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Common Fixed Points for Four Mappings in N-Cone Metric Space Satisfying Generalised Weak Contractive condition

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Abstract:- In this paper, we prove common fixed points of four mappings satisfying a generalised weak contractive condition in an N-cone metric space. Our results are significant extension and generalisations of many recent results in the literature. In addition, we provide some illustrative examples to highlight the realized improvements.

***Keywords**: Cauchy Sequence, Weak Annihilator, Commuting Mapping, Cone Metric Space, N- Cone Metric Space.*

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1. Introduction

In 1922, S. Banach developed the fixed point theorem for contraction mapping in metric space[8]. Subsequently, several fixed point theorems have been demonstrated by various writers, and this theory has been widely generalised. In 1982, S. Sessa introduced weakly commuting maps [9], and Jungck, G. [10] defined compatibility. Jungck and Rhoades [11] first proposed the concept of weakly compatible maps because they claimed that although compatible maps were weakly compatible, the contrary was not true. Cone metric space was established by Huang and Zhang [1]. They achieved certain fixed point solutions under specific contractive circumstances by substituting a complete normed space for the set of real numbers in metric space. The notion of cone metric space over Banach algebras was first presented by Xu and Liu [2]. By eliminating the whole normed space, it is distinguished from metric space. It is shown that, with regard to the existence of fixed points of mappings, cone metric space over Banach algebras is not comparable to metric space. Numerous noteworthy and interesting discoveries on cone metric space over Banach algebras have been made by authors (see [3, 5, 6, 12, 13, 15]). N-cone metric space was established by Malviya and Fisher [7], who also proved asymptotically regular maps for fixed-point theorems. In 2015, Jerolina, Geeta, and Neeraj [4] demonstrated unique fixed-point theorems for contractive mappings in N-cone metric spaces. There has been ongoing study on common fixed points of mappings under contractive type situations.

In this work, we improve and generalize previous fixed-point results published in the literature by developing common fixed-point theorems for two pairs of weakly compatible mappings in N-cone metric space.

Definition 1.1: [1] Let E be a real Banach Space and P a subset of E. The set P is called a cone if and only if

(i) P is closed, non-empty and $P \neq 0$;

(ii) $a, b \in R$, $a, b \geq 0$, $x, y \in P \Rightarrow ax + by \in K$;

(iii) $P \cap (-P) = 0$.

For a given cone $P \subset E$, let $\leq$ be a partial ordering with respect to P written as $x \leq y$ if and only if $y - x \in P$. Also let $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x << y$ will stand for $y - x \in$ Int. P, where Int. P denotes the interior of the set K.

Definition 1.2: [1] Let X be a non-empty set. Let that the mapping $d: X \times X \rightarrow E$ satisfies:

(a) $0 < d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(b) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(c) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then (X, d) is called a cone metric space and d is called a cone metric on X.

Example 1.3: [1] Let $E=R^2$, $P= \{(x, y)\in E: x, y\geq 0\}$, $E = R$ and $d: X \times X \to E$ defined by $d(x, y) = (|x-y|, a|x-y|)$, where $a\geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition 1.4: [7] Let X be a non-empty set. An N-cone metric on X is a function $N : X^3 \to E,$ that satisfies the following conditions for all $x, y, z, a \in X$

(i) $N(x, y, z) \geq 0;$

(ii) $N(x, y, z)=0$ if and only if $x = y = z;$

(iii) $N(x,y, z) \leq N(x, x, a) + N(y, y, a) + N(z, z, a).$

Then, the function N is called an N-cone metric and the pair (X, N) is called an N-cone metric space.

Proposition 1.5: [7] If (X, N) is an N-cone metric space for all $x, y, z \in X$ we have $N(x, x, y)=N(y, y, x)$.

Definition 1.6: [7] Let (X, N) be an N-cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $0 << c$ there is N_0 such that for all $n > N_0, N(x_n, x_n, x) << c$ then $\{x_n\}$ is said to be convergent. $\{x_n\}$ converges to x and x is the limit of $\{x_n\}$. We denote this by $x_n \to x$ as ($n \to \infty$).

