Hypo-edge-Hamiltonian Laceability in some classes of graphs

Shashidhar Shekhar Neelannavar[1], Girisha A[2], P Rajendra[3]

[1]*Department Mathematics, R L Jalappa Institute of Technology,Doddaballapura, Bengaluru.*

[2]*Department of Mathematics, SJB Institute of Technology, Bengaluru.*

[3]*Department of Mathematics, CMR Institute of Technology, Bengaluru.*

Abstract:If u and v are any two vertices in G such that d (u,v) = t and P a non-Hamiltonian path in G, for integer r, thenG is said to be $K^{+r}(K^{-r})$ hypo-edge-Hamiltonian-t*-laceable if $P+re(P-re)$ is a Hamiltonian path between u and v. If G is $K^{+r}(K^{-r})$ hypo-edge-Hamiltonian-t*-laceable for all t such $1 \le t \le diam(G)$ then G is termed $K^{+r}(K^{-r})$ hypo-t*-connected. In this paper, we discuss hypo-edge -Hamiltonian-t-Laceability of Banana tree,prism graph and Cyclo productCy (n , mk) of graphs.

Keywords:*Hamiltonian-t*-laceable graph,* $K^{+r}(K^{-r})$ *hypo edge-Hamiltonian.*

1. Introduction:

Propulsion systems need to be reliable and resilient to failures. Hamiltonian cycles can be related to fault-tolerant designs. Finding minimal spanning subgraph of a graph or network is one of the highly significant and widely researched problems in graph theory. Hamiltonian-*t*-laceable graphs are kind of minimal spanning subgraph of a graph with distinct $t \in Z^{+}$ distances. In [4] Murali R et.al have defined that, Let *G*be a finite, simple connected undirected graph. Let *u*and *v*be two vertices in *G*. The distance between *u*and *v* denoted by $d(u,v)$ is the length of a shortest $u - v$ path in *G*. *G* is Hamiltonian-*t*-laceable if there exists a Hamiltonian path between every pair of vertices *u*and *v*with $d(u,v) = t$ and Hamiltonian-*t**-laceable if there exists at least one such pair with $d(u,v) = t$ where *t* is a positive integer such that $1 \le t \le diamG$. In [1] Girisha A et.al have defined that, A non-Hamiltonian graph *G*is hypo-Hamiltonian if $G - v$ is Hamiltonian for any $v \in V(G)$. If *u* and *v* are any two vertices in *G* such that $d(u,v) = t$ and *P* a non-Hamiltonian path in *G*, then *G* is said to be $K^{+r}(K^{-r})$ hypo-edge-Hamiltonian-*t**-laceable if $P + re(P - re)$ is a Hamiltonian path between *u* and *v*. If *G* is $K^{+r}(K^{-r})$ hypo-edge-Hamiltonian-*t**-laceable for all *t* such $1 \le t \le diamG$ then G is termed $K^{+r}(K^{-r})$ hypo- *t**-connected.

In [2] and [3] we have the following results on Hypo-edge-Hamiltonian Laceability.

Theorem 1: For every odd positive integer, $n \ge 5$ such that $n-1 = 1 \bmod 4, J_n\ is\ K^{+1}$ -Hypo-edge-Hamiltonian- $t^{*} - laceable$ where $1 \le t \le diamJ_n$.

Theorem 2: Let J_n be the flower snark. Then for every odd positive integer, $n \ge 7$ such that $n-1 = 1 \bmod 4, J_n\ is\ K^{+1}$ -Hypo-edge-Hamiltonian- $t^{*} - laceable$ where $1 \le t \le diamJ_n$.

Theorem 3: For every positive integer $n \geq 2$, J_{2n} is K^{+0} Hypo-edge-Hamiltonian $t^{*} - laceable$

For *t=1* and K^{+1} Hypo-edge-Hamiltonianfor $2 \leq t \leq diamJ_{2n}$.

