Tuijin Jishu/Journal of Propulsion Technology
ISSN: 1001-4055
Vol. 44 No. 6 (2023)

Fixed Points of G-MS Contraction in G-Metric Space and Application to Integral Equations

[1]D. Sattemma, [2]S. Vijaya Lakshmi, [3]V. Srinivas Chary, [4]G. Murali, [5]G. Sudhaamsh Mohan Reddy

[1]Research Scholar, Department of mathematics, Osmania University, Hyderabad, INDIA.

[2]Department of mathematics, University College of Science, Osmania University, Hyderabad, INDIA.

[3]School of Sciences, Department of mathematics, Mallareddy University, Hyderabad-500014, INDIA.

[4]Department of mathematics, Sreenidhi University, Hyderabad-501301, INDIA.

[5]Faculty of Science and Technology, Icfai Foundation for Higher Education, Hyderabad-501203, INDIA.

Abstract. We present the concept of G-MS contraction and develop fixed point theorems in the arrangement of G-metric spaces using G-MS contraction with relevant examples and applications to integral equations in this work.

Keywords: *G-MS contraction, arrangement, relevant, applications*

INTRODUCTION

S. Banach presented the famous Banach contraction mapping theorem (BCMP) in 1922 in the arrangement of the complete metric space (Ω,ι).Several generalizations of BCMP utilizing various contractive circumstances in the framework of metric, G-metric space, quasi-metric space, b- metric space have been written in the literature by a diverse range of mathematicians during the last several decades.

Mustafa and Sims [17] proposed the notion of G-metric space and obtained various fixed point findings in the G-metric space. Murthy et al. [16] published current fixed point results on the generalized $(\xi,\omega)-$weak contraction condition in the framing of metric spaces in 2023.

PRELIMINARIES

The terms N and R^+ are used to represent positive integers and non-negative real numbers, accordingly, throughout the following. In the follow-up in 1984, Khan, swaleh, and sessa

[12] enhanced Massa's [14] condition utilized in the context of complete metric space (Ω,ι).by including an altering distance function ξ. Khan et al.'s definition is available at [12].

Later, using cyclic maps, Kirk et al. established the fixed point result, see [13]. At WCNA-2000, Billy Rhodes presented the notion of weak contractive circumstances for the first time. He offered the following ideas:

Definition 2.1. [16] Let (Ω,ι).be a metric space. $\hbar:\Omega\to\Omega$ is weakly contractive if $\iota(\hbar\sigma,\hbar\varsigma)\leq \iota(\sigma,\varsigma) - \xi(\iota(\sigma,\varsigma))$, for all $\sigma,\varsigma\in\Omega$ where ξ is an altering distance function.

Murthy et al. acquired the fixed point solution shown below:

Theorem 2.1. [16] Let (Ω,ι).be a complete metric space and H, I, J, K: $\Omega\to\Omega$ be mapping

Satisfying $\chi(\iota(\mathrm{H}\sigma,\mathrm{I}\varsigma))\leq\chi(\Delta(\sigma,\varsigma))-\Lambda(\Theta(\sigma,\varsigma)$, $\forall\ \sigma,\ \varsigma\in\Omega,\ \ \sigma\neq\varsigma$, where,

$$\Delta(\sigma,\varsigma)=\max\left\{\iota(\mathrm{J}\sigma,\ \mathrm{K}\varsigma),\frac{1}{2}\left[\iota(\mathrm{J}\sigma,\ \mathrm{H}\sigma)+\iota(\ \mathrm{K}\varsigma,\mathrm{I}\varsigma)\right],\frac{1}{2}\left[\iota(\mathrm{J}\sigma,\ \mathrm{I}\varsigma)+\iota(\ \mathrm{K}\varsigma,\mathrm{H}\sigma)\right]\right\}$$

And $$\Theta(\sigma,\varsigma)=\min\left\{\iota(\mathrm{J}\sigma,\ \mathrm{K}\varsigma),\frac{1}{2}\left[\iota(\mathrm{J}\sigma,\ \mathrm{H}\sigma)+\iota(\ \mathrm{K}\varsigma,\mathrm{I}\varsigma)\right],\frac{1}{2}\left[\iota(\mathrm{J}\sigma,\ \mathrm{I}\varsigma)+\iota(\ \mathrm{K}\varsigma,\mathrm{H}\sigma)\right]\right\},$$

$\mathrm{H}(\sigma)\subseteq\ \mathrm{K}(\sigma)$ and $I(\sigma)\subseteq\mathrm{J}(\sigma)$, and $(H,J)(\mathrm{I},\mathrm{K})$ are weak compatible pairs, $\Lambda:[0,\infty)\to[0,\infty)$

is such that $\Lambda(\upsilon)>0$, which is lower semi continuous for all $\upsilon>0$ and Λ is discontinuous at $\upsilon=0$

with $\Lambda(0)=0$, $\chi:[0,\infty)\to[0,\infty)$ is an altering distance function. Then H, I, J, and K have unique

fixed point in Ω.

3. Main Results

The next one is our most recent finding based on Murthy et al. 2022.

Definition 3.1. Let (Ω,G) be a G-metric space and $q\in N,\aleph_1,\aleph_2,\aleph_{3,\ldots\ldots}\aleph_q$ are non-vacuous

selection of Ω and $\Gamma=\bigcap_{i=1}^{q}\aleph_i$ and $\hbar$: $\Gamma\to\Gamma$ is a G-MS Contraction if

$\Gamma=\bigcap_{i=1}^{q}\aleph_i$ is a circular model of Γ connection with $\hbar$;

$(\sigma,\varsigma,\tau)\in\aleph_i\times\aleph_{i+1}\times\aleph_{i+2}$, $i=1,2,3\ldots q$ using $(\aleph_{q+1}=\aleph_1)$,

$$\chi(G(\hbar\sigma,\hbar\varsigma,\hbar\tau))\leq\chi(\Delta(\sigma,\varsigma,\tau))-\Lambda(\Theta(\sigma,\varsigma,\tau) \tag{3.1}$$

Where

$$\Delta(\sigma,\varsigma,\tau)=max\left\{\begin{array}{l}G(\sigma,\varsigma,\tau),\frac{1}{3}\left[G(\sigma,\hbar\sigma,\hbar\sigma)+G(\varsigma,\hbar\varsigma,\hbar\varsigma)+G(\tau,\hbar\tau,\hbar\tau)\right],\\ \frac{1}{3}\left[G(\sigma,\hbar\varsigma,\hbar\varsigma)+G(\varsigma,\hbar\tau,\hbar\tau)+G(\tau,\hbar\sigma,\hbar\sigma)\right]\end{array}\right\}$$

(3.2)

$$\Theta(\sigma,\varsigma,\tau)=min\left\{G(\sigma,\varsigma,\tau),\frac{1}{3}\left[G(\sigma,\hbar\varsigma,\hbar\varsigma)+G(\varsigma,\hbar\tau,\hbar\tau)+G(\tau,\hbar\sigma,\hbar\sigma)\right]\right\}$$

(3.3)

$\chi: R^+ \to R^+$ is an altering distance function and $\Lambda: R^+ \to R^+$ is continuous with $\Lambda(\upsilon)>0,$

assuming and just assuming $\upsilon=0$.

Theorem 3.1. Let (Ω,G) be a complete G-metric space and $q\in N, \aleph_1, \aleph_2, \aleph_{3,\ldots\ldots}\aleph_q$ be non-vacuous selections of Ω and $\Gamma=\bigcap_{i=1}^{q}\aleph_i$. Pretend $\hbar:\Gamma\to\Gamma$ is a G-MS contraction. Then there is a single fixed point for $\hbar$ in $\Gamma=\bigcap_{i=1}^{q}\aleph_i$.

Proof. Pretend $\sigma_0\in\aleph_1$. Assume $\sigma_0\in\Omega$ given by $\sigma_{n+1}=\hbar\sigma_n, \forall n\in N\cup\{0\}$. If $\sigma_n=\sigma_{n+1}$ eventually σ_n is a fixed point of $\hbar$. Presume $\sigma_n\neq\sigma_{n+1}$, $\forall n\in N\cup\{0\}$.

