Estimation of errors of signals (functions) by $(C,2)(E,\delta)$ product means of Fourier series

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Abstract:-In this paper, we establish a new theorem to estimate the errors of signals (functions) $f \in Lip\{\zeta(t), r\}$ class by new $(C,2)(E,\delta)$ product summability method.

Keywords: *Signals(functions), (C,2) means, (E,δ) means, Lipschitz class and (C,2)(E,δ) product summability method, Lip{ζ(t),r} class, Fourier Series.*
Mathematical Classification: 42B05, 42B08

1. Introduction

Summability theory plays a very important role in the field of approximation theory. It became very interesting to find the estimate error (degree of approximation) of functions by using various product summability means. Liendler [20], Rhoades [4], Qureshi and Neha [17], Sahney and Goel [9] have determined the degree of approximation of functions belonging to Lipschitz class by Cesàro, Nörlund and generalized Nörlund (N,p,q) means. Later on Lal and Singh [7], Kushwaha [5] have studied the error estimates by trigonometric Fourier approximation of functions belonging to various Lipschitz classes by (C,1)(E,1) and (C,2)(E,1) product summability means respectively. Recently Nigam and Sharma [16] have determined the degree of approximation of functions belonging to $Lip\{\zeta(t), r\}$ class by using $(C,1)(E,\delta)$product method. But nothing seems to have been done so for to obtain the estimate of the error of signals (functions) belonging to $Lip\{\zeta(t), r\}$ class by using $(C,2)(E,\delta)$product summability method which have second order Cesàro means. The product of Euler means with second order Cesàro means is an advantage over the $(C,1)(E,1)$ and$(C,2)(E,1)$ product summability means. So we can say that the results of Nigam and Sharma [2] and Kushwaha [5] are the particular case of the result that we have determined.

2. Definitions and Notations

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with $\{s_n\}$ for its n^{th} partial sum.

Let $\{t_n^{E_\delta}\}$ denote the sequence of $(E,\delta) = E_n^\delta$ means of the sequence$\{s_n\}$. If the (E,δ) transform of $\{s_n\}$ is defined as

$$t_n^{E_\delta}(f;x) = \frac{1}{(1+\delta)^n} \sum_{k=0}^{n} \binom{n}{k} \delta^{n-k} s_k \to s \quad as \quad n \to \infty \tag{2.1}$$

The series $\sum_{n=0}^{\infty} u_n$ is said to besummable to the number s by the (E,δ) method. (Hardy[1])

Let $\{t_n^{C_2}\}$ denote the sequence of $(C,2) = C_n^2$ mean of the sequence$\{s_n\}$. If the $(C,2)$ transform of s_n is defined as

$$t_n^{C_2}(f;x) = \frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}(n-k+1)s_k(f;x) \to s \qquad as \qquad n\to\infty \qquad (2.2)$$

Then the series $\sum_{n=0}^{\infty} u_n$ is said to be summable to the number s by the $(C,2)$method . (Cesàro)

Thus if the $(C,2)$ transform of (E,δ) transform defines $(C,2)(E,\delta)$ transformation and denoted by $C_n^2.E_n^\delta$.

Thus if

$$t_n^{C_2E_\delta}(f;x) = \frac{2}{(n+1)(n+2)}\left[\sum_{k=0}^{n}(n-k+1)\left\{\frac{1}{(1+\delta)^k}\sum_{\nu=0}^{k}\binom{k}{\nu}\delta^{k-\nu}s_k\right\}\right] \to s \quad as \quad n\to\infty \qquad (2.3)$$

where$t_n^{C_2E_\delta}$ denote the sequence of C_2E_δ means that is $(C,2)(E,\delta)$ product means of the sequence s_n. The series $\sum_{n=0}^{\infty} u_n$ is said to be summable to the number s by $(C,2)(E,\delta)$ method. We know that $(C,2)(E,\delta)$ method is regular.

