

Degree Splitting Graphs of Certain Classes of Graphs with Small Power Domination Number

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Abstract: A set S of vertices is defined to be a power dominating set of a graph G if every vertex and every edge in the system is monitored by the set S (according to a set of rules for power system monitoring). The minimum number of elements of a power dominating set of a graph G is the power domination number $\gamma_p(G)$. When operations on graphs are carried out, the original graphs under consideration give rise to new types of graphs. The degree splitting is one such operation, having some applications as well. In this paper, we compute the power domination number γ_p for degree splitting graphs of certain classes of graphs.

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1. Introduction

Let $G = (V, E)$ be a finite, undirected and simple graph, with the number of vertices $|V(G)| = n$. The neighbourhood of $u \in V(G)$ is $N_G(u) = \{v \in V(G) : uv \in E\}$. The degree of u is $\deg(u) = |N(u)|$. For the standard notions on graphs and unexplained concepts, we refer to [1]. A subset $S \subseteq V$ is a dominating set of G [5,6] if every vertex in $V - S$ has at least one neighbour in S . A dominating set S of G is called a minimum dominating set, if S consists of a minimum number of vertices among all dominating sets of G . Based on the concept of domination in graphs, Haynes et al. [5] developed the concept of power domination while formulating in graph theoretical terms, a problem related to electric power system. There has been a number of studies on the power domination number for common graph classes [6] and also on the relationship between domination number and power domination number. A subset $S \subseteq V$ is a power dominating set [5] of G if all the vertices of V can be observed recursively by the following rules: i) all vertices in the neighbour set $N[S]$ are observed initially and ii) if an observed vertex u has all its neighbours observed except for one non-observed neighbour v , then v is observed (by u).

Given a graph G , while the domination number $\gamma(G)$ represents the number of vertices in a minimum dominating set of G , the power domination number $\gamma_p(G)$ is the minimum number of vertices required for a power dominating set of G . Let $P_n, C_n, K_n, K_{m,n}, K_{1,n}, B(n,n)$, and Y_n denote respectively, the path, cycle, complete graph, complete bipartite graph, star graph, bistar graph and prism graph of order n [1,3,7,9,10,12]. In this paper, we investigate power domination number of degree splitting graphs of different classes of graphs.

2. Main Results

2.1 Degree Splitting Graphs

Ponraj and Somasundaram[8] introduced the concept of degree splitting graph $DS(G)$ of a graph G .

Definition 2.1. Given a graph $G = (V, E)$ with $V = S_1 \cup S_2 \cup \dots \cup S_t \cup T$ where t is an integer ≥ 1 , each S_i , $1 \leq i \leq t$, is a set of at least two vertices of G of the same degree and $T = V - \cup S_i$, then the degree splitting graph $DS(G)$ of the graph G is defined as a graph which is obtained from G by adding vertices $w_1, w_2, \dots, w_t$ and joining w_i , for each i , $1 \leq i \leq t$, to each vertex of S_i . Note that if $V(G) = \bigcup_{i=1}^t S_i$ then $T = \phi$.

Theorem 2.2. If G is a connected graph, then $\gamma_p(DS(G)) \leq \gamma_p(G)$

Proof. Let G be a connected graph with $V(G) = \{v_1, v_2, \dots, v_n\}$. Let $D = \{v_i : 1 \leq i < n\}$ be the minimum power dominating set of G . Then $\gamma_p(G) = |D|$. By the definition of $DS(G)$, we have $V(DS(G)) = S_1 \cup S_2 \cup \dots \cup S_t \cup T$, where t is an integer ≥ 1 and T is as in the definition of $DS(G)$. Introduce $\{w_i : 1 \leq i \leq t\}$ and join w_i , for each i , $1 \leq i \leq t$ to every vertex of S_i .

If $T = \phi$, then $\{w_i : 1 \leq i \leq t\}$ is maximal independent set in $DS(G)$. Since every maximal independent set $\{w_i\}$ is a minimum dominating set it is also a minimum power dominating set. Therefore, $D' = \{w_i : 1 \leq i \leq t\}$ is the minimum power dominating set of $DS(G)$.

