

Conceptualizing Symmetry and Group

^[1]Teke Sachin Ramchandra, ^[2] Dr. Rahul Dixit

Department of Mathematics, Sunrise University, Alwar, Rajasthan

Abstract

The conceptualization of symmetry and groups stands as a foundational exploration at the core of mathematics, transcending disciplinary boundaries and finding profound applications in diverse fields. Symmetry, defined by the invariance under transformation, serves as a unifying principle, revealing the underlying regularities and patterns in the natural world, art, science, and technology. Applications of symmetry and group theory span multiple disciplines. In physics, these concepts are instrumental in describing fundamental forces and particles. Crystallography relies on group theory to understand the symmetries inherent in crystal structures. Moreover, symmetry plays a practical role in computer science, aiding in the development of efficient algorithms and image processing techniques. As we conceptualize symmetry and groups, we delve into the abstract and universal principles that govern order and structure. This exploration not only deepens our theoretical understanding of mathematics but also empowers us to model, interpret, and manipulate the world around us with enhanced precision and insight. The beauty lies in the synthesis of abstract mathematical concepts and their tangible applications, showcasing the profound interconnectedness of symmetry and groups with the fabric of our understanding of the universe.

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INTRODUCTION

One of the most important and beautiful themes unifying many areas of modern mathematics is the study of symmetry. Many of us have an intuitive idea of symmetry, and we often think about certain shapes or patterns as being more or less symmetric than others. A square is in some sense “more symmetric” than a rectangle, which in turn is “more symmetric” than an arbitrary four-sided shape. Can we make these ideas precise? Group theory is the mathematical study of symmetry, and explores general ways of studying it in many distinct settings. Group theory ties together many of the diverse topics we have already explored – including sets, cardinality, number theory, isomorphism, and modular arithmetic – illustrating the deep unity of contemporary mathematics.

SHAPES AND SYMMETRIES

Many people have an intuitive idea of symmetry. The shapes in Figure 1 appear symmetric, some perhaps more so than others. However, despite our general intuitions about symmetry, it may not be clear how to make this statement precise. Can it make sense to discuss “how much” symmetry a shape has?



Figure 1.1 Some symmetric polygons.

There some way to make precise the idea that the regular pentagon is “more symmetric” than the equilateral triangle, or that the circle is “more symmetric” than any regular polygon? In this section we will explore symmetry and the way in which it arises in various contexts with which we are familiar, especially in the geometry of regular polygons (2D) and regular polyhedra (3D), such as the Platonic solids. The study of symmetry is a recurring theme in many disparate areas of modern mathematics, as well as chemistry, physics, and even economics.

Rotation symmetries

An equilateral triangle can be rotated by 120, 240, or 360 angles without really changing it. If you were to close your eyes, and a friend rotated the triangle by one of those angles, then after opening your eyes you would not notice that anything had changed. In contrast, if that friend rotated the triangle by 31 or 87, you would notice that the bottom edge of the triangle is no longer perfectly horizontal.

Many other shapes that are not regular polygons also have rotational symmetries. The shapes illustrated in Figure 1.2, for example, each have rotational symmetries. The first example can be rotated only 180, or else 360 or 0. The third shape can be rotated any integer multiple of 90.

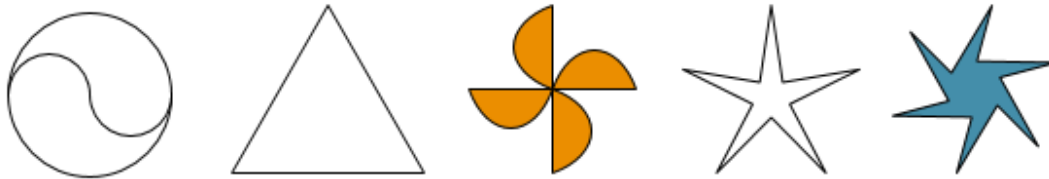


Figure 1.2 Several shapes with rotational symmetries.