(3.4)

We'll demonstrate that $G(\sigma_n,\sigma_{n+1},\sigma_{n+1})=0.$ on the premise, $G(\sigma_n,\sigma_{n+1},\sigma_{n+1})>0,$ we possess $i(n)\in\{1,2,\ldots..q\}, \forall n,$ which means $(\sigma_n,\sigma_{n+1},\sigma_{n+1})\in\aleph_i\times\aleph_{i+1}\times\aleph_{i+2}$. Putting $\sigma=\sigma_n, \varsigma=\sigma_{n+1}$, and $\tau=\sigma_{n+1}$ in the equation (3.1), (3.2) and (3.3), we have

$$\chi\left(G(\hbar\sigma_n,\hbar\sigma_{n+1},\hbar\sigma_{n+1})\right)=\chi\left(G(\sigma_{n+1},\sigma_{n+2},\sigma_{n+2})\right)\le\chi\left(\Delta(\sigma_n,\sigma_{n+1},\sigma_{n+1})\right)-\Lambda\left(\Theta(\sigma_n,\sigma_{n+1},\sigma_{n+1})\right)$$

(3.6)

Where

$$\Delta(\sigma_n,\sigma_{n+1},\sigma_{n+1})=max\left\{\begin{array}{l}G(\sigma_n,\sigma_{n+1},\sigma_{n+1}),\frac{1}{3}\left[\begin{array}{l}G(\sigma_n,\hbar\sigma_n,\hbar\sigma_n)+G(\sigma_{n+1},\hbar\sigma_{n+1},\hbar\sigma_{n+1})\\+G(\sigma_{n+1},\hbar\sigma_{n+1},\hbar\sigma_{n+1})\end{array}\right],\\ \frac{1}{3}\left[\begin{array}{l}G(\sigma_n,\hbar\sigma_{n+1},\hbar\sigma_{n+1})+G(\sigma_{n+1},\hbar\sigma_{n+1},\hbar\sigma_{n+1})\\+G(\sigma_{n+1},\hbar\sigma_n,\hbar\sigma_n)\end{array}\right]\end{array}\right\}$$

$$\Delta(\sigma_n,\sigma_{n+1},\sigma_{n+1})=max\left\{\begin{array}{l}G(\sigma_n,\sigma_{n+1},\sigma_{n+1}),\frac{1}{3}\left[\begin{array}{l}G(\sigma_n,\sigma_{n+1},\sigma_{n+1})+G(\sigma_{n+1},\sigma_{n+2},\sigma_{n+2})\\+G(\sigma_{n+1},\sigma_{n+2},\sigma_{n+2})\end{array}\right],\\ \frac{1}{3}\left[\begin{array}{l}G(\sigma_n,\sigma_{n+2},\sigma_{n+2})+G(\sigma_{n+1},\sigma_{n+2},\sigma_{n+2})\\+G(\sigma_{n+1},\sigma_{n+1},\sigma_{n+1})\end{array}\right]\end{array}\right\}$$

and

$$\Theta(\sigma_n,\sigma_{n+1},\sigma_{n+1}) = \min\left\{G(\sigma_n,\sigma_{n+1},\sigma_{n+1}), \frac{1}{3}\begin{bmatrix} G(\sigma_n,\hbar\sigma_{n+1},\hbar\sigma_{n+1}) + G(\sigma_{n+1},\hbar\sigma_{n+1},\hbar\sigma_{n+1}) \\ +G(\sigma_{n+1},\hbar\sigma_n,\hbar\sigma_n) \end{bmatrix}\right\}$$

$$\Theta(\sigma_n,\sigma_{n+1},\sigma_{n+1}) = \min\left\{G(\sigma_n,\sigma_{n+1},\sigma_{n+1}), \frac{1}{3}\begin{bmatrix} G(\sigma_n,\sigma_{n+2},\sigma_{n+2}) + G(\sigma_{n+1},\sigma_{n+2},\sigma_{n+2}) \\ +G(\sigma_{n+1},\sigma_{n+1},\sigma_{n+1}) \end{bmatrix}\right\}$$

Hence, as a result of the rectangular inequality, we have

$$\Delta(\sigma_n,\sigma_{n+1},\sigma_{n+1}) \le max\left\{\begin{array}{l} G(\sigma_n,\sigma_{n+1},\sigma_{n+1}), \frac{1}{3}\left[G(\sigma_n,\sigma_{n+1},\sigma_{n+1}) + 2G(\sigma_{n+1},\sigma_{n+2},\sigma_{n+2})\right], \\ \frac{1}{3}\begin{bmatrix} G(\sigma_n,\sigma_{n+1},\sigma_{n+1}) + G(\sigma_{n+1},\sigma_{n+2},\sigma_{n+2}) \\ +G(\sigma_{n+1},\sigma_{n+2},\sigma_{n+2}) \end{bmatrix} \end{array}\right\}$$

$$\le max\left\{\begin{array}{l} G(\sigma_n,\sigma_{n+1},\sigma_{n+1}), \frac{1}{3}\left[G(\sigma_n,\sigma_{n+1},\sigma_{n+1}) + 2G(\sigma_{n+1},\sigma_{n+2},\sigma_{n+2})\right], \\ \frac{1}{3}\left[G(\sigma_n,\sigma_{n+1},\sigma_{n+1}) + 2G(\sigma_{n+1},\sigma_{n+2},\sigma_{n+2})\right] \end{array}\right\}$$

$$\le max\left\{G(\sigma_n,\sigma_{n+1},\sigma_{n+1}), \frac{1}{3}\left[G(\sigma_n,\sigma_{n+1},\sigma_{n+1}) + 2G(\sigma_{n+1},\sigma_{n+2},\sigma_{n+2})\right],\right\}$$

(3.7)

$$\Theta(\sigma_n,\sigma_{n+1},\sigma_{n+1}) = \min\left\{G(\sigma_n,\sigma_{n+1},\sigma_{n+1}), \frac{1}{3}\left[G(\sigma_n,\sigma_{n+2},\sigma_{n+2}) + G(\sigma_{n+1},\sigma_{n+2},\sigma_{n+2})\right]\right\}$$

(3.8)

Let some n if at all possible,

$$G(\sigma_n,\sigma_{n+1},\sigma_{n+1}) < G(\sigma_{n+1},\sigma_{n+2},\sigma_{n+2})$$

The rectangular inequality follows

$$0 < G(\sigma_{n+1},\sigma_{n+2},\sigma_{n+2}) - G(\sigma_n,\sigma_{n+1},\sigma_{n+1}) \le G(\sigma_n,\sigma_{n+2},\sigma_{n+2}).$$

Hence, by (3.4) we have

$$G(\sigma_n,\sigma_{n+1},\sigma_{n+1}) > 0.$$

Now, if $G(\sigma_n,\sigma_{n+1},\sigma_{n+1}) < G(\sigma_{n+1},\sigma_{n+2},\sigma_{n+2})$

Then we obtain

$$\Delta(\sigma_n,\sigma_{n+1},\sigma_{n+1}) \le G(\sigma_{n+1},\sigma_{n+2},\sigma_{n+2}) \tag{3.9}$$

and (3.6) indicates that this disparity

$$\chi\left(G(\sigma_{n+1},\sigma_{n+2},\sigma_{n+2})\right) \le \chi\left(\Delta(\sigma_n,\sigma_{n+1},\sigma_{n+1})\right) - \Lambda\left(\Theta(\sigma_n,\sigma_{n+1},\sigma_{n+1})\right)$$
$$\le \chi\left(\Delta(\sigma_n,\sigma_{n+1},\sigma_{n+1})\right)$$

As χ increases monotonically, we obtain

$$G(\sigma_{n+1},\sigma_{n+2},\sigma_{n+2}) < G(\sigma_n,\sigma_{n+1},\sigma_{n+1})$$

(3.10)

from (3.9) and (3.10), we have $\Delta(\sigma_n,\sigma_{n+1},\sigma_{n+1})=G(\sigma_{n+1},\sigma_{n+2},\sigma_{n+2})$

thus using our assumption and (3.6) - (3.8), we have

$$\chi\left(G(\sigma_{n+1},\sigma_{n+2},\sigma_{n+2})\right)\leq\chi\left(G(\sigma_n,\sigma_{n+1},\sigma_{n+1})\right)-\Lambda\left(\Theta(\sigma_n,\sigma_{n+1},\sigma_{n+1})\right)$$
$$<\chi\left(G(\sigma_n,\sigma_{n+1},\sigma_{n+1})\right)$$

which contradicts itself, therefore for all $n\geq 0$,

$$G(\sigma_{n+1},\sigma_{n+2},\sigma_{n+2})<G(\sigma_n,\sigma_{n+1},\sigma_{n+1})$$

(3.11)

By (3.11) we obtain from (3.7) and (3.8), for all $n\geq 0$

$$\Delta(\sigma_n,\sigma_{n+1},\sigma_{n+1})=G(\sigma_n,\sigma_{n+1},\sigma_{n+1}) \tag{3.12}$$

$$\Theta(\sigma_n,\sigma_{n+1},\sigma_{n+1})=\frac{1}{3}\left[G(\sigma_n,\sigma_{n+2},\sigma_{n+2})+G(\sigma_{n+1},\sigma_{n+2},\sigma_{n+2})\right]$$

(3.13)

Putting (3.12) and (3.13) in (3.6), we have $\forall n\geq 0$,

$$\chi\left(G(\sigma_{n+1},\sigma_{n+2},\sigma_{n+2})\right)=\chi\left(G(\sigma_n,\sigma_{n+1},\sigma_{n+1})\right)-\Lambda\left(\frac{1}{3}\left[G(\sigma_n,\sigma_{n+2},\sigma_{n+2})+G(\sigma_{n+1},\sigma_{n+2},\sigma_{n+2})\right]\right)$$

(3.14)

Once more (3.11) suggests that the sequence $G(\sigma_n,\sigma_{n+1},\sigma_{n+1})$ diminishes in a consistent manner.

