TuijinJishu/Journal of Propulsion Technology
ISSN: 1001-4055
Vol. 44 No.6 (2023)

Hamiltonian Laceability in Complete-k-Partite Graph

Sumitra Devi M.R[1],Girisha A[2],KavitaPermi[3]

[1]*Department of Mathematics, SCE, Bengaluru.*

[2]*Department of Mathematics, SJB Institute of Technology, Bengaluru.*

[3]*Department of Mathematics, School of Engineering, Presidency University,Bangalore.*

Abstract:*A graph G is Hamiltonian laceable if there exists a Hamiltonian path between every pair of vertices in G at an odd distance. G is Hamiltonian -t-laceable if there exists a Hamiltonian path between every pair of vertices a and b with* $d(a,b) = t$ *.In this paper we explore Hamiltonian-t-laceabilityproperty inComplete-k-partite graph.*

Keywords:*Hamiltonian-t- laceable,complete-k-partite graph.*

1. Introduction

Hamiltonian cycles, can be used to optimize communication networks, ensuring that signals or commands can be transmitted with minimal delays or energy consumption. The nodes of the graph can be determined by the components incorporated in the power system architecture. The edges of a graph represent interconnections between the various components. A connected simple graph G the distance between any two nodes a and b is denoted as $d(a\ b)$, is the length of the least path joining a and b in G .For standard definitions we refer Hararey[1].A connected simple graph G is Hamiltonian (Hamilton)laceable[3] if there is a Hamilton path joining any two nodes in G at a distance of odd positive integer.$G$is Hamilton-t*-laceable(H-t*-l) [4]if there is a Hamilton path in at least a pair of nodes$u$ and v with distance t, where $1 \le t \le diamG$.

2. Complete-k-partite graph

A complete-k-partite graph is a t is a set of graph vertices decomposed into k-disjoint sets such that no two graph vertices with in the same set are adjacent.For $k = 3$,is a

Complete tripartite graph $K_{p,q,r}$ with $(p+q+r)$number of vertices,$(pq+qr+rp)$ number of edges and the diameter is two.

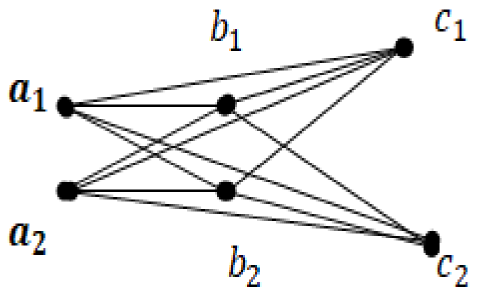


Fig1. $\mathbf{K_{2,2,2}}$

Theorem 2.1: The Complete tripartite graph $G = K_{p,q,r}$ is Hamiltonian-t-laceable for $1 \le t \le diamG$ where $|p| = |q| = |r|$

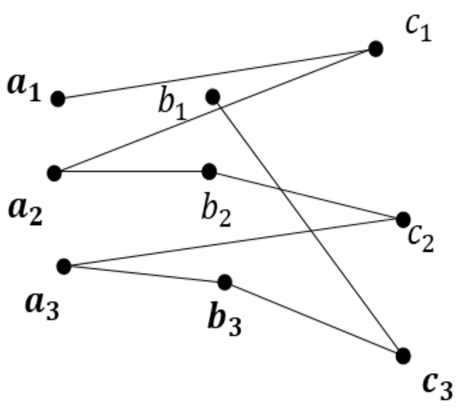
Proof: let $(a_1, a_2, ...a_n), (b_1, b_2, ...b_n), (c_1, c_2, ...c_n)$ denote the vertices in first, second and third partite respectively**.**

In $G = K_{p,q,r}$, $|p| = |q| = |r| = n$, n is the number of nodes in each partites

Case1: For distance $t = 1$ in G

Claim1: The Hamilton path is $a_1\, c_1 \left(\bigcup_{j+1}^{3} a_i b_i\, c_i \right) b_1$ at $\mathrm{d}(a_1\, b_j) = 1\ in\ G,\ j = 1$