Based on Hypo-edge-Hamiltonian Laceable we present the following results

2. Laceability in Banana Trees

Definition 2.1.Banana Trees

Banana tree is graph obtained by connecting one leaf of each of n-copies of an k-star graph with single root vertex that is distinct from all the stars.

Example:

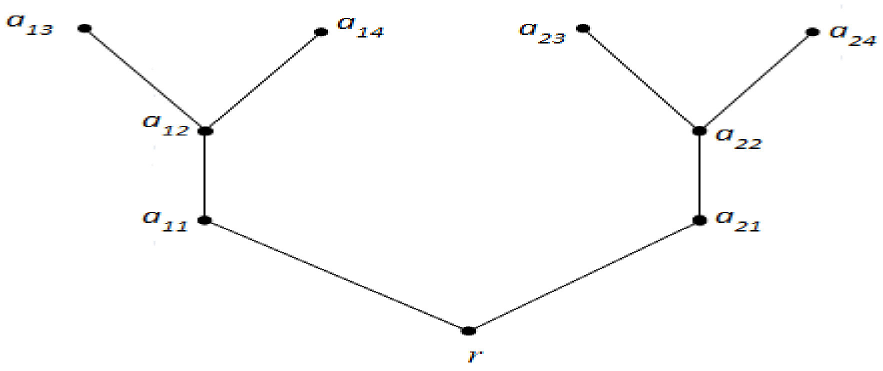

Figure1. Banana tree $B_{2,4}$

Based on this definition we have the following results.

Theorem2.1: The Banana tree $B_{m,n}$ for $m \geq 2$, $n \geq 4$ then $B_{m,n}$ is k^{+1} hypo-Hamiltonian-t^{*}-laceable for distance$t$, *where* $\left(1 \leq t \leq Diam\, B_{m,n}\right)$.

Let $B_{m,n} = G$ is k^{+1} hypo-Hamiltonian-t^{*}-laceable for distance t , *where* $\left(1 \leq t \leq Diam\, G\right)$

The Banana tree $B_{m,n}$ for $m \geq 2$, $n \geq 4$ with *mn*+1 nodes and *mn* edges with diameter 6.

We introduce the following terminologies to prove the results.

$A_n\, p(n) : a_{m1}\ a_{m2}\ a_{m3} * a_{m4} * a_{m5} * - - - - - - - - * a_{mn}$ a path from vertex a_{m1} *to* a_{mn}

$a_m\, p(n)\ : a_{1n} * a_{1(n-1)} * a_{1(n-2)} * - - - - - - - - * a_{14}$ a path from a_{1n} *to* a_{14}