Lemma 1.7: [7] Let (X, N) be an N-cone metric space and P be a normal cone with normal constant k. Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x and $\{x_n\}$ also converges to y then x = y. That is the limit of $\{x_n\}$, if exists, is unique.

Definition 1.8: [7] Let (X, N) be an N-cone metric space and $\{x_n\}$ be a sequence in X. If for any $c \in E$ with $0 << c$ there is N_0 such that for $m, n > N_0, N(x_n, x_n, x_m) << c$ then $\{x_n\}$ is called a Cauchy sequence in X.

Definition 1.9: [7] Let (X, N) be an N-cone metric space. If every Cauchy sequence in X is convergent in X, then X is called a complete N-cone metric space.

Lemma 1.10: [7] Let (X, N) be an N-cone metric space and $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x, then $\{x_n\}$ is a Cauchy sequence.

Definition 1.11: [7] Let (X, N) and (X', N') be N-cone metric spaces. Then, a function $f : X \to X'$ is said to be continuous at a point $x \in X$ if and only if it is sequentially continuous at x, that is, whenever $\{x_n\}$ is convergent to x we have $\{fx_n\}$ is convergent to fx.

Lemma 1.12: [7] Let (X, N) be an N-cone metric space and P be a normal cone with normal constant k. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X and suppose that $x_n \to x, y_n \to y$ as $n \to \infty$. Then $N(x_n, x_n, y_n) \to N(x, x, y)$ as $n \to \infty$.

Remark 1.13: [7] If $x_n \to x$ in an N-cone metric space X, then every subsequence of $\{x_n\}$ converges to x.

Proposition 1.14: [7] Let (X, N) be an N-cone metric space and P be a cone in a real Banach space E. If $u \leq v, v << w$ then $u << w$.

Definition 1.15: [4] Let (X, N) be an N-cone metric space. A map $f : X \to X$ is said to be a contractive mapping if there exists a constant $0 \leq k < 1$ such that $N(fx, fy, fy) \leq kN(x, x, y)$ for all $x, y \in X$.

Lemma 1.16: [7] Let (X, N) be an N-cone metric space and P be an N-cone in a real Banach space E and $k_1, k_2, k_3, k_4, k > 0.$ If $x_n \to x, y_n \to y, z_n \to z$ and $p_n \to p$ in X and

$$ka \leq k_1N(x_n, x_n, x) + k_2N(y_n, y_n, y)+k_3N(z_n, z_n, z) + k_4N(p_n, p_n, p)$$

, then $a = 0$.

Definition 1.17: [14] Let A and S be two self-mappings on a set X. If $Ax = Sx$ for some $x \in X$, then x is called coincidence point of A and S.

Definition 1.18: [15] Let A and S be two self-mappings on a set X. Mappings A and S are said to be commuting if $ASx = SAx$ for all $x \in X$.

2. Fixed Point Theorems

In this part, we prove various fixed-point theorems for two weakly compatible mappings and also introduce the notion of control function in N-cone metric space.

We start with some definitions.

Definition 2.1: Let (X, N) be an N-cone metric space. A pair (f, g) of selfmaps of X is said to be weakly increasing if $fx \leq gfx$ and $gx \leq fgx$ for all $x \in X$.

Definition 2.2: Let (X, N) be an N-cone metric space and f and g be two selfmaps on X. A pair (f, g) of selfmaps of X is said to be partially weakly increasing if $fx \leq gfx$ for all $x \in X$.

Remark 2.3: A pair (f, g) is weakly increasing if and only if the pairs (f, g) and (g, f) are partially weakly increasing.

Definition 2.4: Let (X, N) be an N-cone metric space. A mapping f is called weak annihilator of g if $fgx \leq x$ for all $x \in X$.

Definition 2.5: Let (X, N) be N-cone metric space. A mapping f is dominating if $x \leq fx$ for all $x \in X$.

Definition 2.6: The control functions ψ and ϕ are defined as

(i) $\psi : E \to E$ is a continuous nondecreasing function with $\psi(t) = 0$ if and only if $t = 0$

(ii) $\phi: E \to E$ is a lower semicontinuous function with $\phi(t)=0$ if and only if $t=0$.

Definition 2.7: Let (X, N) be an N-cone metric space, define $d_N : X \times X \to E$ by

$$d_N(x, y) = N(x, y, y) + N(y, x, x).$$

Then (X, d_N) is N-cone metric space.