The Hamilton path is $\left(\bigcup_{i=1}^{j-1} a_i b_i\, c_i \right) a_j\, c_j \left(\bigcup_{j+1}^{n} a_i b_i\, c_i \right) b_j$ at $\mathrm{d}(a_1\, b_j) = 1,\ j \ge 2\ in\ G$



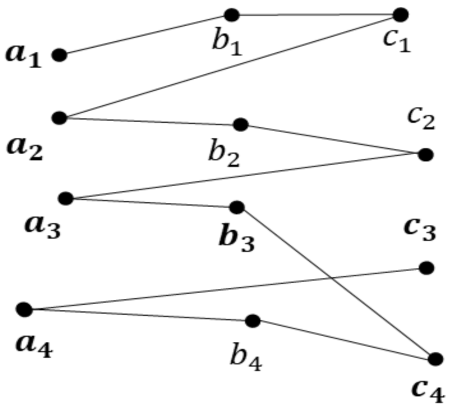


Fig2.Hamilton path from $\mathbf{a_1}$ **to** $\mathbf{b_1}$ **in** $\mathbf{K_{3,3,3}}$**Fig3.Hamilton path from** $\mathbf{a_1}$ **to** $\mathbf{c_3}$ **in** $\mathbf{K_{4,4,4}}$

Claim2: The Hamiltonian path is $\left(\bigcup_{i=1}^{j-1} a_i b_i\, c_i \right) a_j\, b_j \left(\bigcup_{j+1}^{n} c_i b_i\, a_i \right) c_j$ at

$\mathrm{d}(a_1\, c_j) = 1\ in\ G\ for\ j \ge 2$

Case2: For distance $t = 2$ in G

Claim3: The Hamilton path is $\left(\bigcup_{\substack{i=1\\i\neq j}}^{j-1} a_i b_i c_i\right) b_j\, c_j\, a_j$ at distance $d(a_1\, a_j) = 2$ in G for $j \geq 2$

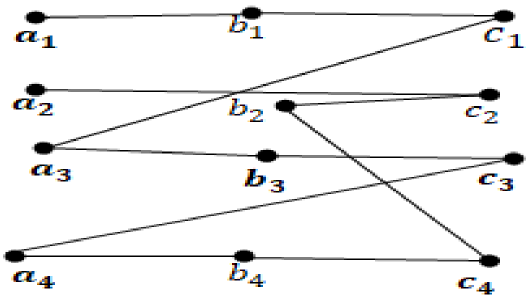


Fig4.Hamilton path from a_1 to a_2 in $K_{4,4,4}$

Claim4: The Hamilton path is $b_1\, a_1\, c_1 \left(\bigcup_{\substack{i=2\\i\neq j}}^{n} a_i b_i c_i\right) a_j\, c_j\, b_j$ at distance $d(b_1\, b_j) = 2$ in G for $j \geq 2$

Claim5: The Hamilton path is $\left(\bigcup_{\substack{i=1\\i\neq j}}^{n} c_i b_i a_i\right) b_j\, a_j\, c_j$ at distance $d(c_1\, c_j) = 2$ in G for $j \geq 2$

Hence from the claim(1 to 5), we conclude a complete tripartite graph $G = K_{p,q,r,} |p| = |q| = |r|$ is Hamiltonian-t-laceable for $1 \leq t \leq diamG$.

Theorem2.2: The Complete tripartite graph $G = K_{p,q,r}$ such that $|p| < |q|$ is Hamiltonian-t-laceable for $1 \leq t \leq diamG$

Proof: Let $(a_1, a_2, \ldots\ldots a_m), (b_1, b_2, \ldots\ldots b_n), (c_1, c_2, \ldots\ldots c_n)$ denote the vertices in first, second and third partite respectively.