Finally there exist $s\geq 0$ so

$$\lim_{n\to\infty}G(\sigma_n,\sigma_{n+1},\sigma_{n+1})=s.$$

Once more, based on the last property of G-metric space, we posses

$$G(\sigma_n,\sigma_{n+2},\sigma_{n+2})\leq G(\sigma_n,\sigma_{n+1},\sigma_{n+1})+G(\sigma_{n+1},\sigma_{n+2},\sigma_{n+2})$$
$$G(\sigma_n,\sigma_{n+2},\sigma_{n+2})-2s\leq G(\sigma_n,\sigma_{n+1},\sigma_{n+1})+G(\sigma_{n+1},\sigma_{n+2},\sigma_{n+2})-2s$$
$$G(\sigma_n,\sigma_{n+2},\sigma_{n+2})-2s\leq[G(\sigma_n,\sigma_{n+1},\sigma_{n+1})-s]$$

As $n\to+\infty$ we have $\lim_{n\to\infty}G(\sigma_n,\sigma_{n+2},\sigma_{n+2})=2s$.

And when $n\to+\infty$ in (3.14), we posses

$$\lim_{n\to\infty}\chi\left(G(\sigma_{n+1},\sigma_{n+2},\sigma_{n+2})\right)=\lim_{n\to\infty}\chi\left(G(\sigma_n,\sigma_{n+1},\sigma_{n+1})\right)-\lim_{n\to\infty}\Lambda\left(\frac{1}{3}\left[\begin{matrix}G(\sigma_n,\sigma_{n+2},\sigma_{n+2})\\+G(\sigma_{n+1},\sigma_{n+2},\sigma_{n+2})\end{matrix}\right]\right)$$

As χ and Λ continuous

$$\chi(s)\leq\chi(s)-\Lambda(s).$$

Therefore $\Lambda(s)=0$ and hence $s=0$. Thus

$$\lim_{n\to\infty}G(\sigma_n,\sigma_{n+1},\sigma_{n+1})=0 \tag{3.15}$$

Now let's presume that $\{\sigma_n\}$ doesn't seem a G- Cauchy sequence in Ω. Subsequently with a certain

$\grave{o} > 0$, that we may identify distinct$\{\sigma_{m(k)}\},\{\sigma_{n(k)}\}$ sub sequences of $\{\sigma_n\}$ so $n(\rho) > m(\rho) > \rho$

For $\rho \in N$ and

$$G\left(\sigma_{m(\rho)}, \sigma_{n(\rho)}, \sigma_{n(\rho)}\right)) \ldots \grave{o} \tag{3.16}$$

Which implies,

$$G(\sigma_{m(\rho)}, \sigma_{n(\rho)-1}, \sigma_{n(\rho)-1}) < \grave{o} \tag{3.17}$$

By using the rectangular inequality and (3.16) and (3.17), we have

$$\grave{o} \le G(\sigma_{m(\rho)}, \sigma_{n(\rho)}, \sigma_{n(\rho)}) \le G(\sigma_{m(\rho)}, \sigma_{n(\rho)-1}, \sigma_{n(\rho)-1}) + G(\sigma_{n(\rho)-1}, \sigma_{n(\rho)}, \sigma_{n(\rho)})$$
$$< \grave{o} + G(\sigma_{n(\rho)-1}, \sigma_{n(\rho)}, \sigma_{n(\rho)})$$

Letting $\rho \to +\infty$ and using (3.15), we obtain

$$\lim_{\rho \to +\infty} G\left(\sigma_{m(\rho)}, \sigma_{n(\rho)}, \sigma_{n(\rho)}\right) = \grave{o} \tag{3.18}$$

Putting $\sigma = \sigma_{m(\rho)-1}, \varsigma = \sigma_{n(\rho)-1}, \tau = \sigma_{n(\rho)-1}$ in (3.1) - (3.3) respectively, for all $\rho \in N$

$$\chi\ (G(\sigma_{m(\rho)-1}, \sigma_{n(\rho)-1}, \sigma_{n(\rho)-1})) = \chi(G(\sigma_{m(\rho)}, \sigma_{n(\rho)}, \sigma_{n(\rho)}))$$
$$\le \chi(\Delta(\sigma_{n(\rho)-1}, \sigma_{n(\rho)-1}, \sigma_{n(\rho)-1})) - \Lambda(\Theta(\sigma_{m(\rho)-1}, \sigma_{n(\rho)-1}, \sigma_{n(\rho)-1}))$$

(3.19)

Where

$$\Delta(\sigma_{m(\rho)-1}, \sigma_{n(\rho)-1}, \sigma_{n(\rho)-1}) = max\left\{\begin{array}{l} G(\sigma_{m(\rho)-1}, \sigma_{n(\rho)-1}, \sigma_{n(\rho)-1}), \frac{1}{3}\left[\begin{array}{l} G(\sigma_{m(\rho)-1}, \hbar\sigma_{m(\rho)-1}, \hbar\sigma_{m(\rho)-1}) \\ +G(\sigma_{n(\rho)-1}, \hbar\sigma_{n(\rho)-1}, \hbar\sigma_{n(\rho)-1}) \\ +G(\sigma_{n(\rho)-1}, \hbar\sigma_{n(\rho)-1}, \hbar\sigma_{n(\rho)-1}) \end{array}\right], \\ \frac{1}{3}\left[\begin{array}{l} G(\sigma_{m(\rho)-1}, \hbar\sigma_{n(\rho)-1}, \hbar\sigma_{n(\rho)-1}) + G(\sigma_{n(\rho)-1}, \hbar\sigma_{n(\rho)-1}, \hbar\sigma_{n(\rho)-1}) \\ +G(\sigma_{n(\rho)-1}, \hbar\sigma_{m(\rho)-1}, \hbar\sigma_{m(\rho)-1}) \end{array}\right] \end{array}\right\}$$

$$\Delta(\sigma_{m(\rho)-1}, \sigma_{n(\rho)-1}, \sigma_{n(\rho)-1}) = max\left\{\begin{array}{l} G(\sigma_{m(\rho)-1}, \sigma_{n(\rho)-1}, \sigma_{n(\rho)-1}), \frac{1}{3}\left[\begin{array}{l} G(\sigma_{m(\rho)-1}, \sigma_{m(\rho)}, \sigma_{m(\rho)}) \\ +G(\sigma_{n(\rho)-1}, \sigma_{n(\rho)}, \sigma_{n(\rho)}) \\ +G(\sigma_{n(\rho)-1}, \sigma_{n(\rho)}, \sigma_{n(\rho)}) \end{array}\right]', \\ \frac{1}{3}\left[\begin{array}{l} G(\sigma\sigma_{m(\rho)-1}, \sigma_{n(\rho)}, \sigma_{n(\rho)}) + G(\sigma_{n(\rho)-1}, \sigma_{n(\rho)}, \sigma_{n(\rho)}) \\ +G(\sigma_{n(\rho)-1}, \sigma_{m(\rho)}, \sigma_{m(\rho)}) \end{array}\right] \end{array}\right\}$$

$$\Theta(\sigma_{m(\rho)-1}, \sigma_{n(\rho)-1}, \sigma_{n(\rho)-1}) = min\left\{\begin{array}{l} G(\sigma_{m(\rho)-1}, \sigma_{n(\rho)-1}, \sigma_{n(\rho)-1}), \frac{1}{3}\left[\begin{array}{l} G(\sigma_{m(\rho)-1}, \hbar\sigma_{n(\rho)-1}, \hbar\sigma_{n(\rho)-1}) \\ +G(\sigma_{n(\rho)-1}, \hbar\sigma_{n(\rho)-1}, \hbar\sigma_{n(\rho)-1}) \\ +G(\sigma_{n(\rho)-1}, \hbar\sigma_{m(\rho)-1}, \hbar\sigma_{m(\rho)-1} \end{array}\right], \\ \frac{1}{3}\left[\begin{array}{l} G(\sigma_{m(\rho)-1}, \sigma_{n(\rho)}, \sigma_{n(\rho)}) + G(\sigma_{n(\rho)-1}, \sigma_{n(\rho)}, \sigma_{n(\rho)}) \\ +G(\sigma_{n(\rho)-1}, \sigma_{m(\rho)}, \sigma_{m(\rho)}) \end{array}\right] \end{array}\right\}$$