Let f be 2π-periodic, Lebesgue integrable function on $[-\pi,\pi]$ and the Fourier series associated with $f(x)$ at a point x is defined by

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} A_n, \qquad n \in N \qquad (2.4)$$

with partial sum $s_n(f;x)$.

We use following notations through out the paper

$$\phi(t) = f(x+t) - 2f(x) + f(x-t)$$

$$K_n(t) = \frac{1}{\pi(n+1)(n+2)}\sum_{k=0}^{n}(n-k+1)\left[\frac{1}{(1+q)^k}\sum_{\nu=0}^{k}\left\{\binom{k}{\nu}q^{k-\nu}\frac{\sin(\nu+1/2)t}{\sin(t/2)}\right\}\right] \qquad (2.5)$$

And L_r- norm is defined by

$$\|f\|_r = \left(\int_0^{2\pi}|f(x)|^r\,dx\right)^{\frac{1}{r}} \quad , \qquad r \ge 1$$

and the estimation of errors which is known as degree of approximation of a function f given by Zygmund [19].

$$E_n(f) = min\|t_n(x) - f(x)\|_r$$

where$t_n(x)$ is some n^{th} degree trigonometric polynomial. This method of approximation is called the trigonometric Fourier approximation.

A function $f \in Lip\alpha$ if

$$f(x+t) - f(x) = O(|t|^\alpha) \qquad for \quad 0 < \alpha \le 1, \qquad t > 0.$$

and function $f \in Lip(\alpha, r)$ if

$$\left(\int_0^{2\pi}|f(x+t) - f(x)|^r dx\right)^{\frac{1}{r}} = O(|t|^\alpha) \qquad for \quad 0 < \alpha \le 1, \qquad r \ge 1.$$

Given a positive increasing function $\zeta(t)$ and an integer $r \ge 1, t \in Lip\{\zeta(t), r\}$ if

TuijinJishu/Journal of Propulsion Technology
ISSN: 1001-4055
Vol. 44 No. 6 (2023)

$$\left(\int_0^{2\pi} |f(x+t)-f(x)|^r dx\right)^{\frac{1}{r}} = O\{\zeta(t)\}$$

If$\zeta(t) = t^{\alpha}$then $Lip\{\zeta(t), r\}$ class coincides with the $Lip(\alpha, r)$ class and if $r \to \infty$ then $Lip(\alpha, r)$ class reduces to the $Lip\alpha$ class.

Kushwaha[5] has proved a theorem on approximation of function by $(C,2)(E,1)$ product summability method as following-

Theorem:- If $f: R \to R$ is 2π-periodic,Lebesgueintegrable on $[-\pi, \pi]$ and belonging to $Lip(\alpha, r)$ class then the estimate error of signals (functions) f by the $(C,2)(E,1)$ product means of Fourier series of f satisfies

$$t_n^{C_2E_1}(f;x) = \frac{2}{(n+1)(n+2)}\left[\sum_{k=0}^{n}(n-k+1)\left\{\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}s_\nu(f;x)\right\}\right] \quad (2.6)$$

of its Fourier series is given by

$$\|C_n^2E_n^1 - f\|_r = O\left\{\zeta\left(\frac{1}{n+1}\right)\right\} \quad (2.7)$$

3. Main Theorem

Theorem-If a function $f$be2π periodic, Lebesgue Integral on $[-\pi, \pi]$ and belonging to $Lip\{\zeta(t), r\}$ class then the estimate error of functions (signals) f by the $(C,2)(E,\delta)$ product means of Fourier series of f is given by

$$\left\|C_n^2E_n^\delta - f\right\|_r = O\left[(n+1)^{1/r}\zeta\left(\frac{1}{n+1}\right)\right] \quad (3.1)$$