In $G = K_{p,q,q}$, $|p| = m \geq 1, |q| = n \geq 2, \ni |m - n| \geq 1$, n is the number of nodes in each partites

Case1: For $t = 1$ in $d(a_1\, b_j) = 1$

Claim6: The Hamilton path is $a_1 c_1 \left(\bigcup_{i=2}^{m} a_i b_i c_i\right)\left(\bigcup_{i=m+1}^{n} b_i\, c_i\right) b_1$ at distance $d(a_1\, b_j) = 1,\ j = 1$

The Hamilton path is $a_1 c_1 b_1 (c_2 a_2)\left(\bigcup_{i=3}^{m} c_i b_i a_i\right)\left(\bigcup_{i=m+1}^{n} b_i\, c_i\right) b_2$ at distance $d(a_1\, b_j) = 1,\ j = 3,4$

The Hamilton path is $a_1c_1b_1\left(\bigcup_{i=2}^{j-i} c_ib_ia_i\right)(c_ja_j)\left(\bigcup_{i=4}^{m} c_ib_ia_i\right)\left(\bigcup_{i=m+1}^{n} b_i\, c_i\right)b_j$

at distance $d(a_1\, b_j) = 1,\ j \geq 5$

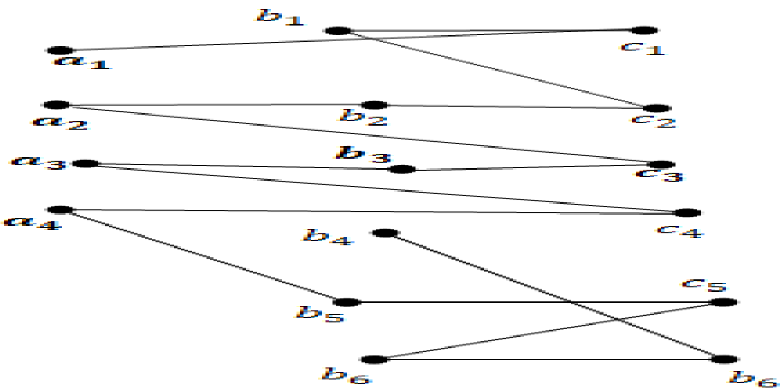
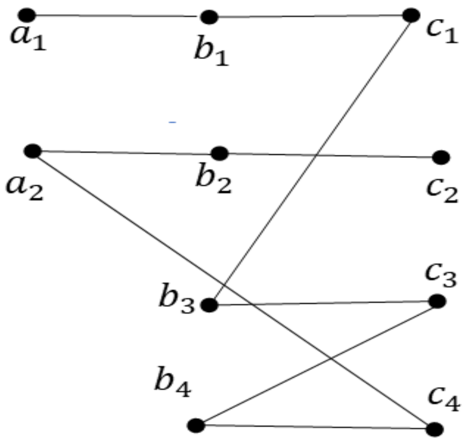


Fig5.Hamilton path from a_1 to b_4 in $K_{4,6,6}$

Claim7: For $t = 1$in $G\, at\, d(a_1\, c_j) = 1,\ j = 1$ the Hamilton path is $a_1b_1\left(\bigcup_{i=2}^{m} c_ib_ia_i\right)\left(\bigcup_{i=m+1}^{n} c_i\, b_i\right)c_1$

The Hamilton path is $\left(\bigcup_{\substack{i=1\\ i\neq j}}^{m} a_ib_ic_i\right)\left(\bigcup_{\substack{i=m+1\\ i\neq j}}^{n} b_i\, c_i\right)a_jb_jc_j \quad d(a_i\, c_j) = 1,\ j \geq 2$



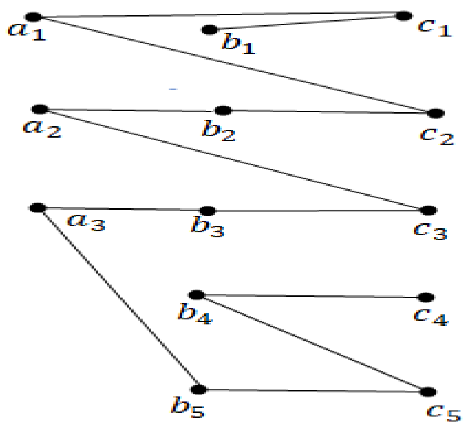


Fig6.Hamilton path from a_1 to b_4 in $K_{2,4,4}$Fig7.Hamilton path from b_1 to c_4 in $K_{3,5,5}$

Claim8: For $t = 1$in $G\, d(b_1\, c_j) = 1$ the Hamilton path is $b_1a_1\left(\bigcup_{\substack{i=2\\ i\neq j}}^{m} c_ib_ia_i\right)\left(\bigcup_{i=m+1}^{n} c_ib_i\right)c_1$