$\forall \rho \in N$, we've got

$$G(\sigma_{m(\rho)-1},\ \sigma_{n(\rho)-1},\ \sigma_{n(\rho)-1}) \le G(\sigma_{m(\rho)-1}, \sigma_{m(\rho)}, \sigma_{m(\rho)}) + G(\sigma_{m(\rho)}, \sigma_{n(\rho)}, \sigma_{n(\rho)}) + G(\sigma_{n(\rho)}, \sigma_{n(\rho)-1}, \sigma_{n(\rho)-1})$$

and

$$G(\sigma_{m(\rho)}, \sigma_{n(\rho)}, \sigma_{n(\rho)}) \le G(\sigma_{m(\rho)}, \sigma_{m(\rho)-1}, \sigma_{m(\rho)-1}) + G(\sigma_{m(\rho)-1},\ \sigma_{n(\rho)-1},\ \sigma_{n(\rho)-1}) + G(\sigma_{n(\rho)-1}, \sigma_{n(\rho)}, \sigma_{n(\rho)})$$

By using (3.15) and (3.18) and letting $\rho \to +\infty$, we have

$$\lim_{\rho \to +\infty} G\left(\sigma_{m(\rho)-1},\ \sigma_{n(\rho)-1},\ \sigma_{n(\rho)-1}\right) = \grave{o} \tag{3.20}$$

For all $\rho \in N$, we've got

$$G(\sigma_{m(\rho)-1},\ \sigma_{n(\rho)},\ \sigma_{n(\rho)}) \le G(\sigma_{m(\rho)-1}, \sigma_{m(\rho)}, \sigma_{m(\rho)}) + G(\sigma_{m(\rho)}, \sigma_{n(\rho)}, \sigma_{n(\rho)})$$

And

$$G(\sigma_{m(\rho)},\ \sigma_{n(\rho)},\ \sigma_{n(\rho)}) \le G(\sigma_{m(\rho)}, \sigma_{m(\rho)-1}, \sigma_{m(\rho)-1}) + G(\sigma_{m(\rho)-1}, \sigma_{n(\rho)}, \sigma_{n(\rho)})$$

Letting $\rho \to +\infty$, from (3.15) and (3.18) we've got

$$\lim_{\rho \to +\infty} G\left(\sigma_{m(\rho)-1}, \sigma_{n(\rho)}, \sigma_{n(\rho)}\right) = \grave{o} \tag{3.21}$$

For all $\rho \in N$, we've got

$$G(\sigma_{n(\rho)-1}, \sigma_{m(\rho)}, \sigma_{m(\rho)}) \le G(\sigma_{n(\rho)-1}, \sigma_{n(\rho)}, \sigma_{n(\rho)}) + G(\sigma_{n(\rho)}, \sigma_{m(\rho)}, \sigma_{m(\rho)})$$

and

$$G(\sigma_{n(\rho)}, \sigma_{m(\rho)}, \sigma_{m(\rho)}) \le G(\sigma_{n(\rho)}, \sigma_{n(\rho)-1}, \sigma_{n(\rho)-1}) + G(\sigma_{n(\rho)-1}, \sigma_{m(\rho)}, \sigma_{m(\rho)})$$

By using (3.15) and (3.18) and letting $\rho \to +\infty$, we have

$$\lim_{\rho \to +\infty} G\left(\sigma_{n(\rho)-1}, \sigma_{m(\rho)}, \sigma_{m(\rho)}\right) = \grave{o} \tag{3.22}$$

With continuous χ and Λ, $\rho \to +\infty$ in (3.19) from (3.15), (3.18)-(3.22), we have

$$\chi(\varepsilon) \le \chi(\varepsilon) - \Lambda(\varepsilon)$$

a paradox caused by $\grave{o} > 0$, which means σ_n is G-Cauchy. There are some $\sigma \in \Omega$ by completeness
of (Ω, G),

$$\lim_{\rho \to +\infty} \sigma_n = \sigma^0 \tag{3.23}$$

Now we assert

$$\sigma^0 \in \bigcap_{i=1}^{q} \aleph_i$$

Since $\sigma_0 \in \aleph_1, \left(\sigma_{nq}\right)_{n\ge 0} \subseteq \aleph_1$, and $\aleph_1$ is closed from (3.23), respectively, we obtained $\sigma^o \in \aleph_1$. Once more, we have $\left(\sigma_{nq+1}\right)_{n\ge 0} \subseteq \aleph_2$. Since $\aleph_2$ is closed, we were able to derive $\sigma^o \in \aleph_2$. from (3.23). Continuing in this manner, we come to

$$\sigma^0 \in \bigcap_{i=1}^{q} \aleph_i$$

Now, we can demonstrate σ^o is fixed point of $\hbar$, from (3.24) since $\forall i(n) \in \{1,2,......q\}$ which
means $\sigma_n \in \aleph_i(n)$, using (3.1) with $\sigma = \sigma_n, \varsigma = \sigma_o$ and $\tau = \sigma_o$, we posses

$$\chi\left(G\left(\hbar\sigma_n, \hbar\sigma^o, \hbar\sigma^o\right)\right) = \chi\left(G\left(\sigma_{n+1}, \hbar\sigma^o, \hbar\sigma^o\right)\right) \le \chi\left(\Delta\left(\sigma_n, \sigma^o, \sigma^o\right)\right) - \Lambda(\chi(\Theta(\sigma_n, \sigma^o, \sigma^o))$$

Where

$$\Delta\left(\sigma_n, \sigma^0, \sigma^0\right) = \max\left\{\begin{array}{l} G\left(\sigma_n, \sigma^0, \sigma^0\right), \frac{1}{3}\left[\begin{array}{l} G\left(\sigma_n, \hbar\sigma_n, \hbar\sigma_n\right) + G\left(\sigma^0, \hbar\sigma^0, \hbar\sigma^0\right) \\ +G\left(\sigma^0, \hbar\sigma^0, \hbar\sigma^0\right) \end{array}\right], \\ \frac{1}{3}\left[G\left(\sigma_n, \hbar\sigma^0, \hbar\sigma^0\right) + G\left(\sigma^0, \hbar\sigma^0, \hbar\sigma^0\right) + G\left(\sigma^0, \hbar\sigma_n, \hbar\sigma_n\right)\right] \end{array}\right\}$$

$$\Theta\left(\sigma_n, \sigma^0, \sigma^0\right) = \min\left\{G\left(\sigma_n, \sigma^0, \sigma^0\right), \frac{1}{3}\left[G\left(\sigma_n, \hbar\sigma^0, \hbar\sigma^0\right) + G\left(\sigma^0, \hbar\sigma^0, \hbar\sigma^0\right) + G\left(\sigma^0, \hbar\sigma_n, \hbar\sigma_n\right)\right]\right\}$$

$$= \min\left\{G\left(\sigma_n, \sigma^0, \sigma^0\right), \frac{1}{3}\left[G\left(\sigma_n, \hbar\sigma^0, \hbar\sigma^0\right) + G\left(\sigma^0, \hbar\sigma^0, \hbar\sigma^0\right) + G\left(\sigma^0, \sigma_{n+1}, \sigma_{n+1}\right)\right]\right\}$$

Now

$$\chi\left(G\left(\sigma^0, \hbar\sigma^0, \hbar\sigma^0\right)\right) \le \chi\left(\frac{2}{3}G\left(\sigma^0, \hbar\sigma^0, \hbar\sigma^0\right)\right)$$

As $n \to +\infty$

$$\chi\left(G\left(\sigma^0, \hbar\sigma^0, \hbar\sigma^0\right)\right) \le \chi\left(\frac{2}{3}G\left(\sigma^0, \hbar\sigma^0, \hbar\sigma^0\right)\right)$$

Since χ increases monotonically,

$$G\left(\sigma^0, \hbar\sigma^0, \hbar\sigma^0\right) \le \frac{2}{3}G\left(\sigma^0, \hbar\sigma^0, \hbar\sigma^0\right)$$

Which is contradictory; as a result, $G\left(\sigma^0, \hbar\sigma^0, \hbar\sigma^0\right) = 0$ and hence $\sigma^0 = \hbar\sigma^0$. Therefore σ^0 is a
fixed point of $\hbar$. Assume for the sake of disparity, ς^0 is one more fixed point and $\hbar\varsigma^0 = \varsigma^0$.