Provided that $\zeta(t)$ satisfies the following conditions:

$$\left\{\int_0^{\frac{1}{n+1}}\left(\frac{t|\phi(t)|}{\zeta(t)}\right)^r dt\right\}^{1/r} = O\left(\frac{1}{(n+1)}\right) \quad (3.2)$$

and

$$\left\{\int_{1/(n+1)}^{\pi}\left(\frac{t^{-\iota}|\phi(t)|}{\zeta(t)}\right)^r dt\right\}^{1/r} = O\left((n+1)^\delta\right) \quad (3.3)$$

whereι is an arbitrary positive number such that $s(1-\iota)-1>0$, $\frac{1}{r}+\frac{1}{s}=1$, $1 \le r < \infty$. These conditions (3.2) and (3.3) hold in $C_n^2E_n^\delta$ that is $(C,2)(E,\delta)$ means of the Fourier series.

4. Lemmas

We prove following lemmas for the proof of main theorems:

Lemma 1- $|\kappa_n(t)| = O(n+1)$; for $0 \le t \le \frac{1}{n+1}$.

Proof:-For $0 \le t \le \frac{1}{n+1}$; $\sin nt \le n \sin t$.

$$|\kappa_n(t)| = \frac{1}{\pi(n+1)(n+2)}\left|\sum_{k=0}^{n}(n-k+1)\left[\frac{1}{(1+\delta)^k}\sum_{\nu=0}^{k}\left\{\binom{k}{\nu}\delta^{k-\nu}\frac{\sin\left(\nu+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\right\}\right]\right|$$

$$\le \frac{1}{\pi(n+1)(n+2)}\left|\sum_{k=0}^{n}(n-k+1)\left[\frac{1}{(1+\delta)^k}\sum_{\nu=0}^{k}\left\{\binom{k}{\nu}\delta^{k-\nu}\frac{(2\nu+1)\sin(t/2)}{\sin(t/2)}\right\}\right]\right|$$

$$= \frac{1}{\pi(n+1)(n+2)}\left|\sum_{k=0}^{n}(n-k+1)\left[\frac{1}{(1+\delta)^k}\sum_{\nu=0}^{k}\left\{\binom{k}{\nu}\delta^{k-\nu}(2\nu+1)\right\}\right]\right|$$

$$= \frac{1}{\pi(n+1)(n+2)}\left|\sum_{k=0}^{n}(n-k+1)\left[(2k+1)\frac{1}{(1+\delta)^k}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}\delta^{k-\nu}\right\}\right]\right|$$

Since $\sum_{\nu=0}^{k}\binom{k}{\nu}\delta^{k-\nu} = (1+\delta)^k$

$$= \frac{1}{\pi(n+1)(n+2)}\left[\sum_{k=0}^{n}(n-k+1)\left\{(2k+1)\frac{1}{(1+\delta)^k}.(1+\delta)^k\right\}\right]$$

$$= \frac{1}{\pi(n+1)(n+2)}\left[\sum_{k=0}^{n}(n-k+1)(2k+1)\right]$$

$$= \frac{1}{\pi(n+1)(n+2)}\sum_{k=0}^{n}\{(n-k+1)(2k+1)\}$$

$$= \frac{1}{\pi(n+1)(n+2)}\sum_{k=0}^{n}\{(n+1)(2k+1)-k(2k+1)\}$$

$$= \frac{(n+1)}{\pi(n+1)(n+2)}\sum_{k=0}^{n}(2k+1) - \frac{1}{\pi(n+1)(n+2)}\sum_{k=0}^{n}k(2k+1)$$

$$= \frac{(n+1)}{\pi(n+1)(n+2)}\sum_{k=0}^{n}(2k+1) - \frac{1}{\pi(n+1)(n+2)}\left[2\sum_{k=0}^{n}k^2 + \sum_{k=0}^{n}k\right]$$

$$= \frac{(n+1)^3}{\pi(n+1)(n+2)} - \frac{1}{\pi(n+1)(n+2)}\left[2.\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right]$$

$$= \frac{(n+1)^3}{\pi(n+1)(n+2)} - \frac{1}{\pi(n+1)(n+2)}\left\{\frac{n(n+1)(4n+5)}{6}\right\}$$

$$\approxeq O(n+1)$$

Lemma 2- $|\kappa_n(t)| = O\left(\frac{1}{t}\right)$; for $\frac{1}{n+1} \le t \le \pi$.