The Hamilton path is $b_1c_1a_1\left(\bigcup_{\substack{i=2\\i\neq j}}^{m}c_ib_ia_i\right)\left(\bigcup_{i=m+1}^{n}b_ic_i\right)a_jb_j\,c_j\ at\ d(b_1\,c_j)=1,\ j\geq 2$

Case2: For $t=2$ in G

Claim9:The Hamilton path is $\left(\bigcup_{\substack{i=1\\i\neq j}}^{m}a_ib_ic_i\right)\left(\bigcup_{i=m+1}^{m}b_ic_i\right)b_jc_ja_j$ at $d(a_1\,a_j)=2,\ j\geq 2$

Claim10:The Hamilton path for $d(b_1\,b_j)=2,\ j\geq 2\,in\ G$ is

$$b_1c_1a_1\left(\bigcup_{\substack{i=2\\i\neq j}}^{j-1}c_ib_ia_i\right)c_j\left(\bigcup_{i=j+1}^{m}a_ib_ic_i\right)\left(\bigcup_{i=m+1}^{n}b_ic_i\right)a_jb_j$$

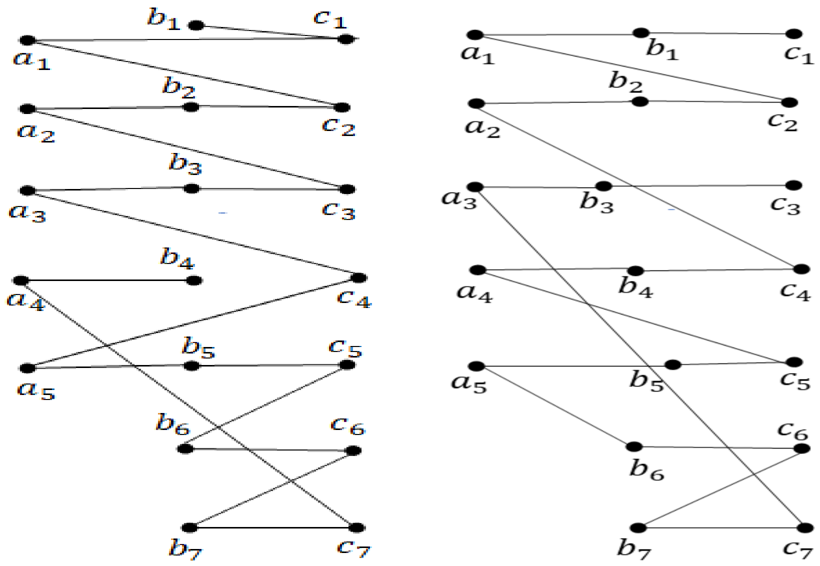


Fig8.Hamilton path from b_1 to b_4 in $K_{5,7,7}$Fig8.Hamilton path from a_1 to b_4 in $K_{5,7,7}$

Claim11:The Hamilton path in $d(c_1\,c_j)=2,\ j\geq 2\,in\ G$ is

$$\left(\bigcup_{\substack{i=1\\i\neq j}}^{m}c_i\,b_i\,a_i\right)\left(\bigcup_{\substack{i=m+1\\i\neq j}}^{n}b_ic_i\right)a_jb_j\,c_j$$

Hence from the claim(6 to11), we conclude a Complete tripartite graph $G=K_{p,q,q}$ is Hamiltonian-t-laceable for $1\leq t\leq diamG$.

Theorem 2.3: A Complete tripartite graph $G=K_{p,q,r}\ where|p|<|q|<|r|$ such that $|p-r|=2$ is Hamiltonian-t-laceable for $1\leq t\leq diamG$

Proof: Let $(a_1,a_2,.......a_m),(b_1,b_2,.......b_n),(c_1,c_2,.......c_s)$ denote the vertices in first, second and third partite respectively.

