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Initial Coefficient Constraints For Certain Subclass Of Bi-Univalent Functions

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Abstract— In this paper, we propose to investigate the coefficient bounds for certain subclasses of bi-univalent functions. Some interesting applications of the results are also obtained.
Keywords— Bi-univalent functions, Starlike functions, Convex functions, bi-starlike functions and bi-convex functions.

1. Introduction

Let A denote the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n \qquad (1.1)$$

which are analytic in the open unit disc $U=\{z:z\in C \; and \; |z|<1\}$ and satisfy the conditions specified by R.M.Ali, S.Devi and W.C.Ma in the references [1,2,4,713,14,15]

and $\quad f(f^{-1}(w))=w \qquad \left(|w|<r_0(f);r_0(f)\geq\frac{1}{4}\right)$

where

$$f^{-1}(w)=w-a_2w^2+(2a_2^2-a_3)w^3-\ldots \qquad (1.2)$$

A function $f\in A$ satisfies bi-univalent definition in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U. It can be found in recent [1,3,5,6,8] and [11,12,13,14,15,16].

With the reference of [7], $f(z)$ be an analytic and univalent function with positive real part one, and symmetric with respect to the real axis. Expansion in Taylor's series

$$\phi(z)=1+B_1z+B_2z^2+\ldots \qquad (1.3)$$

with $B_i>0$

By $S^*(\phi(z))$ and $k(\phi(z))$ derived as [13,14],

The classes $S^*(\phi(z))$ and $k(\phi(z))$ are the delays of a classical sets of a SCF by [7]. Also f and f^{-1} are respectively SCF. These classes are denoted respectively by $S_\Sigma^*(\phi(z))$ and $k_\Sigma(\phi(z))$ (see [1]).

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Similarly, $S^{*}(\gamma,\phi(z))$ and $k(\gamma,\phi(z))$ of SCC order $\gamma(\gamma\in C\setminus\{0\})$, [7,10] and

$$k(\gamma,\phi(z)):=\left\{f:f\in S \text{ and } 1+\frac{1}{\gamma}\left(\frac{zf''(z)}{f'(z)}-1\right)\prec\phi(z);z\in U\right\} \quad (1.4)$$

Also a function f is BS-BC of complex order $\gamma(\gamma\in C\setminus\{0\})$ if both f and f^{-1} are respectively SCC order and derived as $\gamma(\gamma\in C\setminus\{0\})$. $S_{\Sigma}^{*}(\gamma,\phi(z))$ and $k_{\Sigma}(\gamma,\phi(z))$.[17]

Here, ICB for certain subclass of BUF are obtained. Several related classes are derived, and a connected [10].
As per definition $f(z)$ class, then

$$1+\frac{1}{\gamma}\left[(1-\lambda)\frac{f(z)}{z}+\lambda(f'(z)+tzf''(z))-1\right]\prec\phi(z) \quad (1.5)$$

and

$$1+\frac{1}{\gamma}\left[(1-\lambda)\frac{f^{-1}(w)}{w}+\lambda((f^{-1}(w))'+tw(f^{-1}(w)))''-1\right]\prec\phi(w) \quad (1.6)$$, [2 & 4].

Note that the special values of t,γ,λ and $\phi(z)$ leads to the class [3,14]. $W_{\Sigma}(\gamma,\lambda,t,\phi(z))$ leads as follows. [11]

For $\lambda=1$ the class

$W_{\Sigma}(\gamma,1,t,\phi(z))\equiv R_{\Sigma}(\gamma,\lambda,\phi(z))$ is

$$1+\frac{1}{\gamma}[(f'(z)+tzf''(z))-1]\prec\phi(z)$$

and

$$1+\frac{1}{\gamma}[(f^{-1}(w))'+tw(f^{-1}(w))''-1]\prec\phi(w)$$

The class $R_{\Sigma}(\gamma,\lambda,\phi(z))$, [14, 3].