The fourth shape can be rotated any integer multiple of 72. The fifth shape can be rotated any integer multiple of 60. More generally, we say that a shape has **rotational symmetry of order n** if it can be rotated by any multiple of $360/n$ without changing its appearance. We can imagine constructing other shapes with rotational symmetries of arbitrary order. If the only rotations that leaves a shape unchanged are multiples of 360, then we say that the shape has only the trivial (order $n = 1$) symmetry.

Mirror reflection symmetries

Another type of symmetry that we can find in two-dimensional geometric shapes is mirror reflection symmetry. More specifically, we can draw a line through some shapes and reflect the shape through this line without changing its appearance. This is called a mirror reflection symmetry.

Further consideration of the equilateral triangle (cf. Figure 40) shows that there are actually three distinct mirror lines through which we can reflect the shape without changing its appearance. If we were to reflect the triangle through any other line, the shape as a whole would look different.

Rotational symmetries and mirror reflection symmetries are not exclusive, and the same shape can have symmetries of both kinds. The equilateral triangle clearly has both mirror reflection symmetries and rotational symmetries. Likewise, the fourth shape in Figure 3 has five mirror reflection symmetries, along with many five rotational ones. The shapes in Figure, alternatively, have only mirror reflection symmetries but no rotational ones.

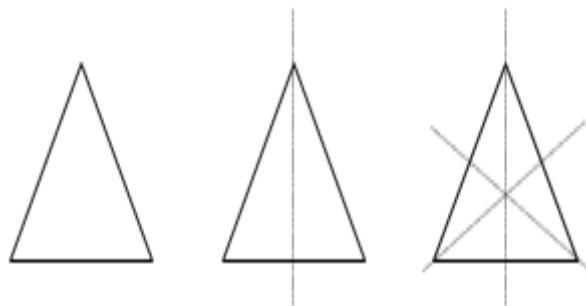


Figure 1.3 A line can be drawn through a triangle to highlight its symmetry. If the shape is reflected through this line, then we obtain the same equilateral triangle, unmoved.

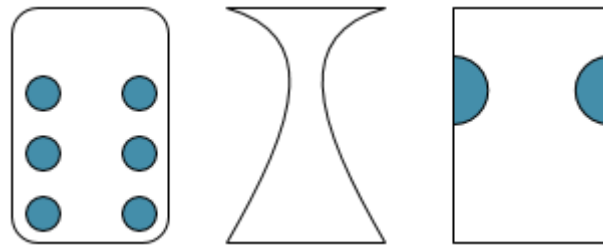


Figure 1. 4 Each of these shapes can be reflected through a vertical line; none of these shapes have any rotational symmetry.

Counting symmetries

One way in which we can quantify the “amount” of symmetry of an object is by counting its number of symmetries. For example, we might count the number of rotational symmetries of an object, along with its mirror reflection symmetries. However, counting the symmetries of a shape can be challenging. It is not immediately clear which symmetries we should count and which, if any, we should not count. To understand why we might not count certain symmetries, consider rotating the equilateral triangle by 120, 240, and 360. Of course the numbers by which we are rotating the triangle are different, and so we might be inclined to count each of them separately. But notice that we can also rotate the triangle by 480, 600, and 720. Should we count those as different symmetries? If we do count them, then what would stop us from counting an infinite number of rotational symmetries for a triangle, or for that matter, any shape?

One way to limit the number of symmetries we count involves coloring, or otherwise labeling, the shape. For example, we can color each edge of the equilateral triangle, as illustrated in Fig. Symmetries can then be captured as changes of colors that leave the uncolored shape fixed. Any triangle in either row can be obtained from any other triangle in that row through a rotation; triangles can be obtained from triangles in the other row through reflections.