Case1: For $t=1$ in G

Claim12: The Hamilton path is $a_1c_1\left(\bigcup_{i=2}^{3} b_ic_{i+1}\right)a_2c_4\left(\bigcup_{i=3}^{p} a_ib_{i+1}c_{i+2}\right)b_1$ for $d(a_1 b_j)=1, j=1$

The Hamilton path for $d(a_1 b_j)=1, j=2$ in G

$$a_1c_1\, b_1c_2\, b_3c_3a_2c_4\left(\bigcup_{i=3}^{p} a_ib_{i+1}c_{i+2}\right)b_2$$

The Hamilton path for $d(a_1 b_j)=1, j=3$ in G

$$a_1c_1\left(\bigcup_{i=1}^{2} b_ic_{i+1}\right)a_2c_4\left(\bigcup_{i=3}^{p} a_ib_{i+1}c_{i+2}\right)b_3$$

The Hamilton path for $d(a_1 b_j)=1$ in G *for* $j\geq 4$

$$a_1c_1\, b_1c_2\, b_3c_3\left(\bigcup_{i=2}^{j-2} a_ib_{i+1}c_{i+2}\right)a_{j-1}c_{j+1}\left(\bigcup_{i=4}^{p} a_ib_{i+1}c_{i+2}\right)b_j$$

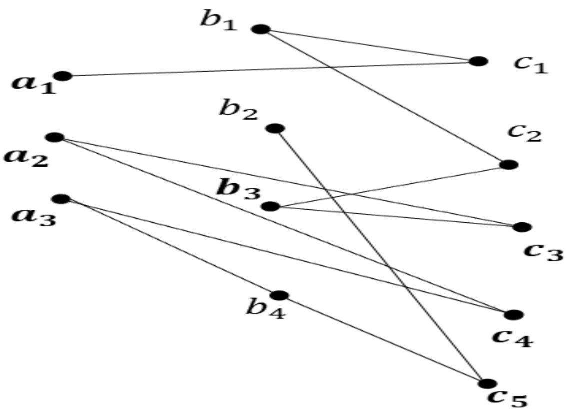
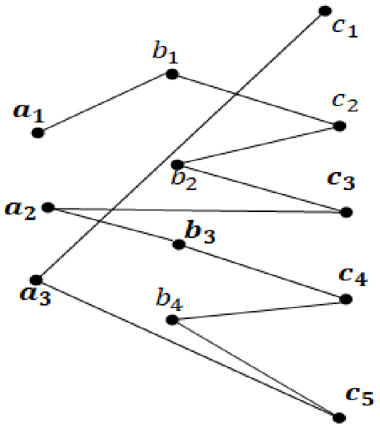


Fig9.Hamilton path from a_1 to b_2 in $K_{3,4,5}$

Claim13: The Hamilton path is $\left(\bigcup_{i=1}^{r-1} a_i\, b_i\, c_i\right) b_r\, c_{r+1}\left(\bigcup_{i=r}^{p-1} a_ib_{i+1}\, c_{i+2}\right)b_q\, c_r\, a_p\, c_j$ for $d(a_1\, c_j)=1$, $j\geq 2$

The Hamilton path is $a_1\, b_1\, c_2b_2\, c_3\left(\bigcup_{i=2}^{p-1} a_ib_{i+1}\, c_{i+2}\right)b_q\, c_r\, a_p\, c_1$ at $d(a_1\, c_j)=1, j=1$



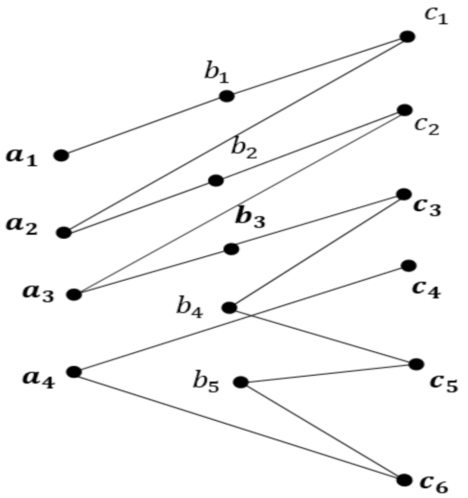


Fig10.Hamilton path from a_1 to c_1 in $K_{3,4,5}$Fig11.Hamilton path from a_1 to c_4 in $K_{4,5,6}$

Claim14: The Hamilton path is $\left(\bigcup_{i=1}^{2} b_i\, c_{i+1}\right)\left(\bigcup_{i=2}^{p} a_i b_{i+1}\, c_{i+2}\right) a_1\, c_1$ at $d(b_1\, c_1) = 1\ in\ G$