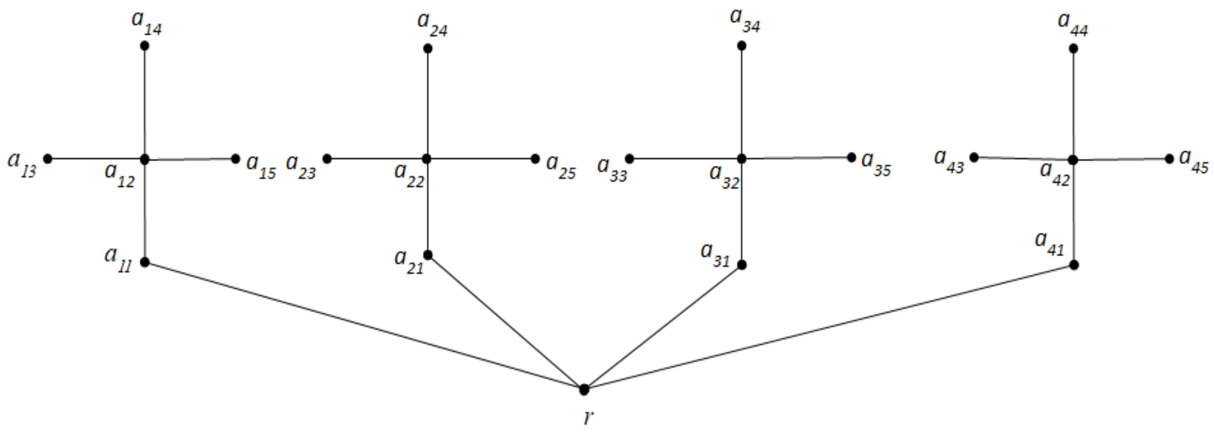

Figure2. *Banana tree* $B_{4,5}$

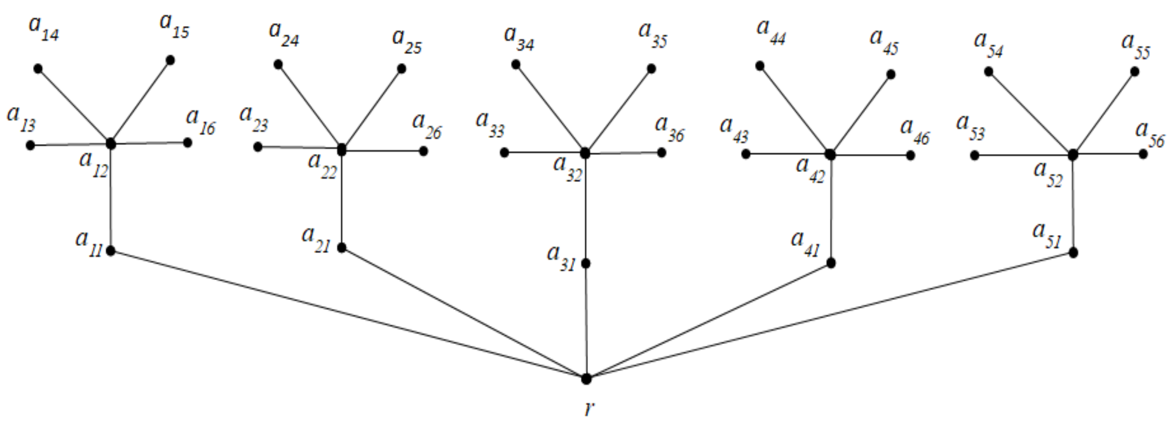

Figure3. *Banana tree* $B_{5,6}$

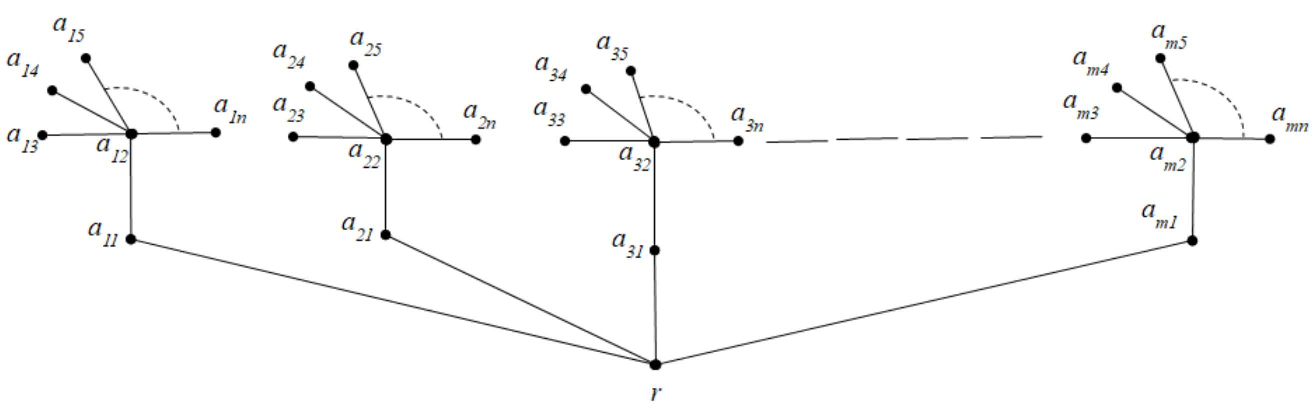

Figure4. *Banana tree* $B_{m,n}$

We establish the following cases

Case i) *For* $t = 1$ *and* $m \geq 2$, $n \geq 4$

In this case we observe that $d(a_{11}a_{12})=1$ in G then the path

$$(a_{11,}\ r)\cup(r\ ,\ a_{m1})\cup A_m p(n)\cup A_{m-1}p(n)\cup A_{m-2}p(n)------\cup A_2 p(n)\cup(a_{2n}\ ,\ a_{1n})\cup a_m p(n)$$