For $\lambda=\alpha,t=0$ the class

$W_{\Sigma}(\gamma,\alpha,0,\phi(z))\equiv B_{\Sigma}(\gamma,\alpha,\phi(z))$ is

$$1+\frac{1}{\gamma}\left[(1-\alpha)\frac{f(z)}{z}+\alpha f'(z)-1\right]\prec\phi(z)$$

and

$$1+\frac{1}{\gamma}\left[(1-\alpha)\frac{f^{-1}(w)}{w}+\alpha(f^{-1}(w))'-1\right]\prec\phi(w)$$

Remark1.1. By [11], [16]). for $\gamma=1$

$$\phi(z)=\frac{1+(1-2\beta)z}{1-z},0\le\beta<1$$

and $\phi(z)=\left(\frac{1+z}{1-z}\right)^{\eta},0<\eta\le 1,$

the classes $B_{\Sigma}\left(\alpha,\frac{1+(1-2\beta)z}{1-z}\right)\equiv B_{\Sigma}(\alpha,\beta)$ and

$B_{\Sigma}\left(\alpha,\left(\frac{1+z}{1-z}\right)^{\eta}\right) \equiv B_{\Sigma}^{\eta}(\alpha)$ were introduced and studied by Frasin and Aouf [5,7].

(1) For $\lambda = 1$ and $t = 0$ the classes

$W_{\Sigma}(\gamma,1,0,\phi(z)) \equiv P_{\Sigma}(\gamma,\phi(z))$ is

$$1+\frac{1}{\gamma}\left[f'(z)-1\right] \prec \phi(z)$$

and

$$1+\frac{1}{\gamma}\left[(f^{-1}(w))'-1\right] \prec \phi(w)$$

Remark1.2.For $\gamma = 1$, [15] ,

$$\phi(z) = \frac{1+(1-2\beta)z}{1-z}, 0 \leq \beta < 1$$

and $\phi(z) = \left(\frac{1+z}{1-z}\right)^{\eta}, 0 < \eta \leq 1,$

the classes

$$P_{\Sigma}\left(\alpha, \frac{1+(1-2\beta)z}{1-z}\right) \equiv P_{\Sigma}(\beta)$$

and

$P_{\Sigma}\left(\left(\frac{1+z}{1-z}\right)^{\eta}\right) \equiv P_{\Sigma}^{\eta}$, introduced in [12] , [6].

For , the following coefficient estimation holds.

In order to derive our results, we shall need the following lemma.[7]

With the reference of Lemmma 1.1. in [7, 9] is consider to find P, where P is the family of all functions p, analytic in U, for which $R\{p(z)\} > 0$ where

.

2. COEFFICIENT BOUNDS

Theorem 2.1. If $f \in W_{\Sigma}(\gamma,\lambda,t,\phi(z))$, then

$$|a_2| \leq \frac{|\gamma| B_1\sqrt{B_1}}{\sqrt{\left|\gamma[1+2\lambda(1+3t)]B_1^2+[1+\lambda(1+2t)]^2(B_1-B_2)\right|}} \tag{2.1}$$

and

$$|a_3| \leq \frac{|\gamma| B_1}{1+2\lambda(1+3t)} + \frac{|\gamma|^2 B_1^2}{[1+\lambda(1+2t)]^2} \tag{2.2}$$

Proof. Two analytic functions, will be existe namely, $r,s : U \to U$, with $r(0) = 0 = s(0),$ [7], from $f \in W_{\Sigma}(\gamma,\lambda,t,\phi(z))$ such that

$$1+\frac{1}{\gamma}\left[(1-\lambda)\frac{f(z)}{z}+\lambda f'(z)-1\right] = \phi(r(z)) \tag{2.3}$$

and

$$1+\frac{1}{\gamma}\left((1-\lambda)\frac{f^{-1}(w)}{w}+\lambda(f^{-1}(w))'-1\right)=\phi(s(z)) \quad (2.4)$$

We observed that $p(z), q(z)$ are analytic and they are constant at $z = 0. r(z)$ and $s(z)$ in terms of $p(z)$ and $q(z)$ from [9].