Using this coloring allows us to count symmetries carefully. If changing the shape in two different ways results in the same coloring, then we should count those two symmetries as the same. For example, rotating the equilateral triangle by 120 or 480 results in the same coloring, so we count those as the same symmetry. Likewise, rotating the triangle by 0 and 360 also result in

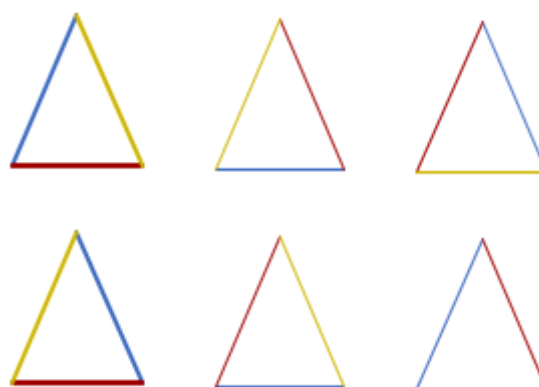


Figure 1. 5 Equilateral triangle with edges colored. Any triangle in either row can be obtained from another triangle in the row through a rotation; triangles can be obtained from triangles in the other row through reflections.

The same coloring, so we count those the same as well. To reduce confusion, we use a number between 0 and 360 (not including 360 itself) to describe the angle of a rotation; thus, we prefer 120 to 480, despite their

equivalence. Likewise, for reasons that will become clearer in the following section, we discussing 0 rotations, or “doing nothing” to 360 rotations, despite their equivalence.

We are therefore left with six symmetries of the triangle – the rotations (0, 120, and 240), and three reflections, one for each of the mirror planes passing through a corner and the center of the triangle. These symmetries can be pictured by how they transform the colored triangle in Figure.

Symmetries of the square

A square is in some sense “more symmetric” than a triangle because it has more symmetries. Figure 1.6 below shows a square with colored edges arranged in different ways. Again, you might notice that any two squares in the same row can be obtained from one another through rotations, whereas those in distinct rows can only be obtained from one another through a reflection. Some thought will show that there are no other rotations or mirror reflection symmetries, and so these figures represent all eight symmetries of the square.

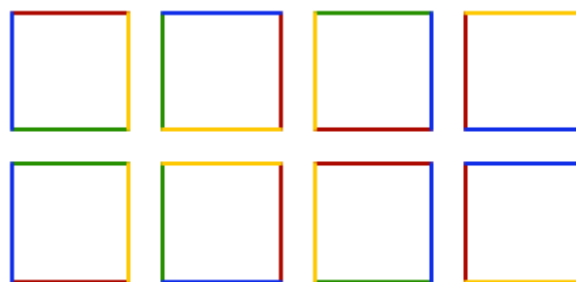


Figure 1.6 Squares.

Although this section is concerned primarily with rotational and mirror reflection symmetries of single objects in two dimensions, other types of symmetries arise in infinite systems and in higher dimensions. We do not consider those symmetries in this section.

Until now we have considered what symmetries are and briefly discussed how to count them. To say that a particular shape is “more symmetric” than another one can be made precise by considering their total number of symmetries. For example, the three shapes in Figure each have a set of four symmetries. However, notice that the first two shapes have the same set of 4 rotational symmetries (0, 90, 180, and 270), but no mirror reflection symmetries. In contrast, the third shape has 2 rotational symmetries and 2 mirror reflection symmetries. How can we distinguish between the first and second shapes, on the

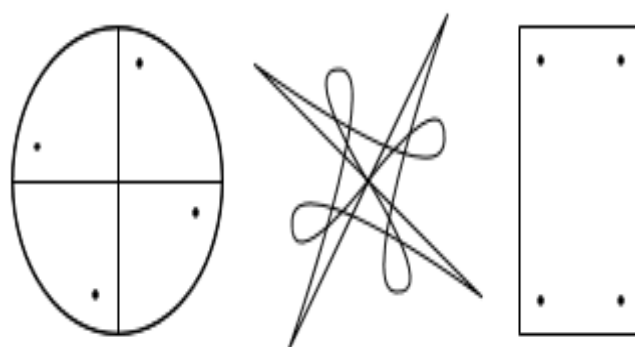


Figure 1.7 Three shapes, each with 4 symmetries.