Now, we have the following fixed point results.

Theorem 2.8: Let (X, d_N) be an N-complete cone metric space. Let f, g, S and T be self-maps on X, (T, f) and (S, g) be partially weakly increasing with $f(x) \subseteq T(X)$ and $g(X) \subseteq S(X)$, dominating maps f and g are weakly annihilators of T and S, respectively. Suppose that there exist control functions ψ and φ such that for every two comparable elements $x, y \in X$,

$$\psi(d_N(fx, gy)) \le \psi(M(x,y)) - \varphi(M(x,y)) \tag{2.1}$$

is satisfied where

$$M(x,y) = \max\left\{ \begin{matrix} d_N(Sx,Ty), d_N(fx,Sx), d_N(gy,Ty), \\ \dfrac{d_N(Sx,gy)+d_N(fx,Ty)}{2} \end{matrix} \right\}.$$

If for a non-decreasing sequence $\{x_n\}$ with $x_n \le y_n$ for all n and $y_n \to u$ implies that $x_n \le u$ and either

(i) $\{f, g\}$ are compatible, f or S is continuous and $\{g, T\}$ are weakly compatible or

(ii) $\{g, T\}$ are compatible, g or T is continuous and $\{f, S\}$ are weakly compatible.

then f, g, S and T have a common fixed point. Moreover, the set of common fixed points of f, g, S and T is well ordered if and only if f, g, S and T have one and only one common fixed point.

Proof: Let x_0 be an arbitrary point in X. We construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_{2n-1} = fx_{2n-2} = Tx_{2n-1}$, and $y_{2n} = gx_{2n-2} = Sx_{2n}$. By given assumptions, $x_{2n-2} \le fx_{2n-2} = Tx_{2n-1} \le fTx_{2n-1} \le x_{2n-1}$ and $x_{2n-1} \le gx_{2n-1} = Sx_{2n} \le Sgx_{2n} \le x_{2n}$. Thus for $n \ge 1$ we have $x_n \le x_{n+1}$. we suppose that $d(y_{2n}, y_{2n+1}) > 0$ for every n. If not then $y_{2n} = y_{2n+1}$, for some n. From (2.1), we obtain

$$\begin{aligned} \psi(d_N(y_{2n+1}, y_{2n+2})) &= \psi(d_N(fx_{2n}, gx_{2n+1})) \\ &\le \psi(M(x_{2n}, x_{2n+1})) - \phi(M(x_{2n}, x_{2n+1})), \end{aligned} \tag{2.2}$$

where

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$$M(x_{2n}, x_{2n+1})$$
$$= \max\left\{\begin{array}{l} d_N(Sx_{2n}, Tx_{2n+1}), d_N(fx_{2n}, Sx_{2n}), \\ d_N(gx_{2n+1}, Tx_{2n+1}), \\ \dfrac{d_N(Sx_{2n}, gx_{2n+1}) + d_N(fx_{2n}, Tx_{2n+1})}{2} \end{array}\right\}$$
$$= \max\left\{\begin{array}{l} d_N(y_{2n}, y_{2n+1}), d_N(y_{2n+1}, y_{2n}), d_N(y_{2n+2}, y_{2n+1}), \\ \dfrac{d_N(y_{2n}, y_{2n+2}) + d_N(y_{2n+1}, y_{2n+1})}{2} \end{array}\right\}$$
$$= \max\left\{0, 0, d_N(y_{2n+2}, y_{2n+1}), \frac{d_N(y_{2n+1}, y_{2n+2})}{2}\right\}$$
$$= d_N(y_{2n+1}, y_{2n+2})$$

Hence, $\psi(d_N(y_{2n+1}, y_{2n+2})) \leq \psi(d_N(y_{2n+1}, y_{2n+2})) - \phi(d_N(y_{2n+1}, y_{2n+2}))$, implies that $\phi(d_N(y_{2n+1}, y_{2n+2})) = 0.$ As, $\phi(t) = 0$ if and only if $t = 0, y_{2n+1} = y_{2n+2}$. Following the similar arguments, we obtain $y_{2n+2} = y_{2n+3}$ and so on. Thus $\{y_n\}$ becomes a constant sequence and y_{2n} is the common fixed point of f, g, S and T.