If $T \neq \phi$. There is at least one vertex in G which is not in $\{S_i : 1 \leq i \leq t\}$. Since G is an induced subgraph of $DS(G)$ to power dominate all the vertices of $DS(G)$, we require at least $|w_i \cup T|$ vertices. Therefore, $D' = \{w_i : 1 \leq i \leq t\} \cup T$, becomes a minimum power dominating set of $DS(G)$ and also noted that $|D'| \leq |D|$. Hence, $\gamma_p(DS(G)) \leq \gamma_p(G)$.

Theorem 2.3. For the complete bipartite graph $K_{m,n}$, $m \neq n$, $m, n \geq 3$, $\gamma_p(DS(K_{m,n})) = 2$.

Proof. Let $V_1 = \{v_1, v_2, \dots, v_m\}$ and $V_2 = \{u_1, u_2, \dots, u_n\}$ be the partition of $V(K_{m,n})$. The complete bipartite graph $K_{m,n}$ contains two types of vertices namely, vertices of degree n and vertices of degree m . Thus $V(K_{m,n}) = S_1 \cup S_2$, where $S_1 = \{v_i : 1 \leq i \leq m\}$ and $S_2 = \{u_j : 1 \leq j \leq n\}$. For obtaining $DS(K_{m,n})$ from $K_{m,n}$, we add w_1 and w_2 corresponding to S_1 and S_2 respectively. We claim that $D' = \{w_1, w_2\}$ is a minimum power dominating set for $DS(K_{m,n})$ because the vertices in v_i ($1 \leq i \leq m$) are dominated and hence power dominated by the vertex w_1 while the vertices of u_j , ($1 \leq j \leq n$) dominated and hence power dominated by the vertex w_2 . Thus $\gamma_p(DS(K_{m,n})) = 2$.

On the other hand, while computing γ_p -set of $K_{m,n}$, note that every vertex of V_1 power dominates every vertex of V_2 and vice versa. Therefore, we choose one vertex from V_1 and another vertex from V_2 for γ_p -set. Thus $\gamma_p(DS(K_{m,n})) = \gamma_p(K_{m,n})$.

Corollary 2.4. For $m, n \geq 3$ with $m = n$, $\gamma_p(DS(K_{n,n})) = 1$.

Proof. In a complete bipartite graph $K_{n,n}$ with a bipartition $V_1 \cup V_2$ of its vertex set, we construct $DS(K_{n,n})$ by introducing a new vertex w and join this to every vertex of $K_{n,n}$. We observe that the vertex w dominates and hence power dominates all the vertices of $DS(K_{n,n})$.

Theorem 2.5. *If G is a k -regular bipartite graph with $k \geq 3$, then $\gamma_p(DS(G)) < \gamma_p(G)$.*

Proof. Let G be a k -regular bipartite graph with bipartition (V_1, V_2) . Since G is k -regular, $k|V_1| = |E| = k|V_2|$ and so, since $k \geq 3$, $|V_1| = |V_2|$. In order to construct $DS(G)$ from G , we add w and join w to each vertex of S_1 . Thus $V(DS(G)) = S_1 \cup \{w\}$. We claim that $D' = \{w\}$ is a γ_p -set for $DS(G)$, because w dominates and hence power dominates all the vertices of $DS(G)$. Hence $\gamma_p(DS(G)) = 1$.

On the other hand, we observe that any single vertex v_i of G cannot power dominate the entire k -regular bipartite graph G . Hence $\gamma_p(G) \geq 2$. Thus $\gamma_p(DS(G)) < \gamma_p(G)$.

Definition 2.6. [2] *The binomial tree of order $t \geq 0$ with root R is the tree B_t defined as follows:*

If $t = 0$, $B_t = B_0 = \{R\}$. That is B_t consists of a single vertex R

If $t > 0$, $B_t = R, B_0, B_1, \dots, B_{t-1}$. That is, B_t comprises the root R and t binomial subtrees $B_0, B_1, \dots, B_{t-1}$.

Binomial trees B_0 , B_1 , B_2 and B_3 are shown in Figure 1.