$$\cup(a_{14,}*a_{13})\cup(a_{13},a_{12})$$

is K^{+1} hypo edge-Hamiltonian-1*-laceable path from a_{11} *to* a_{12}.

Case ii) *For* $t=2$ *and* $m\geq 2\ ,\ n\geq 4$

In this case we observe that $d(a_{11}a_{13})=2$ in G then the path

$$(a_{11,}\ r)\cup(r\ ,\ a_{m1})\cup A_m p(n)\cup A_{m-1}p(n)\cup A_{m-2}p(n)------\cup A_2 p(n)\cup(a_{2n}\ ,\ a_{1n})\cup a_m p(n)$$

$$\cup(a_{14,}a_{12})\cup(a_{12},a_{13})$$

is K^{+1} hypo edge-Hamiltonian-2*-laceable path from a_{11} *to* a_{13}.

Case iii) *For* $t=3$ *and* $m\geq 2\ ,\ n\geq 4$

In this case we observe that $d(a_{13},r\ )=3$ in G then the path

$$(r\ ,\ a_{m1})\cup A_m p(n)\cup A_{m-1}p(n)\cup A_{m-2}p(n)------\cup A_2 p(n)\cup(a_{2n}\ ,\ a_{1n})\cup a_m p(n)$$

$$\cup(a_{14,}a_{12})\cup(a_{12},a_{11})\cup(a_{11},*a_{13})$$

is K^{+1} hypo edge-Hamiltonian-3*-laceablepath from a_{13} *to* r.

Case iv) *For* $t=4$ *and* $m\geq 2\ ,\ n\geq 4$

In this case we observe that $d(a_{13},a_{m1})=4$ in G then the path

$$(a_{m1}\ ,\ a_{m2})\cup[A_m p(n)-(a_{m1}\ ,\ a_{m2})]\cup A_{m-1}p(n)\cup A_{m-2}p(n)------\cup A_2 p(n)\cup(a_{2n}\ ,\ a_{1n})$$

$$\cup a_m p(n)\cup(a_{14,}a_{12})\cup(a_{12},a_{11})\cup(a_{11},r)\cup(r,*a_{13})$$

is K^{+1} hypo edge-Hamiltonian-4*-laceable path from a_{13} *to* a_{m1}.

Case v) *For* $t=5$ *and* $m\geq 2\ ,\ n\geq 4$

In this case we observe that $d(a_{13}, a_{m2}) = 5$ in G then the path

$$(a_{m2}\ ,\ a_{m3}) \cup \{A_m\, p(n) - [(a_{m1}\ ,\ a_{m2}) \cup (a_{m2,}\, a_{m3})]\} \cup A_{m-1}\, p(n) \cup A_{m-2}\, p(n) ------ \cup A_2\, p(n)$$

$$\cup (a_{2n}\ ,\ a_{1n}) \cup a_m\, p(n) \cup (a_{14\,,}\, a_{12}) \cup (a_{12}\, , a_{11}) \cup (a_{11}, r) \cup (r\, , a_{m1}) \cup (a_{m1}, {*}a_{13})$$

is K^{+1} hypo edge-Hamiltonian-5*-laceable path from a_{13} *to* a_{m2}.

3. Laceability in Prism graphs

Definition 3.1: Prism graph

A prism graph, is a graph corresponding to the skeleton of n-prism. A prism graph Y_n has $2n$ vertices and $3n$ edges.