Take, $d(y_{2n}, y_{2n+1}) > 0$ for each n. Since x_{2n} and x_{2n+1} are comparable, from (2.1) we obtain

$$\begin{aligned} &\psi(d_N(y_{2n+2}, y_{2n+1})) = \psi(d_N(y_{2n+1}, y_{2n+2})) \\ &= \psi(d_N(fx_{2n}, gx_{2n+1})) \\ &\leq \psi(M(x_{2n}, x_{2n+1})) - \phi(M(x_{2n}, x_{2n+1})) \\ &\leq \psi(M(x_{2n}, x_{2n+1})). \end{aligned}$$

Therefore

$$d_N(y_{2n+1}, y_{2n+2}) \leq M(x_{2n}, x_{2n+1}), \tag{2.3}$$

where

$$M(x_{2n}, x_{2n+1})$$
$$= \max\left\{\begin{array}{l} d_N(Sx_{2n}, Tx_{2n+1}), d(fx_{2n}, Sx_{2n}), \\ d_N(gx_{2n+1}, Tx_{2n+1}), \\ \frac{d_N(Sx_{2n}, gx_{2n+1}) + d_N(fx_{2n}, Tx_{2n+1})}{2} \end{array}\right\}$$
$$= \max\left\{\begin{array}{l} d_N(y_{2n}, y_{2n+1}), d_N(y_{2n+1}, y_{2n}), d_N(y_{2n+2}, y_{2n+1}), \\ \frac{d_N(y_{2n}, y_{2n+2}) + d_N(y_{2n+1}, y_{2n+1})}{2} \end{array}\right\}$$
$$\le \max\left\{\begin{array}{l} d_N(y_{2n+1}, y_{2n}), d_N(y_{2n+2}, y_{2n+1}), \\ \frac{d_N(y_{2n}, y_{2n+1}) + d_N(y_{2n+1}, y_{2n+2})}{2} \end{array}\right\}$$
$$= \max\{d_N(y_{2n+1}, y_{2n}), d_N(y_{2n+2}, y_{2n+1})\}.$$

If $\max\{d_N(y_{2n+1}, y_{2n}), d_N(y_{2n+2}, y_{2n+1})\} = d_N(y_{2n+2}, y_{2n+1})$, then (2.3) gives that $M(x_{2n}, x_{2n+1}) = d_N(y_{2n+2}, y_{2n+1})$, and

$$\psi(d_N(y_{2n+2}, y_{2n+1}))$$
$$= \psi(M(x_{2n}, x_{2n+1})) - \phi(M(x_{2n}, x_{2n+1}))$$
$$= \psi(d_N(y_{2n+2}, y_{2n+1})) - \phi(d_N(y_{2n+2}, y_{2n+1})),$$ gives a contradiction.

Hence $d_N(y_{2n+2}, y_{2n+1}) \le d_N(y_{2n+1}, y_{2n}).$ Moreover $M(x_{2n}, x_{2n+1}) \le d_N(y_{2n}, y_{2n+1}).$ But since

$$M(x_{2n}, x_{2n+1}) \ge \max\{d_N(y_{2n}, y_{2n+1}), d_N(y_{2n+2}, y_{2n+1})\}$$
$$= M(x_{2n}, x_{2n+1})$$
.

Similarly, $d_N(y_{2n+1}, y_{2n+2}) \le d_N(y_{2n+2}, y_{2n+1}).$ Thus the sequence $\{d_N(y_{2n+1}, y_{2n})\}$ is nonincreasing and so there exists $\lim_{n\to\infty} d_N(y_{2n+1}, y_{2n}) = L \ge 0.$ Suppose that $L > 0$. Then