Or equivalently,

$$r(z) = \frac{p(z)-1}{p(z)+1} \quad \ldots\ldots\ldots(2.5)$$

$$s(z) = \frac{q(z)-1}{q(z)+1} \quad \ldots\ldots\ldots(2.6)$$

$$1+\frac{1}{\gamma}\left[(1-\lambda)\frac{f(z)}{z}+\lambda f'(z)-1\right]=\phi\left(\frac{p(z)-1}{p(z)+1}\right) \quad (2.7)$$

and

$$1+\frac{1}{\gamma}\left((1-\lambda)\frac{f^{-1}(w)}{w}+\lambda(f^{-1}(w))'-1\right)=\phi\left(\frac{q(w)-1}{q(w)+1}\right) \quad (2.8)$$

Using (2.5) and (2.6) together with (1.3), it is evident that

$$\phi\left(\frac{p(z)-1}{p(z)+1}\right)=1+\frac{1}{2}B_1p_1z+\left(\frac{1}{2}B_1\left(p_2-\frac{1}{2}p_1^2\right)+\frac{1}{4}B_2p_1^2\right)z^2+\ldots \quad (2.9)$$

$$\phi\left(\frac{q(w)-1}{q(w)+1}\right)=1+\frac{1}{2}B_1q_1w+\left(\frac{1}{2}B_1\left(q_2-\frac{1}{2}q_1^2\right)+\frac{1}{4}B_2q_1^2\right)w^2+\ldots \quad (2.10)$$

Since is of the form (1.1), a computation shows that its inverse has the expression given by (1.2).
It follows from (2.7), (2.8), (2.9) and (2.11) that

$$\frac{1}{\gamma}(1+\lambda(1+2t))a_2=\frac{1}{2}B_1p_1 \quad (2.11)$$

$$\frac{a_3}{\gamma}(1+2\lambda(1+3t))=\frac{1}{2}B_1\left(p_2-\frac{1}{2}p_1^2\right)+\frac{1}{4}B_2p_1^2 \quad (2.12)$$

$$-\frac{1}{\gamma}(1+\lambda(1+2t))a_2=\frac{1}{2}B_1q_1 \quad (2.13)$$

and

$$\frac{1+2\lambda(1+3t)}{\gamma}(2a_2^2-a_3)=\frac{1}{2}B_1\left(q_2-\frac{1}{2}q_1^2\right)+\frac{1}{4}B_2q_1^2 \quad (2.14)$$

From (2.11) and (2.13), it follows that

$$p_1=-q_1 \quad (2.15)$$

and

$$\frac{4}{\gamma^2}[1+\lambda(1+2t)]^2a_2^2=B_1^2(p_1^2+q_1^2) \quad (2.16)$$

From (2.1) and Lemma 1.1, derived $|p_2|\le 2$ & $|q_2|\le 2$.

Using (2.24) and (2.14), resultant of (2.11) is

$$4a_3=\frac{\gamma B_1(p_2-q_2)}{1+2\lambda(1+3t)}+\frac{\lambda^2B_1^2p_1^2}{(1+\lambda(1+2t))^2} \quad (2.17)$$

With once again, we readily get the bound given in (2.2)

Remark 2.1[7]. Taking $\lambda=1$ in Theorem 2.1, we obtain the for $\lambda=1$ and $\gamma=1$ in Theorem 2.1

If we set $\phi(z)=\frac{1+Az}{1-Bz},-1\le B<A\le 1,$ in the class $W_{\Sigma}(\gamma,\lambda,t,\phi(z))$, we have $W_{\Sigma}(\gamma,\lambda,t,(A,B))$ and defined as

$$1+\frac{1}{\gamma}\left[(1-\lambda)\frac{f(z)}{z}+\lambda(f'(z)+tzf''(z))-1\right]\prec\frac{1+Az}{1+Bz},\quad z\in U \text{ and}$$

$$1+\frac{1}{\gamma}\left[(1-\lambda)\frac{f^{-1}(w)}{w}+\lambda((f^{-1}(w))'+tw(f^{-1}(w)))''-1\right]\prec\frac{1+Aw}{1+Bw},\quad w\in U$$

[1, 10, 14].