This suggests that

$$\varsigma^0 \in \bigcap_{i=1}^{q} p_i$$

By using (3.1) for $\sigma = \sigma^0$, $\varsigma = \varsigma^0$ and $\tau = \varsigma^0$

$$\chi\left(G\left(\hbar\sigma^o, \hbar\varsigma^0, \hbar\varsigma^0\right)\right) = \chi\left(G\left(\sigma^o, \varsigma^0, \varsigma^0\right)\right) \le \chi\left(\Delta\left(\sigma^o, \varsigma^0, \varsigma^0\right)\right) - \Lambda(\chi(\Theta(\sigma^o, \varsigma^0, \varsigma^0)) \quad (3.25)$$

Where

$$\Delta\left(\sigma^0,\varsigma^0,\varsigma^0\right)=max\left\{\begin{array}{l}G\left(\sigma^0,\varsigma^0,\varsigma^0\right),\frac{1}{3}\left[\begin{array}{l}G\left(\sigma^0,\hbar\sigma^0,\hbar\sigma^0\right)+G\left(\varsigma^0,\hbar\varsigma^0,\hbar\varsigma^0\right)\\+G\left(\varsigma^0,\hbar\varsigma^0,\hbar\varsigma^0\right)\end{array}\right],\\ \frac{1}{3}\left[G\left(\sigma^0,\hbar\varsigma^0,\hbar\varsigma^0\right)+G\left(\varsigma^0,\hbar\varsigma^0,\hbar\varsigma^0\right)+G\left(\varsigma^0,\hbar\sigma^0,\hbar\sigma^0\right)\right]\end{array}\right\}$$

$$=max\left\{\begin{array}{l}G\left(\sigma^0,\varsigma^0,\varsigma^0\right),\frac{1}{3}\left[G\left(\sigma^0,\sigma^0,\sigma^0\right)+2G\left(\varsigma^0,\varsigma^0,\varsigma^0\right)\right],\\ \frac{1}{3}\left[G\left(\sigma^0,\varsigma^0,\varsigma^0\right)+G\left(\varsigma^0,\varsigma^0,\varsigma^0\right)+G\left(\varsigma^0,\sigma^0,\sigma^0\right)\right]\end{array}\right\}$$

$$=max\left\{G\left(\sigma^0,\varsigma^0,\varsigma^0\right),0,\ \frac{2}{3}\left[G\left(\sigma^0,\varsigma^0,\varsigma^0\right)\right]\right\}=G\left(\sigma^0,\varsigma^0,\varsigma^0\right)$$

(3.26)

$$\Theta\left(\sigma^0,\varsigma^0,\varsigma^0\right)=min\left\{G\left(\sigma^0,\varsigma^0,\varsigma^0\right),\frac{1}{3}\left[G\left(\sigma^0,\hbar\varsigma^0,\hbar\varsigma^0\right)+G\left(\varsigma^0,\hbar\varsigma^0,\hbar\varsigma^0\right)+G\left(\varsigma^0,\hbar\sigma^0,\hbar\sigma^0\right)\right]\right\}$$

$$=min\left\{G\left(\sigma^0,\varsigma^0,\varsigma^0\right),\frac{1}{3}\left[G\left(\sigma^0,\varsigma^0,\varsigma^0\right)+G\left(\sigma^0,\varsigma^0,\varsigma^0\right)+G\left(\varsigma^0,\sigma^0,\sigma^0\right)\right]\right\}$$

(3.27)

In light of the fact that σ^0,ς^0 are both fixed points of $\hbar$ from (3.25) to (3.27) we have

$\chi\left(G\left(\sigma^o,\varsigma^0,\varsigma^0\right)\right)\leq\chi\left(G\left(\sigma^o,\varsigma^0,\varsigma^0\right)\right)-\Lambda(\chi(G(\sigma^o,\varsigma^0,\varsigma^0))$

Which implies that $G\left(\sigma^o,\varsigma^0,\varsigma^0\right)=0$. Hence $\sigma^0=\varsigma^0$.

Example 3.1. Let $\Omega=R^3$ and $\hbar:\Omega\times\Omega\times\Omega\to R^+$ be given by

$$G\left(\left(\sigma_1,\varsigma_1,\tau_1\right),\left(\sigma_2,\varsigma_2,\tau_2\right),\left(\sigma_3,\varsigma_3,\tau_3\right)\right)=\left|\sigma_1-\sigma_2\right|+\left|\sigma_2-\sigma_3\right|+\left|\sigma_3-\sigma_1\right|+\left|\varsigma_1-\varsigma_2\right|+\left|\varsigma_2-\varsigma_3\right|$$
$$+\left|\varsigma_3-\varsigma_1\right|+\left|\tau_1-\tau_2\right|+\left|\tau_2-\tau_3\right|+\left|\tau_3-\tau_1\right|$$

(Ω,G) is complete and let $\aleph_1=\left\{(h,0,0):0\leq h\leq 1\right\}$, $\aleph_2=\{(0,i,0):0\leq i\leq 1\}$ and $\aleph_3=\{(j,0,0):\ 0\leq j\leq 1\}$ be the closed subsets of Ω.. Let $\hbar:\aleph_1\cup\aleph_2\cup\aleph_3\to\aleph_1\cup\aleph_2\cup\aleph_3$ be a map, such that $\hbar=(h,0,0)=\left(0,0,\frac{h}{3}\right)$, $\hbar=(0,i,0)=\left(0,\frac{i}{3},0\right)$ and $\hbar=(0,0,j)=\left(\frac{j}{3},0,0\right)$.

Clearly $\Gamma=\aleph_1\cup\ \aleph_2\cup\ \aleph_3$ is a cyclic representation of Γ in relation to $\hbar$.

affirm: $\hbar$ adheres to (3.1). Let $\chi(l)=l$ and $\Lambda(l)=\frac{l}{3}$ such that $\chi(l)=l$. Let $\sigma=(h,0,0)$, $\varsigma=(0,i,0)$

and $\tau=(0,0,j)$ then $\hbar\sigma=\hbar(h,0,0)=\left(0,0,\frac{h}{3}\right)$, $\hbar\varsigma=\hbar(0,i,0)=\left(0,\frac{i}{3},0\right)$ and $\hbar\tau=\hbar(0,0,j)=\left(\frac{j}{3},0,0\right)$

and $G(\hbar\sigma,\hbar\varsigma,\hbar\tau)=\frac{1}{3}(h+i+j), G(\sigma,\varsigma,\tau)=(h+i+j)$ and

$$\frac{1}{3}\left[G(\sigma,\hbar\sigma,\hbar\sigma)+G(\varsigma,\hbar\varsigma,\hbar\varsigma)+G(\tau,\hbar\tau,\hbar\tau)\right]=\frac{1}{3}\left[\begin{array}{l} G\left((h,0,0),\left(0,0,\frac{h}{3}\right),\left(0,0,\frac{h}{3}\right)\right) \\ +G\left((0,i,0),\left(0,\frac{i}{3},0\right),\left(0,\frac{i}{3},0\right)\right) \\ +G\left((0,0,j),\left(\frac{j}{3},0,0\right),\left(\frac{j}{3},0,0\right)\right)\end{array}\right]$$

$$\leq\frac{2}{3}(h+i+j)+\frac{2}{9}(h+i+j)$$

$$\frac{1}{3}\left[G(\sigma,\hbar\varsigma,\hbar\varsigma)+G(\varsigma,\hbar\tau,\hbar\tau)+G(\tau,\hbar\sigma,\hbar\sigma)\right]=\frac{1}{3}\left[\begin{array}{l} G\left((h,0,0),\left(0,\frac{i}{3},0\right),\left(0,\frac{i}{3},0\right)\right) \\ +G\left((0,i,0),\left(\frac{j}{3},0,0\right),\left(\frac{j}{3},0,0\right)\right) \\ +G\left((0,0,j),\left(0,0,\frac{h}{3}\right),\left(0,0,\frac{h}{3}\right)\right)\end{array}\right]$$

$$\leq\frac{2}{3}(h+i+j)+\frac{2}{9}(h+i+j)$$

Hence

$$\chi\left(G(\hbar\sigma,\hbar\varsigma,\hbar\tau)\right)=\frac{2}{3}(h+i+j)=\chi\left(G(\sigma,\varsigma,\tau)\right)\leq\chi\left(\Delta(\sigma,\varsigma,\tau)\right)-\Lambda\left(\Theta(\sigma,\varsigma,\tau)\right)$$

Theorem (3.1) is fully met, and $\hbar$ has an unique fixed point $(0,0,0)\in\aleph_1\cup\aleph_2\cup\aleph_3$.