The Hamilton path is $b_1\, c_1 b_2\, c_3 \left(\bigcup_{i=2}^{p} a_i b_{i+1}\, c_{i+2}\right) a_1\, c_2$ at $d(b_1\, c_2) = 1\ in\ G$

The Hamilton path is $\left(\bigcup_{i=1}^{2} b_i\, c_i\right)\left(\bigcup_{i=2}^{p} a_i b_{i+1}\, c_{i+2}\right) a_1\, c_3$ at $d(b_1\, c_3) = 1$

The Hamilton path is $\left(\bigcup_{i=1}^{2} b_i\, c_i\right)\left(\bigcup_{i=2}^{j-2} a_i b_{i+1}\, c_{i+1}\right)\left(\bigcup_{i=j-1}^{p} a_i b_{i+1}\, c_{i+2}\right) a_1\, c_j$ at

$d(b_1\, c_j) = 1 \quad for \quad j \geq 4$

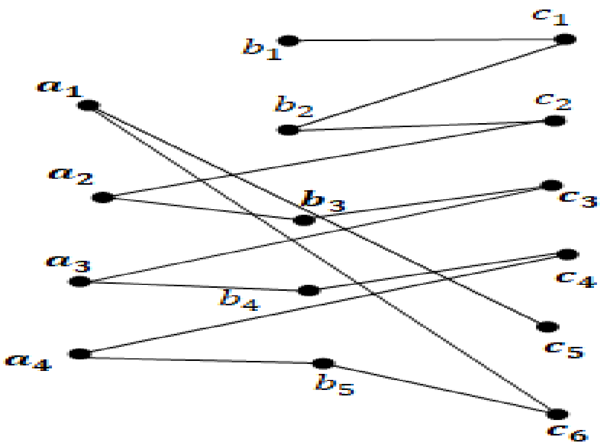
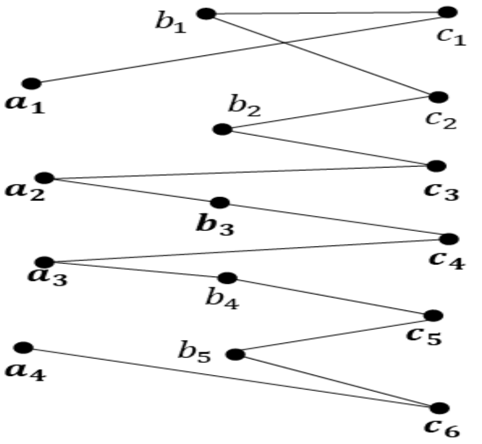


Fig12.Hamilton path from b_1 to c_5 in $K_{4,5,6}$

Case2: For t=2 in G

Claim15: The Hamilton path is $a_1\left(\bigcup_{i=1}^{3} c_i b_i\right) c_4 \left(\bigcup_{i=3}^{p} a_i b_{i+1}\, c_{i+2}\right) a_2$ at $d(a_1\, a_j)=2,\ j=2$

The Hamilton path is $a_1\left(\bigcup_{i=1}^{2} c_i\, b_i\right) c_3 \left(\bigcup_{i=2}^{j-1} a_i b_{i+1}\, c_{i+2}\right) b_{j+1}\, c_{j+2}\left(\bigcup_{i=j+1}^{p} a_i b_{i+1}\, c_{i+2}\right) a_j$ at $d(a_1\, a_j)=2,\ j\geq 3$



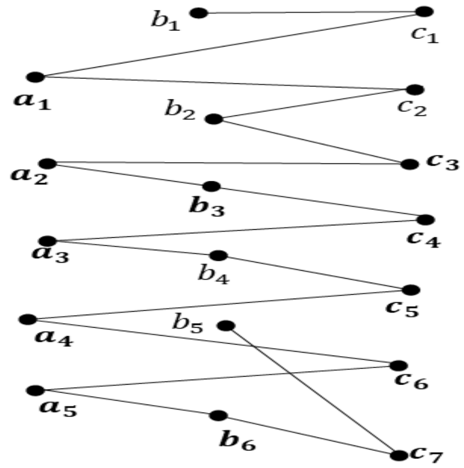


Fig13.Hamilton path from a_1 to a_4 in $K_{4,5,6}$Fig14.Hamilton path from b_1 to b_5 in $K_{5,6,7}$