The first two have 4 rotational symmetries (0, 90, 180, and 270) and no mirror reflection symmetries. The third has 2 rotational symmetries (0 and 180), and two mirror reflection symmetries.

One hand, and the third shape, on the other? The mathematical development of group theory provides rigorous tools to describe symmetries of shapes. Before consider the actual definition of a *group*, we first consider a more general topic of binary operators.

GROUPS

The study of symmetry has undergone tremendous change in the late 19th and early 20th centuries with the development of group theory, a part of an area called algebra (people who study algebra are called algebraists). Algebra and group theory has found applications in geometry, graph theory, physics, chemistry, architecture, crystallography, and countless other areas of modern science. There is hardly a discipline in which the study of symmetry, often with tools provided by group theory, has not played an important role. In the previous two sections we have discussed shapes and their symmetries, and binary operators and several of their properties. The theory of groups will provide the link between these two topics, which might appear otherwise unrelated.

Remember that in Section 1.2 we considered several properties that a binary operator could have when acting on a given set. For example, closure describes the property of being able to combine two elements in a set to obtain another element also in the same set. We also considered identity elements and inverses, as well as the associative property. An important point that we made then is that not every set and binary operator possesses all of these properties. We saw some sets that were closed under an operator, for example, but which do not possess inverses, and other sets in which we could find an identity element, but for which not all elements have inverses.

A group is merely a choice of set S and binary operator? That satisfies four conditions.

Definition 7. A *group* is a set G and operator? Such that:

- (closure) G is closed under?; i.e., if $a, b \in G$, then $a ? b \in G$.
- (identity) There exists an identity element $e \in G$; i.e., for all $a \in G$ we have $a ? e = e ? a = a$.
- (inverses) Every element $a \in G$ has an inverse in G ; i.e., for all $a \in G$, there exists an element $a^0 \in G$ such that $a ? a^0 = a^0 ? a = e$.
- (associativity) The operator $?$ acts associatively; i.e., for all $a, b, c \in G$, $a ? (b ? c) = (a ? b) ? c$.

Although this definition sounds complicated, and perhaps even arbitrary, it turns out that many of the examples we have already considered are in fact groups; for the sake of time and focus we will generally not spend much time discussing the associative property.

Let us consider several examples. Most important for our connections to symmetries, it turns out that the set of symmetries of any geometric shape constitute a group when the binary operator is defined by defining $a ? b$ as “do a and then do b ”. (We briefly note, for the sake of completeness, that our conventions are in contrast to mathematical convention. In particular, most mathematicians would interpret $a ? b$ to mean first do b and then do a .)

Example 1. Let us consider the set $S = \{0, 1, 2, 3\}$ under addition mod 4. It is straightforward to see that this choice of set and binary operator constitute a group. (1) The set is closed under addition mod 4, as for any pair of numbers $a, b \in S$, their sum mod 4 is also an element of S . (2) S , we have The element/number 0 here $a + 0 = 0 + a = a$. is an identity element, since for any element

(3) Confirming inverses is slightly less straightforward, but it is not difficult to confirm. The inverse of 0 is 0 (itself), since $0 + 0 \equiv 0 \pmod{4}$; the inverse of 1 is 3, the inverse of 2 is 2, and the inverse of 3 is 1, and combining any of these elements with its inverse (through addition mod 4) gives us the identity element 0. (4) Finally, modular addition is associative.

We can generalize this example to $\{0, 1, \dots, m-1\}$ and addition modulo m , where m is a natural number. It is straightforward to see that addition modulo m is closed on this set, and that 0 can serve as the identity element, for any choice of m . The inverse of any element a in this set is $m - a \pmod{m}$. For example, in mod 17, the inverse of 5 is $17 - 5 = 12$, which when added to 5 is congruent to 0 mod 17. Finally, as noted before, modular arithmetic is always associative.

Example 2. The set of all integers $\mathbb{Z}$ under addition is an example of a group, albeit one with an infinite number of elements in it. This choice of set and binary operator satisfies all four conditions to constitute a set.