APPLICATION

The generated one is used to discover a solution to the following:

$$-\frac{d^2\varsigma}{d\sigma^2}=\begin{cases} E\left(\sigma,\varsigma(\sigma)\right),\sigma\in[0,1], \\ \varsigma(0)=\varsigma(1)=0.\end{cases}$$

(4.1)

Where $E:[0,1]\times R$ is continuous mapping. The integral equation is comparable to this matter.

$$\varsigma(\sigma)=\int_0^1 G'(\sigma,\upsilon)E\left(\upsilon,\varsigma(\upsilon)\right)d\upsilon \qquad (4.2)$$

For $\sigma\in[0,1], G'(\sigma,\upsilon)$ is the Green's function determined by

$$G'(\sigma,\upsilon)=\begin{cases}\upsilon(1-\sigma); if\ 0\le\upsilon\le\sigma;\\ \sigma(1-\upsilon); if\ \sigma\le\upsilon\le 1\end{cases}$$

Assume $\Omega=C([0,1],R^{+})$ is an array that includes continuous non-negative real graded operations defined on $[0,1]$. Now that we've determined the generalized metric G on Ω,

$$G(\sigma,\varsigma,\tau)=max[|\sigma(\upsilon)-\varsigma(\upsilon)|+|\varsigma(\upsilon)-\tau(\upsilon)|+|\tau(\upsilon)-\sigma(\upsilon)|].$$

For $\sigma,\varsigma,\tau\in\Omega$. Then (Ω,G) is a G-metric space. Allow $\aleph_1=\aleph_2=\Omega=C([0,1],R^{+})$. It is obvious

so $\aleph_1,\aleph_2$ represent closed subsets of Ω. Let $\hbar:\ \aleph_1\cup\aleph_2\to\aleph_1\cup\aleph_2$. be defined by

$$\hbar(\varsigma(\sigma))=\int_0^1 G'(\sigma,\upsilon)E(\upsilon,\varsigma(\upsilon))d\upsilon,\ \ \upsilon\in[0,1]$$

Clearly, $\hbar(\aleph_1)\subseteq\aleph_2, \hbar(\aleph_2)\subseteq\aleph_3$ and $\hbar(\aleph_3)\subseteq\aleph_1$. This, $\hbar$ is cyclic map on $\aleph_1\cup\aleph_2\cup\aleph_3$

.

Imagine the following proposition is accurate: $|E(\upsilon,\theta)-E(\upsilon,\vartheta)|\ \le\ |\theta-\vartheta|$, for all $\upsilon\in[0,1]$ and $\theta,\vartheta\in R^{+}$. Then (4.2) presents a distinctive solution $\varsigma^{*}\in\Omega$. So, now we show that for each $\varsigma\in\aleph_1$ and $\varsigma\in\aleph_1,\mu\in\aleph_2,\nu\in\aleph_3$. We have

$$\chi(G(\hbar\varsigma,\hbar\mu,\hbar\nu))\le\chi(\Delta(\varsigma,\mu,\nu))-\Lambda(\Theta(\varsigma,\mu,\nu))$$

For $\chi(\tilde{n})=\tilde{n}$ and $\Lambda(\tilde{n})=\frac{1}{2}\tilde{n}$. Now, let $(\varsigma,\mu,\nu)\in\aleph_1\times\aleph_2\times\aleph_3$. Therefore, by (3.1), We conclude that every one $\tilde{n}\in[0,1]$.

$$\begin{aligned}G(\hbar\varsigma,\hbar\mu,\hbar\nu)&=max|\hbar\varsigma(\tilde{n})-\hbar\mu(\tilde{n})|+\max|\hbar\mu(\tilde{n})-\hbar\nu(\tilde{n})|+\max|\hbar\mu(\tilde{n})-\hbar\varsigma(\tilde{n})|\\ &=max\left|\int_0^1 G'(\tilde{n},\upsilon)E(\upsilon,\varsigma(\upsilon))d\upsilon-\int_0^1 G'(\tilde{n},\upsilon)E(\upsilon,\mu(\upsilon))d\upsilon\right|\\ &+\ \max\left|\int_0^1 G'(\tilde{n},\upsilon)E(\upsilon,\mu(\upsilon))d\upsilon-\int_0^1 G'(\tilde{n},\upsilon)E(\upsilon,\nu(\upsilon))d\upsilon\right|\\ &+\ \max\left|\int_0^1 G'(\tilde{n},\upsilon)E(\upsilon,\nu(\upsilon))d\upsilon-\int_0^1 G'(\tilde{n},\upsilon)E(\upsilon,\varsigma(\upsilon))d\upsilon\right|\end{aligned}$$

$$= max\left|\int_0^1 G'(\tilde{n},\upsilon)[E(\upsilon,\varsigma(\upsilon)) - E(\upsilon,\mu(\upsilon))]d\upsilon\right| + \max\left|\int_0^1 G'(\tilde{n},\upsilon)[E(\upsilon,\mu(\upsilon))d\upsilon - E(\upsilon,\nu(\upsilon))]d\upsilon\right|$$

$$+ \max\left|\int_0^1 G'(\tilde{n},\upsilon)[E(\upsilon,\nu(\upsilon))d\upsilon - E(\upsilon,\varsigma(\upsilon))]d\upsilon\right|$$

$$\le max\int_0^1 G'(\tilde{n},\upsilon)|E(\upsilon,\varsigma(\upsilon)) - E(\upsilon,\mu(\upsilon))|d\upsilon + \max\int_0^1 G'(\tilde{n},\upsilon)|E(\upsilon,\mu(\upsilon))d\upsilon - E(\upsilon,\nu(\upsilon))|d\upsilon$$

$$+ \max\int_0^1 G'(\tilde{n},\upsilon)|E(\upsilon,\nu(\upsilon))d\upsilon - E(\upsilon,\varsigma(\upsilon))|d\upsilon$$

$$\le max\int_0^1 G'(\tilde{n},\upsilon)\left[|\varsigma(\upsilon)-\mu(\upsilon)|+|\mu(\upsilon)-\nu(\upsilon)|+|\nu(\upsilon)-\varsigma(\upsilon)|\right]d\upsilon$$

$$\le max\int_0^1 G'(\tilde{n},\upsilon)\max\left[|\varsigma(\upsilon)-\mu(\upsilon)|+|\mu(\upsilon)-\nu(\upsilon)|+|\nu(\upsilon)-\varsigma(\upsilon)|\right]d\upsilon$$

$$= G(\varsigma,\mu,\nu)\ \max\int_0^1 G'(\tilde{n},\upsilon)d\upsilon$$

(4.3)

It is easy to verify that $\int_0^1 G'(\tilde{n},\upsilon)d\upsilon = \frac{\tilde{n}}{2} - \frac{\tilde{n}^2}{2}$ and thus, $\int_0^1 G'(\tilde{n},\upsilon)d\upsilon = \frac{1}{8}$. Consider the above facts, (4.3) gives us

$$G(\hbar\varsigma,\hbar\mu,\hbar\nu) \le \frac{1}{8}G(\varsigma,\mu,\nu) \tag{4.4}$$