Claim16: The Hamilton path is $b_1\left(\bigcup_{i=1}^{2} c_i\, a_i\right) c_3\ b_3\, c_4 \left(\bigcup_{i=3}^{p} a_i b_{i+1}\, c_{i+2}\right) b_2$ at $d(b_1\, b_j)=2,\ j=2\, in\ G$

The Hamilton path is $b_1\, c_1 a_1\, c_2 b_2 c_3\ \ a_2\, c_4\left(\bigcup_{i=3}^{p} a_i b_{i+1}\, c_{i+2}\right) b_3$ at $d(b_1\, b_j)=2\ ,\ j=3 in\ G$

The Hamilton path is $b_1\, c_1 a_1\, c_2 b_2\, c_3\left(\bigcup_{i=2}^{j-2} a_i b_{i+1}\, c_{i+2}\right) a_{j-1}\, c_{j+1}\left(\bigcup_{i=r}^{p} a_i b_{i+1}\, c_{i+2}\right) b_j$ a $d(b_1\, b_j)=2,\ j\geq 4 in\ G$

Claim17: The Hamilton path is $c_1 b_1\left(\bigcup_{i=1}^{p} c_{i+2}\, b_{i+1}\, a_i\right) c_2$ at $d(c_1\, c_j),\ j=2 in\ G$

The Hamilton path is $c_1 b_1 c_2\ \ a_1\, b_2\left(\bigcup_{i=2}^{p} c_{i+2}\, b_{i+1}\, a_i\right) c_3$ at $d(c_1\, c_j),\ j=3 in\ G$

The Hamilton path is $c_1 b_1 c_2\left(\bigcup_{i=1}^{j-3} a_i b_{i+1}\, c_{i+2}\right) a_{j-2}\, b_{j-1}\left(\bigcup_{i=j-1}^{p} c_{i+2}\, b_{i+1}\, a_i\right) c_j$ at $d(c_1\, c_j),\ j\geq 4 in\ G$

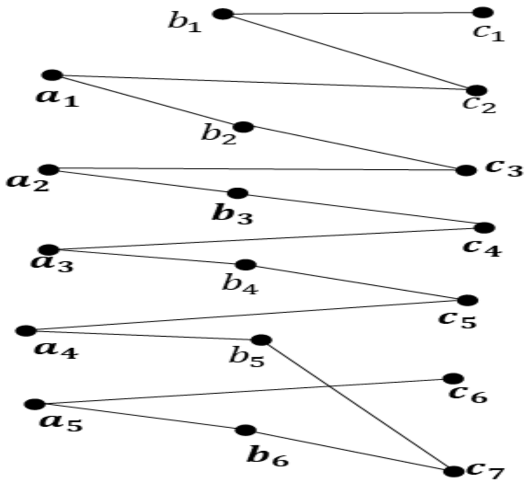


Fig15.Hamilton path from c_1 to c_7 in $K_{5,6,7}$

Hence from the claim (12 to 17), we conclude a Complete tripartite graph $G = K_{p,q,r}$ $p < q < r$ is Hamiltonian-t-laceablefor $1 \leq t \leq diamG$.

3. References

[1] F. Harary1969, *Graph Theory* Addison-Wesley Publishing Company.

[2] S. E. Goodman and S. T. Hedetniemi, 1974, *On Hamiltonian walks in graphs*, *SIAMJ*. Comput.**3**, pp 214-221.

[3] Girisha.A and Murali.R2013,HamiltonianLaceability in a class of 4-regular graphs ISOR journal of Mathematics4(1) pp 07-12.

[4] R. Murali and K.S. Harinath, Hamiltonian-n*-laceable graphs, Far East Journal of Applied Mathematics, 3(1)(1999), pp. 69-84.

[5] B. Alspach and C.Q. Zhang, 1989, *Hamilton cycles in cubic Cayley graphs on dihedral groups*, ArsCombinatoria**28**, pp 101-108.

[6] M. Behzad, G. Chartrand and L. Lesniak-Foster, Graphs and Digraphs (Prindle, Weber & Schmidt, Boston, 1979).