Example 3. The same set of set might not be a group under a different operator. For example, the integers do not constitute a group under multiplication. Although 1 is good choice of identity element, almost no elements have an inverse. For example, the integer $a = 5$ has no “inverse” a^0 so that $a \rightarrow a^0 = 1$.

Example 4. Likewise, the same operator might not be a group if the set is changed. For example, even though $\mathbb{Z}$ constitutes a group under addition, the set of natural numbers $\mathbb{N}$ does not. Since every element is positive, there is 0, does not constitute certainly no identity element if we add the number 0 to a group since although it has an identity element, it does not have inverses for $\mathbb{N}$, i.e., even the set e such that $a + e = e\mathbb{N} + \{a\} = a$ for all $a \in \mathbb{N}$. Even almost any of its elements.

Group order

Occasionally we will want to have some way of measuring the “size” of a group. We use the word order to denote the number of elements in the associated set.

Definition 8. The **order** of a group given by a set G and binary operator? is the number of elements in G , i.e., the order of G , sometimes written as $|G|$.

We have seen several examples of finite groups, including sets $\{0, 1, 2, \dots, m-1\}$ under addition modulo m . The order of such a group is m . A group that has only one element in it, such as $\{0\}$ under addition, is called a *trivial group*.

Groups of symmetries

The ultimate goal of this section was to see that symmetries of shapes can be studied carefully, using the tools of group theory. It turns out that many sets of symmetries constitute a group when the binary operator is defined as $a \circ b = \text{“do } a \text{ and then do } b\text{”}$. Let us look at several examples.

Example 1. Let us reconsider the set of all rotations of the equilateral triangle: $S = \{\text{rotate } 0, \text{ rotate } 120, \text{ rotate } 240\}$. This is *not* the set of all symmetries, but it is a set of all rotational symmetries. Notice that we can combine any two of these symmetries to form a symmetry in this set. Notice also that rotating by 0 serves as the identity element, and that each of the rotations have an inverse. Finally, rotations in space are always associative. Using the definition of order, we can say that the order of the group of rotational symmetries of the equilateral triangle is 3. More generally, if we consider all n rotations of a regular polygon with n sides, then we get a group of order n .

Example 2. The set of all symmetries of a square also constitute a group under the operator of doing one symmetry and then doing another one. You might recall that the square has 8 different symmetries, four rotational ones and four mirror reflections. It might take some thinking to realize that combining any two of these symmetries will give us another symmetry in the group. It is also straightforward to see that the “do-nothing” rotation is an identity element, and also that that every symmetry can be reversed. Rotations are reversed by other rotations, and mirror reflection symmetries are always reverse themselves – if you take a reflection of a reflection (through the same mirror), then you always come back to the shape from which you began. More generally, if we consider all n rotations and all n reflections of a regular polygon with n sides, then we get a group with order $2n$.

Example 3. We can’t always combine arbitrary symmetries to form a group. Consider for example the set of all mirror reflection symmetries of an equilateral triangle, or of a square. You will notice that combining any two mirror reflection symmetries will give us a symmetry not in the group. In fact, combining two mirrors will always give us a rotation. If you don’t understand or believe me, take a square and label its four edges. Next, “reflect” it through one of the four mirror lines going through its center, and then reflect it again through another mirror line. You will see that the result is indeed a rotation. If you use the same mirror, then the result will be the same as the 0 rotation.

Example 4. The Platonic solids introduce symmetry groups that are substantially more complicated. In class we only considered rotational symmetries of these polyhedra, and we will not be concerned with the full group of symmetries. Let us begin by considering the cube. We can rotate the cube about axes