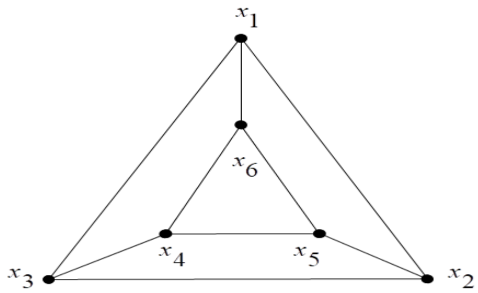

Figure5. *Prism Graph Y_3*

Based on this definition we have the following results.

Theorem 3.1 : Let graph $G = Y_n$, for $n \geq 3$ where n is odd integer is K^{+1} hypo-edge-Hamiltonian-t*-laceable for distance t, where $1 \leq t \leq 3$.

Proof: Consider Prism Graph $G = Y_n$

Let x_i and x_j be any two vertices of G at a distance t i.e $d(x_i , x_j) = t$

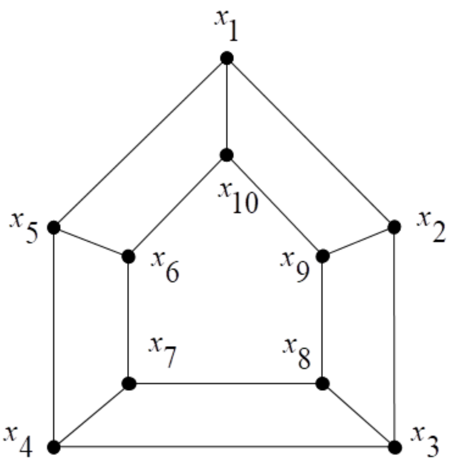

Figure6. *Prism Graph* Y_5

We introduce the following terminologies to prove the results.

$$P_1 = x_{2n-1} - x_{2n} - x_{n+1}\{(x_{n+1} - x_n - x_{n-1} - x_{n+2} - x_{n+3}) \cup$$

$$(x_{n+3} - x_{n-2} - x_{n-3} - x_{n+4} - x_{n+5}) --- \cup (x_{2n-4} - x_5 - x_4 - x_{2n-3} - x_{2n-2})\}$$

$$P_2 = x_{2n-2} - x_{2n-1} - x_{2n} - x_{n+1} - \{(x_{n+1} - x_n - x_{n-1} - x_{n+2} - x_{n+3}) \cup$$

$$(x_{n+3} - x_{n-2} - x_{n-3} - x_{n+4} - x_{n+5}) --- \cup (x_{2n-6} - x_7 - x_6 - x_{2n-5} - x_{2n-4})\}$$

We have the following cases.

Case i) For $t = 1$ and $n \geq 3$

In this case, we observe that $d(x_i, x_j) = 1$ for $i=1$, $j = 2n$ in G then the path

$$\bigcup_{k=1}^{2n-1} (x_k, x_{(k+1)})$$

is a Hamiltonian-1^* -laceable path between x_i and x_j

Case ii) For $t = 2$ *and* $n \geq 3$

In this case, we observe that $d(x_i, x_j) = 2$ for $i=1, j=2n-1$ in G then the path

$$P_1 \cup (x_{2n-2}, x_3) \cup (x_3, x_2) \cup (x_2, x_1)$$

is a K^{+1} hypo edge-Hamiltonian-2* - laceable path between the vertices x_i and x_j .

Case iii) For $t = 3$and$n\geq 5$

In this case, we observe that$d(x_i , x_j)=3$ *where* $i=1$, $j = 2n-2$ in G and the path

$$P_2 \cup (x_{2n-4}, x_5) \cup (x_5{}^*, x_{2n-3}) \cup (x_{2n-3}, x_4) \cup (x_4, x_3) \cup (x_3, x_2) \cup (x_2, x_1)$$

is a K^{+1} hypo edge-Hamiltonian-3*- laceable path between the vertices x_i and x_j .

Theorem3.2: Let graph $G = Y_n$,for $n\geq 4$where n is even integer is K^{+1} hypo-edge-Hamiltonian-t*-laceable for distance$t$,where$1 \leq t \leq 3$.