$\psi(d_N(y_{2n+2}, y_{2n+1})) \le \psi(M(x_{2n+1}, x_{2n})) - \phi(M(x_{2n+1}, y_{2n})),$ and lower semicontinuity of ϕ gives

$$\limsup_{n\to\infty} \psi(d_N(y_{2n+2}, y_{2n+1}))$$
$$\le \limsup_{n\to\infty} \psi(M(x_{2n+1}, x_{2n})) - \liminf_{n\to\infty} \phi(M(x_{2n+1}, x_{2n}))$$

which implies that $\psi(L) \le \psi(L) - \phi(L),$ a contradiction. Therefore, $L = 0$. So, we conclude that

$$\lim_{n\to\infty} d_N(y_{2n+1}, y_{2n}) = 0 \tag{2.4}$$

Now we shall show that $\{y_n\}$ is a Cauchy sequence. For this it is sufficient to show that $\{y_{2n}\}$ is Cauchy in X. If not, there is $\varepsilon > 0$ and there exist even integers $2n_k$ and $2m_k$ with $2m_k > 2n_k > k$ such that

$$d_N(y_{2m_k}, y_{2n_k}) \ge \varepsilon \tag{2.5}$$

and $d_N(y_{2m_k-2}, y_{2n_k}) < \varepsilon.$ Since

$$\varepsilon \le d_N(y_{2m_k}, y_{2n_k}) \le d_N(y_{2n_k}, y_{2m_k-2})$$
$$+d_N(y_{2m_k-1}, y_{2m_k-2}) + d_N(y_{2m_k-1}, y_{2m_k})$$

Now (2.4) and (2.5) implies that

$$\lim_{n\to\infty} d_N(y_{2m_k}, y_{2n_k}) = \varepsilon \tag{2.6}$$

Also (2.4) and inequality $d_N(y_{2m_k}, y_{2n_k}) \le d_N(y_{2m_k}, y_{2m_k-1}) + d_N(y_{2m_k-1}, y_{2n_k})$ gives that $\varepsilon \le \lim_{k\to\infty} d_N(y_{2m_k-1}, y_{2n_k})$, while (2.4) and inequality $d_N(y_{2m_k-1}, y_{2n_k}) \le d_N(y_{2m_k-1}, y_{2m_k}) + d_N(y_{2m_k}, y_{2n_k})$ yields $\lim_{k\to\infty} d_N(y_{2m_k-1}, y_{2n_k}) \le \varepsilon$ and hence

$$\lim_{k\to\infty} d_N(y_{2m_k-1}, y_{2n_k}) = \varepsilon \tag{2.7}$$

As

$$M(x_{2n_k}, x_{2m_k-1})$$
$$= \max\left\{\begin{array}{l} d_N(Sx_{2n_k}, Tx_{2m_k-1}), d_N(fx_{2n_k}, Sx_{2n_k}), \\ d_N(gx_{2m_k-1}, Tx_{2m_k-1}), \\ \dfrac{d_N(Sx_{2n_k}, gx_{2m_k-1}) + d_N(fx_{2n_k}, Tx_{2n_k})}{2} \end{array}\right\}$$

$$= \max\left\{\begin{array}{l} d_N(y_{2n_k}, y_{2m_k-1}), d_N(y_{2n_k+1}, y_{2n_k}), \\ d_N(y_{2m_k}, y_{2m_k-1}), \\ \dfrac{d_N(y_{2n_k}, y_{2m_k}) + d_N(y_{2n_k}, y_{2m_k})}{2} \end{array}\right\},$$

Thus $\lim_{k\to\infty} M(y_{2n_k}, y_{2m_k-1}) = \max\left\{\varepsilon, 0, 0, \dfrac{\varepsilon}{2}\right\}.$ From (2.1), we obtain

$$\psi(d_N(y_{2n_k+1}, y_{2m_k})) = \psi(d_N(fx_{2n_k}, gx_{2m_k-1}))$$
$$\le \psi(M(y_{2n_k+1}, y_{2m_k-1})) - \phi(M(x_{2n_k}, x_{2m_k-1})).$$

Taking limit as $k \to \infty$ implies that $\psi(\varepsilon) \le \psi(\varepsilon) - \phi(\varepsilon),$ which is a contradiction as $\varepsilon > 0$.