Proof:-For $\frac{1}{n+1} \le t \le \pi$; by applying Jordans lemma $\sin(t/2) \ge t/\pi$. and $\sin nt \le 1$.

$$|\kappa_n(t)| = \frac{1}{\pi(n+1)(n+2)}\left|\sum_{k=0}^{n}(n-k+1)\left[\frac{1}{(1+\delta)^k}\sum_{\nu=0}^{k}\left\{\binom{k}{\nu}\delta^{k-\nu}\frac{\sin\left(\nu+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\right\}\right]\right|$$

$$\le \frac{1}{\pi(n+1)(n+2)}\left[\sum_{k=0}^{n}(n-k+1)\frac{1}{(1+\delta)^k}\sum_{\nu=0}^{k}\left\{\binom{k}{\nu}\delta^{k-\nu}\frac{1}{t/\pi}\right\}\right]$$

$$= \frac{\pi}{\pi(n+1)(n+2)t}\left[\sum_{k=0}^{n}(n-k+1)\left\{\frac{1}{(1+\delta)^k}\sum_{\nu=0}^{k}\binom{k}{\nu}\delta^{k-\nu}\right\}\right]$$

$$= \frac{1}{(n+1)(n+2)t}\left[\sum_{k=0}^{n}(n-k+1)\left\{\frac{1}{(1+\delta)^k}(1+\delta)^k\right\}\right]$$

$$=\frac{1}{(n+1)(n+2)t}\left\{\sum_{k=0}^{n}(n-k+1)\right\}$$

$$=\frac{1}{(n+1)(n+2)t}\left\{\sum_{k=0}^{n}(n+1)-\sum_{k=0}^{n}k\right\}$$

$$=\frac{n+1}{(n+1)(n+2)t}\sum_{k=0}^{n}(1)-\frac{1}{(n+1)(n+2)t}\sum_{k=0}^{n}k$$

$$=\frac{n(n+1)}{(n+1)(n+2)t}-\frac{n(n+1)/2}{(n+1)(n+2)t}$$

$$\approxeq O\left(\frac{1}{t}\right)$$

5. Proof of the theorem

Following Titchmarsh [6] and using Riemann-Lebesgue theorem $s_n(f;x)n^{th}$ partial sum of the series is given by

$$s_n(f;x)-f(x)=\frac{1}{2\pi}\int_0^{\pi}\phi(t)\,\frac{\sin(n+1/2)t}{\sin t/2}$$

Using the (E,δ) transform of $s_n(f;x)$ is given by

$$t_n^{(E,\delta)}-f(x)=\frac{1}{2\pi}\int_0^{\pi}\frac{\phi(t)}{\sin(t/2)}\left\{\frac{1}{(1+\delta)^k}\sum_{\nu=0}^{k}\binom{k}{\nu}\delta^{k-\nu}\sin(\nu+1/2)t\right\}dt$$

Now denoting $(C,2)(E,\delta)$ transformation of $s_n(f;x)$ is given by

$$t_n^{(C,2)(E,\delta)}-f(x)=\frac{1}{\pi(n+1)(n+2)}\sum_{k=0}^{n}(n-k+1)\left[\frac{1}{(1+\delta)^k}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}\delta^{k-\nu}\int_0^{\pi}\phi(t)\frac{\sin(\nu+1/2)t}{\sin(t/2)}dt\right\}\right]$$

$$=\int_0^{\pi}\phi(t)\kappa_n(t)dt$$

$$=\left(\int_0^{\frac{1}{n+1}}+\int_{\frac{1}{n+1}}^{\pi}\right)\phi(t)\kappa_n(t)dt$$