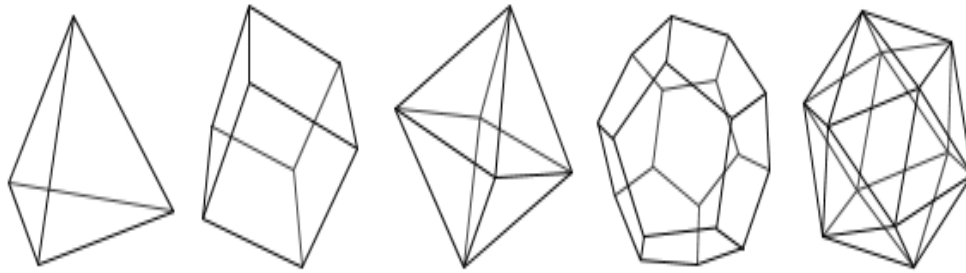


Figure 1.8 The five regular polyhedra, also known as the Platonic solids. Below is listed the number of vertices v , edges e , and faces f of each regular polyhedron, as well as the number of edges per face n and degree d of each vertex.

That pass through two opposite face centers. Each of these axes support four distinct rotations, by 0, 90, 180, or 270. There are two different kinds of axes that also allow for rotations. In particular, we can also draw a line through opposite pairs of corners, allowing us to rotate the cube about them by 0, 120, or 240. Finally, we can draw lines passing through centers of opposite edges. We can rotate the cube about these lines/axes either 0 or 180.

Commutative and non-commutative groups

One important idea that is not obvious at all is that the order of operations can matter, not always but often. To highlight the importance of this point, consider multiplication on the real numbers. For every pair of real numbers $x, y \in \mathbb{R}$ it is always the case that $x \rightarrow y = y \rightarrow x$. The same is true for addition and many other groups we have considered.

However, in many groups, the order in which we combine the elements matters. To see one such example, consider an equilateral triangle and its rotations. We have seen before that the set of symmetries of an equilateral triangle contain three rotations (including the one by 0) and three mirror reflections. Does the order of applying these symmetries matter? Sometimes it does not. For example, consider the rotation by 120 and the rotation by 240. The order in which we apply these symmetries does not matter. However, consider the 120 rotation and a reflection through a vertical mirror.

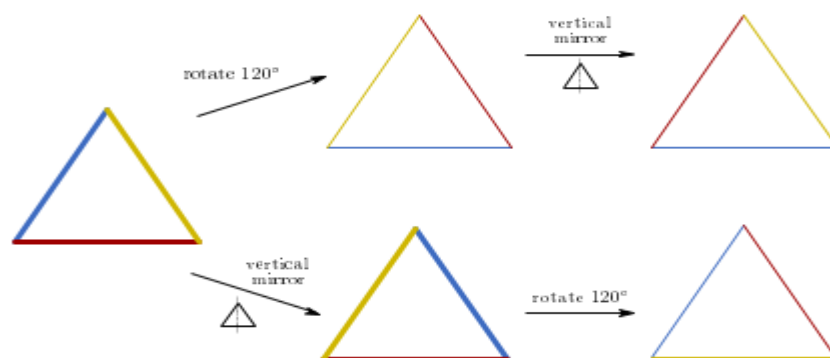


Figure 1.9 Equilateral triangle changed with a 120 rotation and with a reflection through a vertical mirror; the order in which these two operations are performed matters.

Groups in which the order does not matter, such as the integers, rationals, real numbers under addition or multiplication, the order does not matter, and $a + b = b + a$ and $a \rightarrow b = b \rightarrow a$ for any two elements. Such groups are called **commutative**, or Abelian, in honor of Niels Abel, a founding father of group theory. If we

consider the set of rotational symmetries about a single axis of rotation, such as all rotations of a triangle, then that set will form a group which is commutative.

A more complete exploration of groups, even those associated with the Platonic solids, is beyond the scope of these notes. Additional information about this material can be found in the homework assignments and the posted solutions.

1.6 SYMMETRY GROUPS OF SHAPES

One of the primary applications of group theory is the study of symmetries of shapes of different kinds. Symmetries of shapes form groups, and this section will explore many such examples, including those associated with regular polygons and polyhedra.