It follows that $\{y_{2n}\}$ is a Cauchy sequence and since X is complete, there exists a point z in X, such that $\{y_{2n}\}$ converges to z. Therefore,

$$\lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} fx_{2n} = z \text{ and } \lim_{n\to\infty} y_{2n+2} = \lim_{n\to\infty} Sx_{2n+2} = \lim_{n\to\infty} gx_{2n+1} = z.$$

Assume that S is continuous. Since $\{f, S\}$ are compatible, we have

$$\lim_{n\to\infty} fSx_{2n+2} = Sz.$$

Also, $x_{2n+1} \le gx_{2n+1} = Sx_{2n+2}.$ Now

$$\psi(d_N(fSx_{2n+2}, gx_{2n+1})) \le \psi(M(Sx_{2n+2}, x_{2n+1})) - \phi(M(Sx_{2n+2}, x_{2n+1})), \tag{2.8}$$

where

$$M(Sx_{2n+2}, x_{2n+1}) = \max\left\{\begin{array}{l} d_N(SSx_{2n+2}, Tx_{2n+1}), d_N(fSx_{2n+2}, SSx_{2n+2}), \\ d_N(gx_{2n+1}, Tx_{2n+1}), \\ \dfrac{d_N(SSx_{2n+2}, gx_{2n+1}) + d_N(fSx_{2n+2}, Tx_{2n+1})}{2} \end{array}\right\}.$$

On taking limit as $n \to \infty$, we obtain $\psi(d_N(Sz, z)) \le \psi(d_N(Sz, z)) - \phi(d_N(Sz, z)),$ and $Sz = z$.

Now, $x_{2n+1} \le gx_{2n+1}$ and $gx_{2n+1} \to z$ as $n \to \infty$, $x_{2n+1} \le z$ and (2.1) becomes $\psi(d_N(fz, gx_{2n+1})) \le \psi(M(z, x_{2n+1})) - \phi(M(z, x_{2n+1})),$ where

$$M(z, x_{2n+1}) = \max\left\{\begin{array}{l} d_N(Sz, Tx_{2n+1}), d_N(fz, Sz), d_N(gx_{2n+1}, Tx_{2n+1}), \\ \dfrac{d_N(Sz, gx_{2n+1}) + d_N(fz, Tx_{2n+1})}{2} \end{array}\right\}.$$

On taking limit as $n \to \infty$, we obtain $\psi(d_N(fz, z)) \le \psi(d_N(fz, z)) - \phi(d_N(fz, z)),$ and $fz = z$.

Since $f(X) \subseteq T(X)$, there exists a point $fz = Tw.$ such that $gw \ne Tw.$ Since $z \le fz = Tw \le fTw \le w$ implies that $z \le w,$ From (2.1), we obtain

$$\psi(d_N(Tw, gw)) = \psi(d_N(fz, gw)) \le \psi(M(z, w)) - \phi(M(z, w)), \tag{2.9}$$

where

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$$M(z,w)=\max\left\{\begin{array}{l} d_N(Sz,Tw), d_N(fz,Sz), d_N(gw,Tw), \\ \dfrac{d_N(Sz,gw)+d_N(fz,Tw)}{2} \end{array}\right\}$$
$$=\max\left\{\begin{array}{l} d_N(z,z), d_N(z,z), d_N(gw,Tw), \\ \dfrac{d_N(Tw,gw)+d_N(Tw,Tw)}{2} \end{array}\right\}$$
$$=d_N(Tw,gw)$$

Now (2.9) becomes $\psi(d_N(Tw,gw)) \le \psi(d_N(Tw,gw)) - \phi(d_N(Tw,gw)),$ a contradiction. Hence, $Tw=gw.$ Since g and T are weakly compatible, $gz=gfz=gTw=Tgw=Tfz=Tz.$ Thus z is a coincidence point of g and T.

Now, since $x_{2n} \le fx_{2n}$ and $fx_{2n} \to z$ as $n \to \infty$ implies that $x_{2n} \le z$, from (2.1) $\psi(d_N(Tx_{2n},gz)) \le \psi(M(x_{2n},z)) - \phi(M(x_{2n},z)),$ where

$$M(x_{2n},z)$$
$$=\max\left\{\begin{array}{l} d_N(Sx_{2n},Tz), d_N(fx_{2n},Sx_{2n}), \\ d_N(gz,Tz), \\ \dfrac{d_N(Sx_{2n},gz)+d_N(fx_{2n},Tz)}{2} \end{array}\right\}$$
$$=\max\left\{\begin{array}{l} d_N(z,gz), d_N(z,z), d_N(gz,gz), \\ \dfrac{d_N(z,gz)+d_N(z,gz)}{2} \end{array}\right\}$$
$$=d_N(z,gz)$$

On taking limit as $n \to \infty$, we obtain $\psi(d_N(z,gz)) \le \psi(d_N(z,gz)) - \phi(d_N(z,gz)),$ and $gz=z$. Therefore, $fz=gz=Sz=Tz=z.$ The proof is similar when f is continuous.