Now

$$\Delta(\varsigma,\mu,\nu) = max\left\{\begin{array}{l} G(\varsigma,\mu,\nu), \frac{1}{3}\left[G(\varsigma,\hbar\varsigma,\hbar\varsigma)+G(\mu,\hbar\mu,\hbar\mu)+G(\nu,\hbar\nu,\hbar\nu)\right], \\ \frac{1}{3}\left[G(\varsigma,\hbar\mu,\hbar\mu)+G(\mu,\hbar\nu,\hbar\nu)+G(\nu,\hbar\varsigma,\hbar\varsigma)\right] \end{array}\right\}$$

$$= \max\{\max\left\lfloor \left|\varsigma(\upsilon)-\mu(\upsilon)\right|+\left|\mu(\upsilon)-\nu(\upsilon)\right|+\left|\nu(\upsilon)-\varsigma(\upsilon)\right| \right\rfloor,$$

$$\frac{1}{3}\max\begin{bmatrix} \left|\varsigma(\upsilon)-\int_0^1 G'(\tilde{n},\upsilon)E(\upsilon,\varsigma(\upsilon))d\upsilon\right| + \left|\hbar\varsigma-\hbar\varsigma\right| + \left|\int_0^1 G'(\tilde{n},\upsilon)E(\upsilon,\varsigma(\upsilon))d\upsilon-\varsigma(\upsilon)\right| \\ +\left|\mu(\upsilon)-\int_0^1 G'(\tilde{n},\upsilon)E(\upsilon,\mu(\upsilon))d\upsilon\right| + \left|\hbar\mu-\hbar\mu\right| \\ +\left|\int_0^1 G'(\tilde{n},\upsilon)E(\upsilon,\mu(\upsilon))d\upsilon-\mu(\upsilon)\right| + \left|\nu(\upsilon)-\int_0^1 G'(\tilde{n},\upsilon)E(\upsilon,\nu(\upsilon))d\upsilon\right| \\ +\left|\hbar\nu-\hbar\nu\right| + \left|\int_0^1 G'(\tilde{n},\upsilon)E(\upsilon,\nu(\upsilon))d\upsilon-\nu(\upsilon)\right| \end{bmatrix},$$

$$\frac{1}{3}\begin{bmatrix} \left|\varsigma(\upsilon)-\int_0^1 G'(\tilde{n},\upsilon)E(\upsilon,\mu(\upsilon))d\upsilon\right| + \left|\hbar\mu-\hbar\mu\right| \\ +\left|\int_0^1 G'(\tilde{n},\upsilon)E(\upsilon,\mu(\upsilon))d\upsilon-\varsigma(\upsilon)\right| + \left|\mu(\upsilon)-\int_0^1 G'(\tilde{n},\upsilon)E(\upsilon,\nu(\upsilon))d\upsilon\right| \\ +\left|\hbar\nu-\hbar\nu\right| + \left|\int_0^1 G'(\tilde{n},\upsilon)E(\upsilon,\nu(\upsilon))d\upsilon-\mu(\upsilon)\right| \\ +\left|\nu(\upsilon)-\int_0^1 G'(\tilde{n},\upsilon)E(\upsilon,\varsigma(\upsilon))d\upsilon\right| + \left|\hbar\varsigma-\hbar\varsigma\right| + \left|\int_0^1 G'(\tilde{n},\upsilon)E(\upsilon,\varsigma(\upsilon))d\upsilon-\nu(\upsilon)\right| \end{bmatrix}\}$$

$$= \max\{\max\left\lfloor \left|\varsigma(\upsilon)-\mu(\upsilon)\right|+\left|\mu(\upsilon)-\nu(\upsilon)\right|+\left|\nu(\upsilon)-\varsigma(\upsilon)\right| \right\rfloor,$$

$$\frac{1}{3}\max\begin{bmatrix} \left|\varsigma(\upsilon)-\mu(\upsilon)\right|+\left|\mu(\upsilon)-\nu(\upsilon)\right|+\left|\nu(\upsilon)-\varsigma(\upsilon)\right| \\ -\left(\left|\int_0^1 G'(\tilde{n},\upsilon)E(\upsilon,\varsigma(\upsilon))d\upsilon-\int_0^1 G'(\tilde{n},\upsilon)E(\upsilon,\mu(\upsilon))d\upsilon\right|\right) \\ -\left(\left|\int_0^1 G'(\tilde{n},\upsilon)E(\upsilon,\mu(\upsilon))d\upsilon-\int_0^1 G'(\tilde{n},\upsilon)E(\upsilon,\nu(\upsilon))d\upsilon\right|\right) \\ -\left(\left|\int_0^1 G'(\tilde{n},\upsilon)E(\upsilon,\nu(\upsilon))d\upsilon-\int_0^1 G'(\tilde{n},\upsilon)E(\upsilon,\varsigma(\upsilon))d\upsilon\right|\right) \end{bmatrix},$$

$$\frac{1}{3}\max\left[\begin{array}{l}|\varsigma(\upsilon)-\mu(\upsilon)|+|\mu(\upsilon)-\nu(\upsilon)|+|\nu(\upsilon)-\varsigma(\upsilon)| \\ -\left(\left|\int_0^1 G'(\tilde{n},\upsilon)E(\upsilon,\nu(\upsilon))d\upsilon-\int_0^1 G'(\tilde{n},\upsilon)E(\upsilon,\varsigma(\upsilon))d\upsilon\right|\right) \\ -\left(\left|\int_0^1 G'(\tilde{n},\upsilon)E(\upsilon,\mu(\upsilon))d\upsilon-\int_0^1 G'(\tilde{n},\upsilon)E(\upsilon,\varsigma(\upsilon))d\upsilon\right|\right) \\ -\left(\left|\int_0^1 G'(\tilde{n},\upsilon)E(\upsilon,\nu(\upsilon))d\upsilon-\int_0^1 G'(\tilde{n},\upsilon)E(\upsilon,\mu(\upsilon))d\upsilon\right|\right)\end{array}\right]\}$$

$$=\max\{\max\left[|\varsigma(\upsilon)-\mu(\upsilon)|+|\mu(\upsilon)-\nu(\upsilon)|+|\nu(\upsilon)-\varsigma(\upsilon)|\right],$$

$$\frac{1}{3}\max\left[\begin{array}{l}|\varsigma(\upsilon)-\mu(\upsilon)|+|\mu(\upsilon)-\nu(\upsilon)|+|\nu(\upsilon)-\varsigma(\upsilon)|-\int_0^1 G'(\tilde{n},\upsilon)\left|E(\upsilon,\varsigma(\upsilon))-E(\upsilon,\mu(\upsilon))\right|d\upsilon \\ -\int_0^1 G'(\tilde{n},\upsilon)\,|\,E(\upsilon,\mu(\upsilon))-E(\upsilon,\nu(\upsilon))\,|\,d\upsilon-\int_0^1 G'(\tilde{n},\upsilon)\left|E(\upsilon,\nu(\upsilon))-E(\upsilon,\varsigma(\upsilon))\right|d\upsilon\end{array}\right],$$

$$\frac{1}{3}\max\left[\begin{array}{l}|\varsigma(\upsilon)-\mu(\upsilon)|+|\mu(\upsilon)-\nu(\upsilon)|+|\nu(\upsilon)-\varsigma(\upsilon)|-\int_0^1 G'(\tilde{n},\upsilon)\left|E(\upsilon,\nu(\upsilon))-E(\upsilon,\varsigma(\upsilon))\right|d\upsilon \\ -\int_0^1 G'(\tilde{n},\upsilon)\left|E(\upsilon,\mu(\upsilon))-E(\upsilon,\nu(\upsilon))\right|d\upsilon-\int_0^1 G'(\tilde{n},\upsilon)\left|E(\upsilon,\varsigma(\upsilon))-E(\upsilon,\mu(\upsilon))\right|d\upsilon\end{array}\right]\}$$

$$=\max\{\max\left[|\varsigma(\upsilon)-\mu(\upsilon)|+|\mu(\upsilon)-\nu(\upsilon)|+|\nu(\upsilon)-\varsigma(\upsilon)|\right],$$

$$\frac{1}{3}\max\left[\begin{array}{l}|\varsigma(\upsilon)-\mu(\upsilon)|+|\mu(\upsilon)-\nu(\upsilon)|+|\nu(\upsilon)-\varsigma(\upsilon)|-\int_0^1 G'(\tilde{n},\upsilon)[\left|E(\upsilon,\varsigma(\upsilon))-E(\upsilon,\mu(\upsilon))\right| \\ +\left|E(\upsilon,\mu(\upsilon))-E(\upsilon,\nu(\upsilon))\right|+\left|E(\upsilon,\nu(\upsilon))-E(\upsilon,\varsigma(\upsilon))\right|d\upsilon]\end{array}\right],$$

$$\frac{1}{3}\max\left[\begin{array}{l}|\varsigma(\upsilon)-\mu(\upsilon)|+|\mu(\upsilon)-\nu(\upsilon)|+|\nu(\upsilon)-\varsigma(\upsilon)|-\int_0^1 G'(\tilde{n},\upsilon)[\left|E(\upsilon,\nu(\upsilon))-E(\upsilon,\varsigma(\upsilon))\right| \\ +\left|E(\upsilon,\mu(\upsilon))-E(\upsilon,\varsigma(\upsilon))\right|+\left|E(\upsilon,\varsigma(\upsilon))-E(\upsilon,\mu(\upsilon))\right|]d\upsilon\end{array}\right]\}$$