Cyclic Groups

Consider the set of rotations of an equilateral triangle that we considered before. We have the set:

$$S = \{ \text{rotate } 0, \text{rotate } 120, \text{rotate } 240 \}, \quad (85)$$

which as we have seen before forms a group under the binary operation defined by performing one rotation and then another. For reasons that will become clear soon, from now onwards we will call this group C_3 . Likewise, C_4 will be the group:

$$C_4 = \{ \text{rotate } 0, \text{rotate } 90, \text{rotate } 180, \text{rotate } 270 \}. \quad (86)$$

One thing we might notice about these two groups is that all elements of the group can be obtained by taking one element of the set, and combining it different numbers of times. For example, let us use r to denote the rotation by 90. We can then rewrite C_4 as:

$$C_4 = \{ r^0, r^1, r^2, r^3 \}, \quad (87)$$

where powers of r indicate performing the same geometric operation (in this case rotations by 90) multiple times. If we s to denote a rotation by 120, then we can likewise describe C_3 as the set $\{s^0, s^1, s^2\}$.

Both of these examples illustrate the possibility of “generating” certain groups by using a single element of the group, and combining it different numbers of times. We have a special name for such groups:

Definition 9. A **cyclic group** is a group that can be “generated” by combining a single element of the group multiple times. A cyclic group with n elements is commonly named C_n .

Figure 11 illustrates several shapes with symmetry groups that are cyclic.

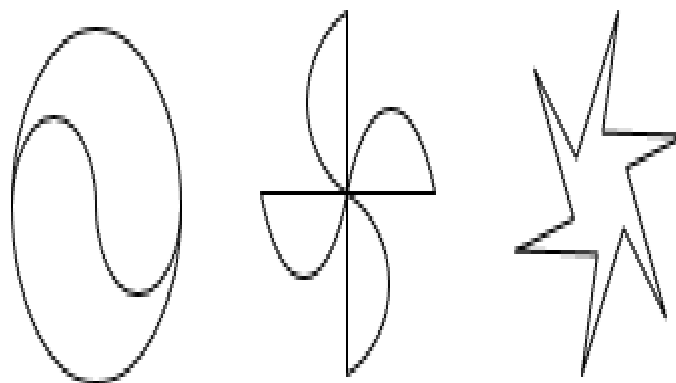


Figure 1.10 Shapes with associated symmetry groups C_2 , C_4 , and C_6 .

The examples above might lead us to wonder whether all symmetry group can in fact be generated by repeatedly combining a single element. Is every symmetry group in fact cyclic? Simple consideration will show us that this is not the case.

Dihedral Groups

Let us reconsider, for example, the set of all symmetries of a square. In addition to four rotational symmetries (0, 90, 180, 270), the square also has four mirror reflection symmetries; the effects of applying these symmetries to a colored square can be seen in Figure. If the first square is identified with the identity.

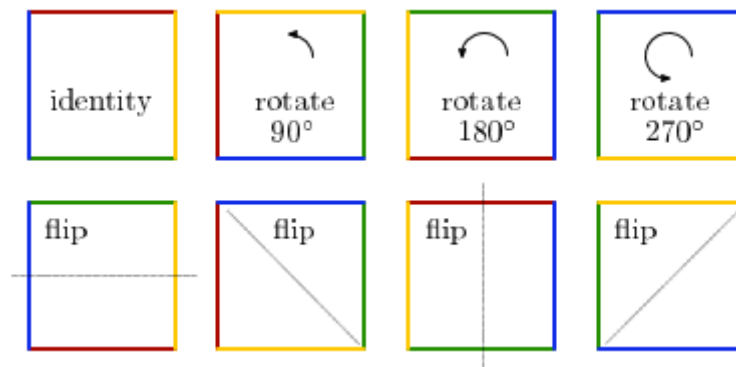


Figure 1.11 A single colored square transformed by rotations and mirror reflections; the set of all n rotation symmetries and n mirror reflection symmetries of a regular polygon with n sides make up the symmetry group of that polygon.