Similarly, the result follows when (ii) holds.

Now, suppose that the set of common fixed points of f, g, S and T is well ordered. We claim that common fixed point of f, g, S and T is unique. Assume on contrary that $fu=gu=Su=Tu=u$ and $fv=gv=Sv=Tv=v$ but $u \ne v$. By supposition, we can replace x by u and y by v in (2.1) to obtain

$$\psi(d(u,v)) = \psi(d_N(fu,gv))$$
$$\le \psi(M(u,v)) - \phi(M(u,v)),$$

where

$$M(u,v)=\max\left\{\begin{array}{l} d_N(Su,Tv), d_N(fu,Su), d_N(gv,Tv), \\ \frac{d_N(Su,gv)+d_N(fu,Tv)}{2} \end{array}\right\}$$

$$=\max\left\{d_N(u,v),0,0,\frac{d_N(u,v)+d_N(u,v)}{2}\right\}$$

$$=d_N(u,v),$$

and $\psi(d_N(u,v))\leq\psi(d_N(u,v))-\phi(d_N(u,v)),$ a contradiction. Hence $u=v.$ Conversely, if f,g,S and T have only one common fixed point then the set of common fixed point of f,g,S and T being singleton is well ordered.

Corollary 2.9: Let (X, d_N) be an N-complete cone metric space. Let f,S and T be self-maps on X, (T,f) and (S,f) be partially weakly increasing with $f(X)\subseteq T(X)$ and $f(X)\subseteq S(X)$, dominating maps f is weakly annihilators of T and S, respectively. Suppose that there exist control functions ψ and φ such that for every two comparable elements $x,y\in X$,

$$\psi(d_N(fx,fy))\leq\psi(M(x,y))-\varphi(M(x,y)) \tag{2.10}$$

where

$$M(x,y)=\max\left\{\begin{array}{l} d_N(Sx,Ty), d_N(fx,Sx), d_N(fy,Ty), \\ \frac{d_N(Sx,fy)+d_N(fx,Ty)}{2} \end{array}\right\}.$$

is satisfied. If for a non-decreasing sequence $\{x_n\}$ with $x_n\leq y_n$ for all n and $y_n\to u$ implies that $x_n\leq u$ and either

(i) $\{f,S\}$ are compatible, f or S is continuous and $\{f,T\}$ are weakly compatible or

(ii) $\{f,T\}$ are compatible, f or T is continuous and $\{f,S\}$ are weakly compatible.

then f,S and T have a common fixed point. Moreover, the set of common fixed points of f,S and T is well ordered if and only if f,S and T have one and only one common fixed point.

Proof: If we take $f=g$ in theorem 2.7, then the required result is obtained.

Corollary 2.10: Let (X, d_N) be an N-complete cone metric space. Let f,g and T be self-maps on X, (T,f) and (T,g) be partially weakly increasing with $f(X)\subseteq T(X)$ and $g(X)\subseteq T(X)$, dominating maps f and g are weakly annihilators of T. Suppose that there exist control functions ψ and φ such that for every two comparable elements $x,y\in X$,

$$\psi(d_N(fx,gy))\leq\psi(M_1(x,y))-\varphi(M_1(x,y)) \tag{2.11}$$

where

$$M_1(x,y)=\max\left\{\begin{array}{l} d_N(Tx,Ty), d_N(fx,Tx), d_N(gy,Ty), \\ \dfrac{d_N(Tx,gy)+d_N(fx,Ty)}{2} \end{array}\right\}.$$

is satisfied. If for a non-decreasing sequence $\{x_n\}$ with $x_n \le y_n$ for all n and $y_n \to u$ implies that $x_n \le u$ and either

(i) $\{f,T\}$ are compatible, f or T is continuous and $\{g,T\}$ are weakly compatible or

(ii) $\{g,T\}$ are compatible, g or T is continuous and $\{f,T\}$ are weakly compatible.

then f,g and T have a common fixed point. Moreover, the set of common fixed points of f,g and T is well ordered if and only if f,g and T have one and only one common fixed point.

