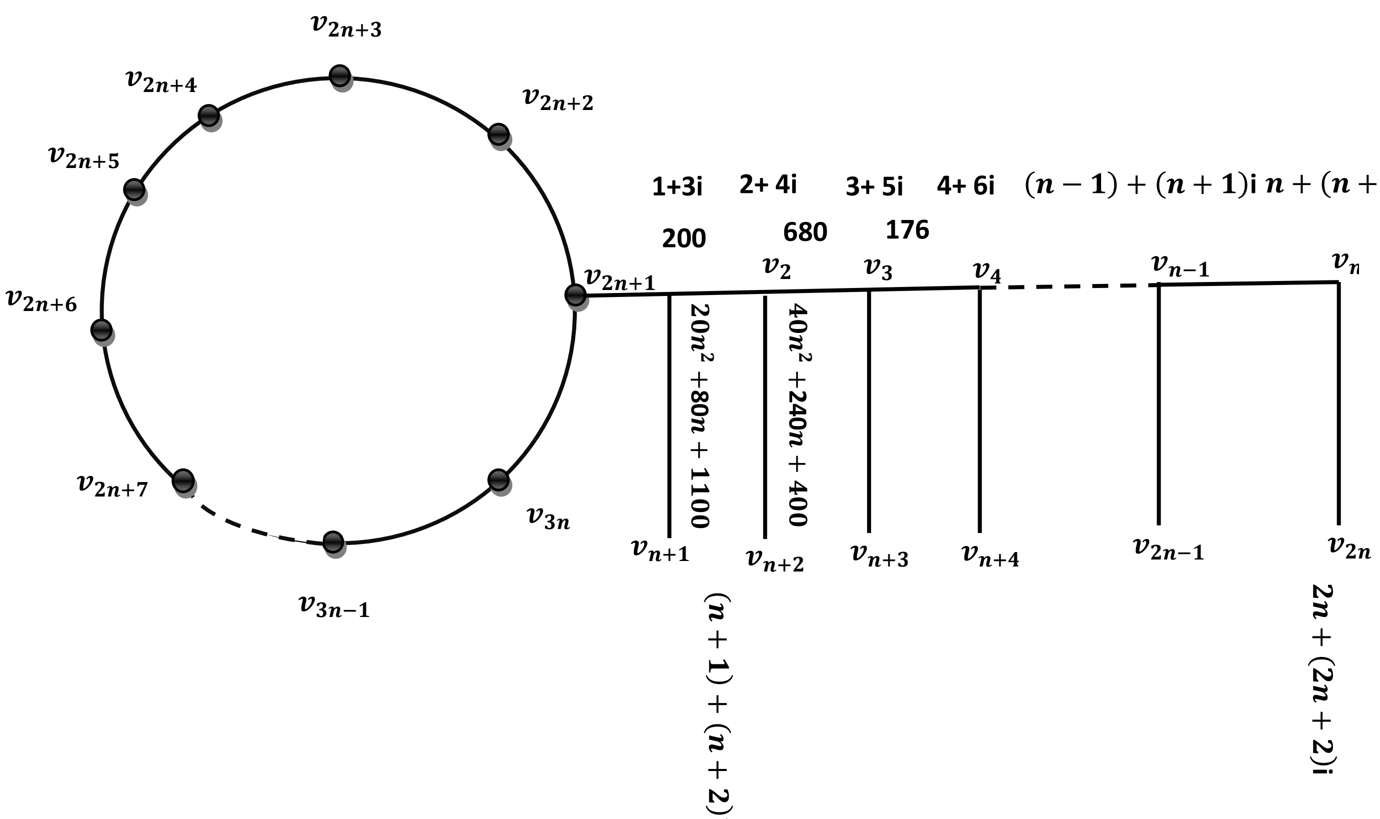


**Figure 3.18.** The Gaussian even antimagic labeling for Key graph with 3n vertices.

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**Theorem 3.15:** The star graph admits Gaussian product even antimagic labeling.

Proof: Let V={$v_1$, $v_2$, …,$v_n$} be the vertices and E = {$v_n v_i$ / $1 \le i \le n-1$} be the edges of the star graph.

Define a function ψ : V → { α + iβ / α, β ∈ N }and such that $\psi(v_e)$ = e+i(e+2) , 1≤e≤ n

Define the induced function τ : E → N such that $\rho(v_j v_{j+1}) = |\psi(v_j)|^2 \; |\psi(v_{j+1})|^2$.

The edge labels are obtained as follows:

ρ ($v_n$ $v_k$) = $4(k^2+2k+2)\ (n^2+2n+2)$,1≤k ≤ n-2 , ρ ($v_n$ $v_{n-1}$)= $4n^4+8n^3+12n^2+8n+8$

Thus ρ (E) ={$20n^2+40n+40$, $40n^2+80n+80$, …,($4n^4+8n^3+12n^2+8n+8$) } in which all the elements are even distinct.

Thus, the Star Graph admits Gaussian product even antimagic labeling.

Example 3.19: The Gaussian even antimagic labeling for Star graph with $n$ vertices is given in figure – 3.19.

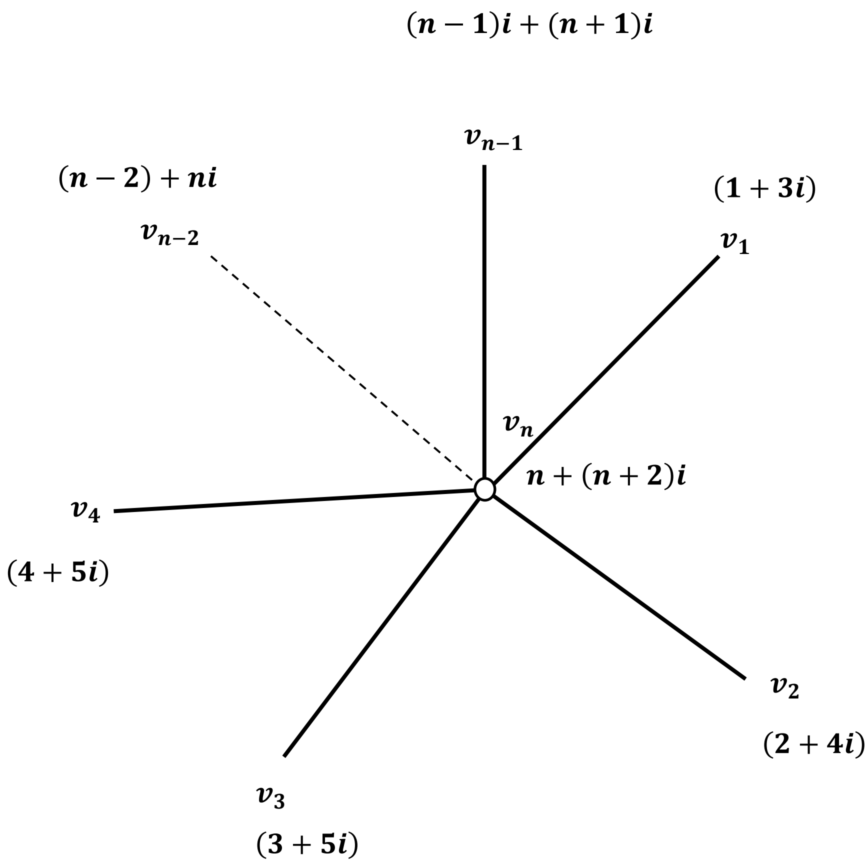


**Figure 3.19.** The Gaussian even antimagic labeling for Star graph with $n$ vertices.

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