Corollary 2.1. If $W_{\Sigma}(\gamma,\lambda,t,(A,B)),$ then

$$|a_2|\le\frac{|\gamma|(A-B)}{\sqrt{|\gamma[1+2\lambda(1+3t)](A-B)+[1+\lambda(1+2t)]^2(A+B)|}}$$

and

$$|a_3|\le\frac{|\gamma|(A-B)}{1+2\lambda(1+3t)}+\frac{|\gamma|^2(A-B)^2}{[1+\lambda(1+2t)]^2}$$

[7], Taking $\phi(z)=\frac{1+(1-2\beta)z}{1-z},0\le\beta<1$ in the class $W_{\Sigma}(\gamma,\lambda,\phi(z)),$ we have $W_{\Sigma}(\gamma,\lambda,t,B)$ and if $f\in W_{\Sigma}(\gamma,\lambda,t,B)$ then the following conditions are satisfied:

$$R\left[1+\frac{1}{\gamma}\left((1-\lambda)\frac{f(z)}{z}+\lambda(f'(z)+tzf''(z))-1\right)\right]>\beta,\quad z\in U$$

and

$$R\left[1+\frac{1}{\gamma}\left((1-\lambda)\frac{f^{-1}(w)}{w}+\lambda((f^{-1}(w))'+tw(f^{-1}(w)))''-1\right)\right]>\beta,\quad z\in U$$

Corollary 2.2. If $f\in W_{\Sigma}(\gamma,\lambda,t,\beta)$, then

$$|a_2|\le|\gamma|\sqrt{\frac{2(1-\beta)}{|\gamma(1+2\lambda(1+3t))|}}$$

and

$$|a_3|\le\frac{2|\gamma|(1-\beta)}{1+2\lambda(1+3t)}+\frac{4|\gamma|^2(1-\beta)^2}{(1+\lambda(1+2t))^2}$$

Remark 2.2. Taking $\lambda=\alpha,\gamma=1$ and $t=0$ in corollary 2.2, our results same as [5, Theorem 3.2, p.1572] , $\lambda=1,\ \gamma=1$ and $t=0$ in corollary 2.2, as appeared in Srivatsava et al. [7 and 12, Theorem 2, p.1191]

Taking $\phi(z)=\left(\frac{1+z}{1-z}\right)^n,0<\eta\le 1$ in the class [7, 14] $W_{\Sigma}(\gamma,\lambda,t,\phi(z)),$ we have $W_{\Sigma}^{\eta}(\gamma,\lambda,t)$ and $f\in W_{\Sigma}^{\eta}(\gamma,\lambda,t)$ if the following conditions are satisfied:

$$\arg\left|1+\frac{1}{\gamma}\left[(1-\lambda)\frac{f(z)}{z}+\lambda(f'(z)+tzf''(z))-1\right]\right|<\eta,\quad z\in U$$

and

$$\arg\left|1+\frac{1}{\gamma}\left[(1-\lambda)\frac{f^{-1}(w)}{w}+\lambda((f^{-1}(w))'+tw(f^{-1}(w)))''-1\right]\right|<\eta,\quad w\in U$$

Corollary 2.3[7]. If $f\in W_{\Sigma}^{\eta}(\gamma,\lambda,\alpha)$, then

$$|a_2|\leq\frac{2|\gamma|\eta}{\sqrt{|2\gamma(1+2\lambda(1+3t))\eta+(1+\lambda(1+2t))^2(1-\eta)|}}$$

and

$$|a_3|\leq\frac{2|\gamma|\eta}{1+2\lambda(1+3t)}+\frac{4|\gamma|^2\eta^2}{(1+\lambda(1+2t))^2}$$

Remark 2.3. Taking $\lambda=\alpha,\gamma=1$ and $t=0$ in corollary 2.3, our results same as appeared in [5,Theorem 2.2, p.1570], $\lambda=1,\gamma=1$ and $t=0$ in corollary 2.6, same as [11] and Srivatsava et al. [12, Theorem 1, p.1190]

Nomenclature
SCF – Starlike and convex function
BS – BC – Bistarlike – Bi Convex function
SCC – Starlike and convex function of complex order
ICB – Initial coefficient bounds
BUF – Bi univalent functions

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