$$=I_1+I_2(say)$$

We consider

$$|I_1|\le\int_0^{1/(n+1)}|\phi(t)||\kappa_n(t)|dt$$

Using Hölder's inequality and the fact that $\phi(t)\in Lip\{\zeta(t),r\}$and using the lemma (1)-

$$|I_1|\ \le\ \left[\int_0^{\frac{1}{(n+1)}}\left\{\frac{t\phi(t)}{\zeta(t)}\right\}^r dt\right]^{\frac{1}{r}}.\ \left[\int_0^{\frac{1}{(n+1)}}\left\{\frac{\zeta(t)|\kappa_n(t)|}{t}\right\}^s dt\right]^{1/s}$$

$$=\ O\left(\frac{1}{n+1}\right)\left[\int_0^{1/(n+1)}\left\{\frac{\zeta(t)|\kappa_n(t)|}{t}\right\}^s dt\right]^{1/s}$$

$$= \quad O\left(\frac{1}{n+1}\right)\left[\int_0^{\frac{1}{(n+1)}}\left\{\frac{(n+1)\zeta(t)}{t}\right\}^s dt\right]^{\frac{1}{s}} \qquad by\ lemma\ 1.$$

Since $\zeta(t)$ is a positive increasing function and using second mean value theorem for integrals-

$$I_1 \quad = \quad O\left(\frac{1}{n+1}\right)\zeta\left(\frac{1}{n+1}\right)\left\{\int_\epsilon^{1/(n+1)}\frac{1}{t^s}dt\right\}^{1\backslash s}$$

$$= \quad O\left\{\zeta\left(\frac{1}{n+1}\right)\right\}\left[\left\{\frac{t^{-s+1}}{-s+1}\right\}_\epsilon^{\frac{1}{n+1}}\right]^{1/s}$$

For some $0 \le \epsilon \le \frac{1}{n+1}$.

$$= \quad O\left\{\zeta\left(\frac{1}{n+1}\right)\right\}\left\{(n+1)^{1-\frac{1}{s}}\right\}$$

$$= \quad O\left\{(n+1)^{\frac{1}{r}}\zeta\left(\frac{1}{n+1}\right)\right\} \qquad Since\ \frac{1}{r}+\frac{1}{s}=1$$

Now consider

$$|I_2| \quad \le \quad \int_{1/(n+1)}^{\pi}|\phi(t)||\kappa_n(t)|$$

Using Hölders inequality

$$|I_2| \quad \le \left[\int_{1/(n+1)}^{\pi}\left\{\frac{t^{-\iota}|\phi(t)|}{\zeta(t)}\right\}^r dt\right]^{1/2}$$

$$= \left[\int_{\frac{1}{(n+1)}}^{\pi}\left\{\frac{\zeta(t)|\kappa_n(t)|}{t^{-\iota}}\right\}^s dt\right]^{1/s}$$

$$= \quad O\{(n+1)^\iota\}\left[\int_{\frac{1}{(n+1)}}^{\pi}\left\{\frac{\zeta(t)|\kappa_n(t)|}{t^{-\iota}}\right\}^s dt\right]^{1/s}$$

$$= \quad O\{(n+1)^\iota\}\left[\int_{\frac{1}{(n+1)}}^{\pi}\left\{\frac{\zeta(t)}{t^{1-\iota}}\right\}^s dt\right]^{1/s} \qquad ,by\ lemma\ 2$$

Now putting $t = \frac{1}{x}$, and $dt = -\frac{1}{x^2}dx$

$$I_2 \quad = \quad O\{(n+1)^\iota\}\left[\int_{1/\pi}^{(n+1)}\left\{\frac{\zeta(1/x)}{x^{\iota-1}}\right\}^s\frac{dx}{x^2}\right]^{1/s}$$