Proof: Consider Prism Graph $G = Y_n$.

Let x_i and x_j be any two vertices of G at a distance t i.e $d(x_i , x_j)=t$

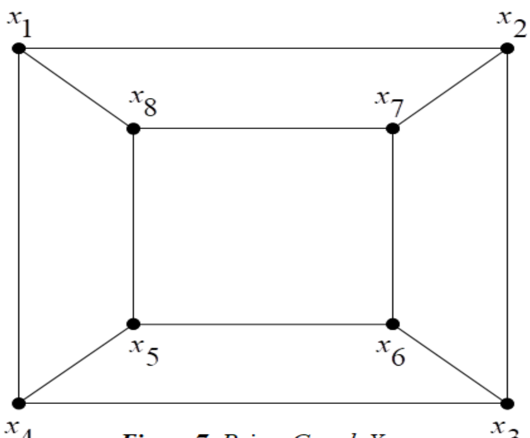

Figure7. *Prism Graph* Y_4

We introduce the following terminologies to prove the results.

$$P_3 = x_{2n-1} - x_{2n} - x_{n+1}\{(x_{n+1} - x_n - x_{n-1} - x_{n+2} - x_{n+3}) \cup$$
$$(x_{n+3} - x_{n-2} - x_{n-3} - x_{n+4} - x_{n+5}) --- \cup (x_{2n-5} - x_6 - x_5 - x_{2n-4} - x_{2n-3})\}$$

$$P_4 = x_{2n-2} - x_{2n-1} - x_{2n} - x_{n+1} - \{(x_{n+1} - x_n - x_{n-1} - x_{n+2} - x_{n+3}) \cup$$
$$(x_{n+3} - x_{n-2} - x_{n-3} - x_{n+4} - x_{n+5}) --- \cup (x_{2n-5} - x_6 - x_5 - x_{2n-4} - x_{2n-3})\}$$

We have the following cases.

Case i) For $t = 1$and$n \geq 4$

In this case, we observe that $d(x_i , x_j)=1$ for $i=1$, $j= 2n$ in G then the path

$$\bigcup_{k=1}^{2n-1} (x_k , x_{(k+1)})$$

is a Hamiltonian-1^* -laceable path between x_i and x_j

Case ii) For $t = 2$ *and* $n \geq 4$

In this case, we observe that $d(x_i , x_j)=2$ for $i=1, j=2n-1$ in G then the path

$$P_3 \cup (x_{2n-3}, x_4) \cup (x_4{}^*, x_{2n-2}) \cup (x_{2n-2}, x_3)(x_3, x_2) \cup (x_2 , x_1)$$

is a K^{+1} hypo edge-Hamiltonian-2^* - laceable path between the vertices x_i and x_j .

Case iii) For $t = 3$*and* $n \geq 4$

In this case, we observe that $d(x_i , x_j)=t$ for $i=1, j=2n-2$ in G then the path

$$P_4 \cup (x_{2n-3}, x_4) \cup (x_4, x_3) \cup (x_3 , x_2) \cup (x_2, x_1)$$

is a K^{+1} hypo edge-Hamiltonian-3^* - laceable path between the vertices x_i and x_j .

4. Laceability in Cyclo Product of Graphs

Definition4.1: Let m, n and k be positive integers such that $m \geq 2$, $n \geq 4$. TheCyclo product $C_y(n , mk)$ is obtained by joining each vertex a_i $(1 \leq i \leq n-1)$ *in* C_n *at* a_{i+mk} under modulo n. where $k = \dfrac{n-2}{m}$.

Cyclo product graphs are subclasses of Circulant graphs.

By definition $C_y(n , mk)$ is Hamiltonian-1^*-laceable for all $n \geq 4$.

In [4] Girisha A et.al proved the results on Laceability in Cyclo Product of Graphs

Theorem4. 1: The graph $C_y(n\ ,\ 2k), n \geq 6$ is Hamiltonian-*t*-laceable for *t =1, 2*.