Proof: If we take $f=g$ and $S=T$ in theorem (2.8) and follow the similar proof we get the required.

Corollary 2.11: Let (X, d_N) be an N-complete cone metric space. Let f and T be self-maps on X, (T,f) be partially weakly increasing with $f(X)\subseteq T(X)$, dominating maps f is weakly annihilators of T. Suppose that there exist control functions ψ and φ such that for every two comparable elements $x,y\in X$,

$$\psi(d_N(fx,gy)) \le \psi(M(x,y)) - \varphi(M(x,y)) \tag{2.12}$$

where

$$M(x,y)=\max\left\{\begin{array}{l} d_N(Tx,Ty), d_N(fx,Tx), d_N(fy,Ty), \\ \dfrac{d_N(Tx,fy)+d_N(fx,Ty)}{2} \end{array}\right\}.$$

is satisfied. If for a non-decreasing sequence $\{x_n\}$ with $x_n \le y_n$ for all n and $y_n \to u$ implies that $x_n \le u$. If $\{f,T\}$ are compatible, f or T is continuous and $\{f,T\}$ are weakly compatible then f and T have a common fixed point. Moreover, the set of common fixed points of f and T is well ordered if and only if f and T have one and only one common fixed point.

Proof: If we take $f=g=I$ (identity) in the theorem (2. 8) and follow the similar proof we get the required result.

Example 2.11: Consider $X=[0,1]\cup\{2,3,4,\ldots\}$ with usual ordering and $E=[0,\infty)$ and

$$d_N(x,y)=\begin{cases} |x-y|, \text{ if } x,y\in[0,1] \text{ and } x\ne y \\ x+y, \text{ if at least one of } x \text{ or } \\ \qquad y\notin[0,1] \text{ and } x\ne y \\ 0, \text{ if } x=y \end{cases}$$

Then (X, d_N) is a complete N-cone metric space. Let $\psi,\phi: E\to E$ be defined by

$$\psi(x)=\begin{cases} 2x,\ if\ \ 0\le x\le \frac{1}{2} \\ 1,\ if\ \ x\in(0,1] \\ 0,\ otherwise \end{cases}$$

and

$$\phi(x)=\begin{cases} \frac{1}{4}-x^2,\ if\ \ 0\le x<\frac{1}{2} \\ 0,\ otherwise \end{cases}$$

and selfmaps $f,g,S\ and\ T$ on X be given by

$$f(x)=\begin{cases} 0,\ if\ \ x=0 \\ \frac{1}{2},\ if\ \ x\in\left(0,\frac{1}{2}\right] \\ 1,\ if\ \ x\in\left(\frac{1}{2},1\right] \\ x,\ if\ \ x\in\{2,3,4,....\} \end{cases}$$

$$g(x)=\begin{cases} 0,\ if\ \ x=0 \\ \frac{1}{2},\ if\ \ x\in\left(0,\frac{1}{2}\right] \\ x,\ if\ \ x\in\left(\frac{1}{2},1\right]\cup\{2,3,4,....\} \end{cases}$$

$$T(x)=\begin{cases} 0,\ if\ \ x\le\frac{1}{2} \\ \frac{1}{2},\ if\ \ x\in\left(\frac{1}{2},1\right] \\ x-1,\ if\ \ x\in\{2,3,4,...\} \end{cases}$$

and

$$S(x)=\begin{cases} 0,\ if\ \ x\le\frac{1}{2} \\ 2x-1,\ if\ \ x\in\left(\frac{1}{2},1\right] \\ x,\ if\ \ x\in\{2,3,4,...\} \end{cases}$$

Then $f, g, S \text{ and } T$ satisfy all conditions of theorem 2.8. Moreover 0 is a unique common fixed point of $f, g, S \text{ and } T$.

3. Conclusion

The concept of weakly compatible mapping in N-cone metric space is presented in this study. Furthermore, it was demonstrated that two weakly compatible mappings had fixed points and that they were unique.

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