Since ζ(t) is a positive increasing function and using second mean value theorem for integrals

$$I_2 \quad = \quad O\left\{(n+1)^\iota\zeta\left(\frac{1}{n+1}\right)\right\}\left[\int_\eta^{(n+1)}\frac{dx}{x^{s(\iota-1)+2}}\right]^{1/s}, \qquad for\ some\ \ 1/\pi \le \eta \le (n+1)$$

$$= \quad O\left\{(n+1)^\iota\zeta\left(\frac{1}{n+1}\right)\right\}\left[\int_1^{(n+1)}\frac{dx}{x^{s(\iota-1)+2}}\right]^{1/s}, \qquad for\ some\ \ 1/\pi \le 1 \le (n+1)$$

$$= O\left\{(n+1)^{\iota}\zeta\left(\frac{1}{n+1}\right)\right\}\left[\left\{\frac{x^{s(1-\iota)-1}}{s(1-\iota)-1}\right\}_1^{n+1}\right]^{1/s}$$

$$= O\left\{(n+1)^{\iota}\zeta\left(\frac{1}{n+1}\right)\right\}\left[(n+1)^{(1-\iota)-1/s}\right]$$

$$= O\left\{\zeta\left(\frac{1}{n+1}\right)\right\}\left[(n+1)^{1-1/s}\right]$$

$$= O\left\{(n+1)^{1/r}\zeta\left(\frac{1}{n+1}\right)\right\}, \quad Since \quad \frac{1}{r}+\frac{1}{s}=1$$

Combining I_1 and I_2 yields-

$$\left|C_n^2E_n^\delta - f\right| = O\left\{(n+1)^{1/r}\zeta\left(\frac{1}{n+1}\right)\right\}$$

Now using L_r- norm, we get

$$\left\|C_n^2E_n^\delta - f\right\|_r = \left\{\int_0^{2\pi}\left|C_n^2E_n^\delta - f\right|^r dx\right\}^{1/r}$$

$$= \left[\int_0^{2\pi}\left\{(n+1)^{1/r}\zeta\left(\frac{1}{n+1}\right)\right\}^r dx\right]^{1/r}$$

$$= \left\{(n+1)^{1/r}\zeta\left(\frac{1}{n+1}\right)\right\}\left\{\left(\int_0^{2\pi}dx\right)^{1/r}\right\}$$

$$= \left\{(n+1)^{1/r}\zeta\left(\frac{1}{n+1}\right)\right\}$$

This completes the proof of the theorem.

6. Some particular cases

1. If $\zeta(t) = t^\alpha$, $0 \le \alpha \le 1$ then $Lip\{\zeta(t), r\}$ class $r \ge 1$, reduces to $Lip(\alpha, r)$ class, then the estimate error of function (signals) by $(C,2)(E,\delta)$ means is given by

$$\left\|C_n^2E_n^\delta - f\right\| = O\left(\frac{1}{n^{\alpha-1/r}}\right), \quad 1/r < \alpha < 1. \tag{6.1}$$

2. If $\delta = 1$ then the estimate error of function (signals) belonging to $Lip\{\zeta(t), r\}$ class by $(C,2)(E,1)$ means is given by

$$\|C_n^2E_n^1 - f\| = O\left(n^{1/r}\zeta\left(\frac{1}{n}\right)\right). \tag{6.2}$$

3. If $r \to \infty$ in case (I), then $lip(\alpha, r)$ class reduces to the class $lip\alpha$, then the estimate error of function (signals) by $(C,2)(E,\delta)$ means is given by

$$\left\|C_n^2E_n^\delta - f\right\| = O\left(\frac{1}{n^\alpha}\right), \quad 0 < \alpha < 1 \tag{6.3}$$

7. Acknowledgement

The authors are grateful to Professor ShyamLal, Department of Mathematics, Institute of Science, Banaras Hindu University (BHU) for the encouragement to this work.

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