Theorem 4.2: The graph $C_y(n\ ,\ 3k), n \geq 6$ is Hamiltonian-*t*-laceable for *t =1, 2*.

Based on Laceability in Cyclo Product of Graphs we prove the following results

Theorem 4.3: The graph $C_y(n, mk), n \geq 5, m \geq 4$. is Hamiltonian-*t*-laceable for distance t, where $(1 \leq t \leq Diam\,G)$

Proof: Let $G = C_y(n\ ,\ mk)$ be a graph with *n*-numbers of vertices and *m*-edges

Let this vertex set be $v = \{a_0, a_1, ...a_{n-1}\}$

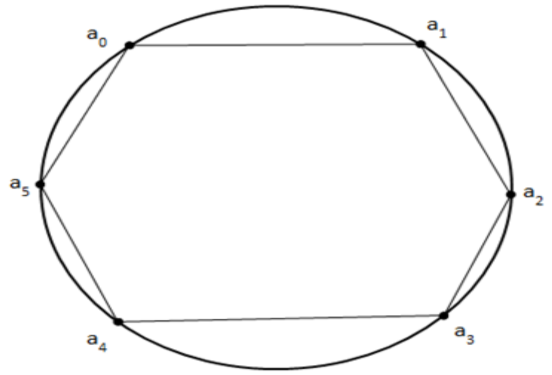

Figure 8. *Cyclo product Cy(6 , 5k)*

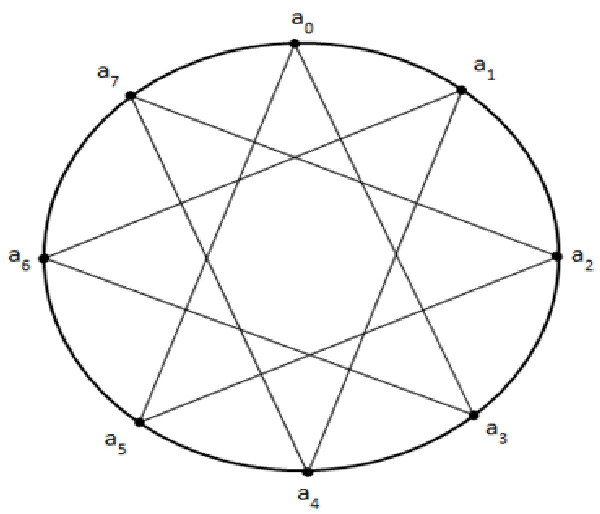

Figure 9. *Cyclo product Cy(8 , 5k)*

We have the following cases

Case i) For $n = ml+2$, $m \geq 4$ and $l \geq 1$

$$The\ path\ p:\ (a_0, a_{n-1}) \cup (.a_{n-1}, a_{n-2}) .... \cup (a_5, a_4) \cup (a_4, a_2) \cup (a_2, a_1) \cup (a_1, a_3)$$

is the Hamiltonian path from a_0 to a_3.

Case ii)For $n = ml+3$, $m \geq 4$ and $l \geq 1$

$$The\ path\ p:\ (a_0, a_1) \cup (.a_1, a_4) \cup (a_4, a_5) \cup (a_5, a_6) .... \cup (a_{n-3}, a_{n-2}) \cup (a_{n-2}, a_{n-1})$$

$$\cup (a_{n-1}, a_3) \cup (a_3, a_2)$$

is the Hamiltonian path from a_0 to a_2.

Case iii)For $n = ml+4$, $m \geq 4$ and $l \geq 1$

$$The\ path\ p:\ (a_0, a_{n-4}) \cup (a_{n-4}, a_{n-3}) .... \cup (a_4, a_3) \cup (a_3, a_{n-1}) \cup (a_{n-1}, a_{n-2})$$

$$\cup (a_{n-2}, a_{n-3}) \cup (a_{n-3}, a_1) \cup (a_1, a_2)$$

isthe Hamiltonian path from a_0 to a_2.

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