$$\geq\max\{\max\left[|\varsigma(\upsilon)-\mu(\upsilon)|+|\mu(\upsilon)-\nu(\upsilon)|+|\nu(\upsilon)-\varsigma(\upsilon)|\right],$$

$$\frac{1}{3}\left[\begin{array}{l}|\varsigma(\upsilon)-\mu(\upsilon)|+|\mu(\upsilon)-\nu(\upsilon)|+|\nu(\upsilon)-\varsigma(\upsilon)| \\ -\int_0^1 G'(\tilde{n},\upsilon)[\left|E(\upsilon,\varsigma(\upsilon))-E(\upsilon,\mu(\upsilon))\right|+\left|E(\upsilon,\mu(\upsilon))-E(\upsilon,\nu(\upsilon))\right| \\ +\left|E(\upsilon,\nu(\upsilon))-E(\upsilon,\varsigma(\upsilon))\right|d\upsilon]\end{array}\right]\}$$

$$\geq \max\left\{G(\varsigma,\mu,\nu),\ \frac{1}{3}\left[G(\varsigma,\mu,\nu)-G(\varsigma,\mu,\nu)\int_0^1 G'(\tilde{n},\upsilon)\,d\upsilon\right]\right\}$$

$$\geq \max\left\{G(\varsigma,\mu,\nu),\ \frac{1}{3}\left[G(\varsigma,\mu,\nu)-\frac{1}{8}G(\varsigma,\mu,\nu)\right]\right\}\geq \max\left\{G(\varsigma,\mu,\nu),\ \frac{7}{24}G(\varsigma,\mu,\nu)\right\}$$

Consequently, we have $$\Delta(\varsigma,\mu,\nu)\geq G(\varsigma,\mu,\nu) \tag{4.5}$$

Furthermore, there is $$\Theta(\varsigma,\mu,\nu)\leq \frac{7}{24}G(\varsigma,\mu,\nu) \tag{4.6}$$

Moreover, from (4.3) - (4.6), we have $\chi\left(G(\hbar\varsigma,\hbar\mu,\hbar\nu)\right)\leq \chi\left(\Delta(\varsigma,\mu,\nu)\right)-\Lambda\left(\Theta(\varsigma,\mu,\nu)\right)$

Hence, according to theorem (3.1), $\hbar$ possesses a single fixed point $\varsigma^{*}\in\aleph_1\cap\aleph_2\cap\aleph_3$. Therefore (4.2) presents a distinctive solution.

References

[1] Abkar, A.; Choudhury, B.S., Fixed point result in partially ordered metric spaces using weak contractive inequalities, Facta Univ. Ser. Math. Inform. 2012, 27, 1–11.

[2] Ahmad, J.; Arshad, M.; Vetro, P, coupled coincidence point results for (ψ, ψ)-contractive mappings in partially ordered metric spaces, Georgian Math. J. 2014, 21, 113–124.

[3] Banach.S, Sur les operations dans les ensembles abstraits et leur application auxequations integrales, Fund. Math., 3(1922), 133-181.

[4] Brouwer, L.E.J. Ü ber Abbildungen von Mannigfaltigkeiten, Math. Ann. 1912, 71, 97–115.

[5] Derafshpour, M.; Rezapur, S.; Shahzad, N, Best Proximity point of cyclic ϕ- contractions in ordered metric spaces, Topol. Methods Nonlinear Anal. 2011, 37, 193–202.

[6] ¯Dori c, D., Common fixed point for generalized (ψ, ϕ)-weak contractions, Appl. Math. Lett. 2009, 22, 1896–1900.

[7] Dutta, P.N.; Choudhury, B.S. A generalization of contraction principle in metric spaces, Fixed Point Theory Appl. 2008, 2008, 406368.

[8] George, R.; Rajagopalan, R, Cyclic Contractions and fixed points in a dislocated metric space, Int. J. Math. Anal. 2013, 7, 403– 411.

[9] Hosseini, V.R, Common fixed for generalized (ϕ, ψ)-weak contractions mappings condition of integral type, Int. J. Math. Anal. 2010, 4, 1535–1543.

[10] Hussain, N.; Arshad, M.; Shoaib, A. Fahimuddin, Common fixedpoint results for α−ψ - contractions on a metric space endowed with graph, J. Inequal. Appl. 2014, 1, 1–14.

[11] Karpagam, S.; Agrawal, S, Best Proximity Points theorems for Cyclic Meir

Keeler Contraction Maps, Nonlinear Anal. 2011, 74, 1040–1046.
[12] Khan, M.S.; Swaleh, M.; Sessa, S., Fixed point theorems by altering distances between the points, Bull. Aust. Math. Soc. 1984, 30, 1–9.

[13] Kirk, W.A.; Srinivasan, P.S.; Veeramani, P, Fixed points for mapping satisfying cyclical contractive conditions, Fixed Point Theory 2003, 4, 79–89.
[14] Massa, S, U nosservazione su un teorema di Browder-Roux-Soardi, Boll. U. M. I. 1973,
[15] 4, 151–155.
[16] Murthy, P.P.; Tas, K.; Patel, U.D., Common fixedpoint theorems for generalized (ξ, ω)-weak contraction condition in complete metric spaces, J. Inequal. Appl. 2015, 139.
[17] Murthy, P.P.; Sahu, P.; Amr Elsonbaty, khizar Hyatt Khan, Ramaswamy. R, Radenovi´c.S, Common fixedpoint theorems for generalized (ξ, ω)-weak contraction condition in complete metric spaces, Mathematics. 2023, 11,166.
[18] Mustafa. Z, B. Sims, A new approach to generalized metric spaces, J. Nonlinear Convex Anal., 7(2) (2006), 289-297.
[19] Mustafa. Z, W. Sathanawi, M. Bataineh, Existence of fixedpoint results in G-metric spaces, Internat. J. Math. Mathematical Sci., (10) (2009).
[20] Nashine, H.K.; Golubovic, Z.; Kadelburg, Z., Nonlinear cyclic weak contractions in G-metric spaces and applications to boundary value problems, Fixed Point Theory Appl. 2012, 2012, 227.
[21] Okeke, G.A.; Francis, D.; Gibali, A, On fixed point theorems for a class of$\alpha - \nu$-Meir-Keeler-type contraction mapping in modular extended b-metric spaces, J. Anal. 2022, 30, 1257–1282.
[22] Okeke, G.A.; Francis, D, Fixed point theorems for Geraghty-type mappings applied to solving nonlinear Volterra-Fredholm integral equations in modular G-metric spaces, Arab. J. Math. Sci. 2021, 27, 214–234.
[23] Okeke, G.A.; Francis, D, Fixed point theorems for asymptotically T-regular mappings in preordered modular G-metric spaces applied to solving nonlinear integral equations, J. Anal. 2022, 30, 501–545.
[24] Pacurar M.; Rus, I.A, Fixed Point Theory for ϕ-contractions, Nonlinear Anal. 2010, 72, 1181–1187.
[25] Petrushel, G, Cyclic representations and Periodic points, Stud. Univ. Babes-Bloyai Math 2005, 50, 107–112.
[26] Popescu, O, Fixed points for (ψ, ϕ)-weak contractions, Appl. Math. Lett. 2011, 24, 1–4.
[27] Rhoades, B.E, Some theorems on weakly contractive maps, Nonlinear Anal. 2001, 47, 2683–2693.
[28] Srinivas Chary. V, Sudhaamsh Mohan Reddy. G, Srinivasa Chary. D, Hseyin Isik and Aydi Hassen, Some fixedpoint theorems for modified JS-G-contractions and

an application to integral equation, Journal of Applied Mathematics and Informatics. 38, No. 5-6, 507-518, 2020.

[29] Srinivas Chary. V, Sudhaamsh Mohan Reddy. G, Srinivasa Chary. D, Hseyin Isik and Aydi Hassen, Some fixedpoint theorems on α−β-G-complete G-metric spaces, Carpathian Mathematical Publications. 2021, 13, 1, 58–67.

[30] Srinivas Chary. V, Sudhaamsh Mohan Reddy. G, Srinivasa Chary. D, Stojan Radenovic, Slobodanka Mitrovic, Coupled fixed point theorems of JS-G-contraction on G-metric spaces, Boletim da Sociedade Paranaense de Matematica, 2023, 41, 1–10.

[31] Srinivas Chary. V, Sudhaamsh Mohan Reddy. G, Srinivasa Chary. D, Stojan Radenovic, Existence of fixed points in G-metric spaces, Boletim da Sociedade Paranaense de Matematica, 2023, 41, 1–18