CONCLUSION

The conceptualization of symmetry and groups represents a fascinating and fundamental aspect of mathematics and its applications across various disciplines. Symmetry, in its broadest sense, refers to the invariance or unchanged appearance of an object under a transformation. This notion plays a crucial role in understanding the regularities and patterns that exist in the natural world, as well as in the realms of art, science, and technology. Groups, in the context of symmetry, provide a rigorous mathematical framework for studying and classifying symmetrical structures. A group is a set equipped with an operation that satisfies specific properties, such as closure, associativity, identity element, and inverses. In the study of symmetry, groups serve as powerful tools for analyzing transformations that preserve the essential features of an object. In conclusion, the conceptualization of symmetry and groups not only provides a powerful theoretical framework for understanding the inherent order in various phenomena but also has far-reaching practical implications across disciplines. As we delve deeper into these mathematical concepts, we uncover the beauty and universality of symmetry, enriching our perspective on the world and enhancing our ability to model and manipulate it effectively.

REFERENCES

- [1] Alcántara, E., & Serrano-Évora, R. (2014). On hook lengths, Durfee rectangles, and the lengths of rows in Young diagrams. *Linear and Multilinear Algebra*, 62(3), 354-365.
- [2] Alghamdi, Ahmad & Otera, Daniele & Russo, Francesco G.. (2010). On some recent investigations of probability in group theory. *Bollettino di Matematica Pura e Applicata*. 3.
- [3] Alperin, J. L., & Bell, R. B. (1995). *Groups and Representations*. Springer.
- [4] Alperin, R. C. (1986). *An introduction to local representation theory*. Cambridge University Press.
- [5] Atiyah, M., & Macdonald, I. (1969). *Introduction to Commutative Algebra*. Westview Press.
- [6] Borodin, Alexei & Olshanski, Grigori. (2016). Representations of the Infinite Symmetric Group. 10.1017/CBO9781316798577.
- [7] Brion, M. (1997). Equivariant cohomology and equivariant intersection theory. *Séminaire Bourbaki*, 813, 1-21.
- [8] Brualdi, R. A., & Cao, H. (2018). The hook-lengths formula revisited. *Ars Combinatoria*, 138, 59-68.
- [9] Brualdi, R. A., & Ryser, H. J. (2006). *Combinatorial Matrix Theory*. Cambridge University Press.
- [10] Brundan, Jonathan & Kleshchev, S.. (2000). Representations Of The Symmetric Group Which Are Irreducible Over Subgroups. *Journal fur die Reine und Angewandte Mathematik*. 530. 10.1515/crll.2001.003.
- [11] Bump, D. (1998). *Lie Groups (Graduate Texts in Mathematics)*. Springer.
- [12] Carter, R. W. (1985). *Simple Groups of Lie Type*. John Wiley & Sons.

- [13] Carter, R. W. (2005). Finite Groups of Lie Type: Conjugacy Classes and Complex Characters (Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts). John Wiley & Sons.
- [14] Cohen, Frederick & Hemmer, David & Nakano, Daniel. (2010). On the cohomology of Young modules for the symmetric group. *Advances in Mathematics*. 224. 1419-1461. 10.1016/j.aim.2010.01.004.
- [15] Craven, David. (2019). Representation Theory of Finite Groups: a Guidebook. 10.1007/978-3-030-21792-1.
- [16] Curtis, C. W., & Reiner, I. (1988). Methods of Representation Theory: With Applications to Finite Groups and Orders. John Wiley & Sons.
- [17] Dabbaghian, Vahid. (2006). On constructing representations of the symmetric groups. *Bulletin of the Korean Mathematical Society*. 43. 10.4134/BKMS.2006.43.1.119.
- [18] Diaconis, P., & Fulman, J. (2005). Carries, Shuffling, and An Amazing Matrix. *arXiv preprint math/0512384*.
- [19] Diaconis, P., & Ram, A. (2002). Analysis of systematic scan Metropolis algorithms using Iwahori-Hecke algebra techniques. *Michigan Mathematical Journal*, 48(1), 157-190.
- [20] Diaconis, P., & Shahshahani, M. (1981). Generating a random permutation with random transpositions. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 57(2), 159-179.