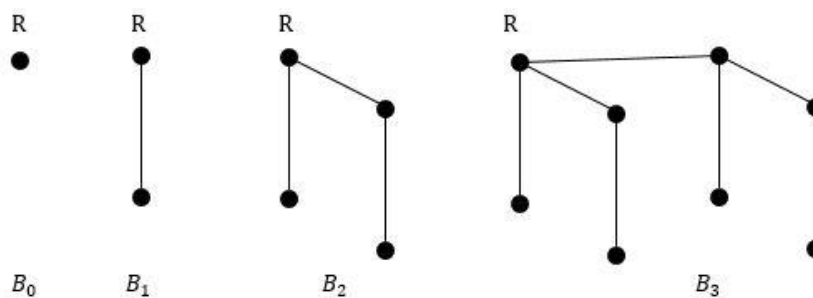


Figure 1: Binomial trees B_0 , B_1 , B_2 and B_3

Theorem 2.7. *Let B_t be a binomial tree with $t \geq 2$. Then $\gamma_p(DS(B_t))$ is 1.*

Proof. Let B_t be binomial tree with $V(B_t) = S_1 \cup S_2 \cup \dots \cup S_t \cup T$ where t is an integer ≥ 1 and $T = \emptyset$. Here $S_1 = \{v_i \in V \mid \deg(v_i) = t\}$, $S_2 = \{v_j \in V \mid \deg(v_j) = t-1\}$, $S_3 = \{v_k \in V \mid \deg(v_k) = t-2\}$, ..., $S_t = \{v_t \in V \mid \deg(v_t) = 1\}$. By the definition of degree splitting graph of B_t , we introduce $\{w_i : 1 \leq i \leq t\}$ and join $\{w_i : 1 \leq i \leq t\}$ to every element of $\{S_i : 1 \leq i \leq t\}$. We claim that $D' = \{w_t\}$ is minimum power dominating set for $DS(B_t)$ because the vertices of the set $\{S_t\}$ are dominated and power dominated by the vertex w_t while the vertices of sets $\{S_1, S_2, \dots, S_{t-1}\}$ are power dominated by the vertex w_t . Thus $\gamma_p(DS(B_t)) = 1$.

Definition 2.8. [11] *A binary tree is a tree in which all but one of the vertices are of degree one or three, including the root vertex v_0 , and there is only one vertex of degree two. A complete binary tree is a binary tree*

in which all leaves are on the same level or all leaves have same distance to the root vertex v_0 . We denote a complete binary tree with diameter $2k$ by $BT(k)$, where $k \geq 1$.

A graph $BT(k)$ can be constructed recursively from two copies of $BT(k-1)$ by joining their root vertices to a new vertex v_0 .

Theorem 2.9. Let $BT(k)$ be a complete binary tree with height $k \geq 3$. Then $\gamma_p(DS(BT(k)))$ is 2.

Proof. Let $BT(k)$ be a complete binary tree with diameter $2k$, where $k \geq 1$. Each of the vertices $v_1, v_2, \dots, v_n$ of $BT(k)$, other than the root vertex v_0 is of degree one or three. The vertex set of $BT(k)$ is $V = S_1 \cup S_2 \cup T$ where $T = \{v_0\}$, $S_1 = \{v_i \in V \mid \deg(v_i) = 3\}$ and $S_2 = \{v_i \in V \mid \deg(v_i) = 1\}$. The degree splitting graph $DS(BT(k))$ is constructed by introducing the new vertices w_1 and w_2 corresponding to S_1 and S_2 respectively. Let $D' = \{w_1, w_2\}$ is γ_p -set for $DS(BT(k))$ because every vertex in S_1 and S_2 are dominated by w_1 and w_2 respectively, while the root vertex v_0 is power dominated by S_2 . Thus $\gamma_p(DS(BT(k))) = 2$.

Remark 2.10. Note that for a complete binary tree $BT(k)$ with height $k \geq 3$, the power domination number $\gamma_p(BT(k)) = 2^{n-1}$.

Definition 2.11. [12] The bistar $B_{n,n}$ is the graph obtained by joining the center vertices of two copies of $K_{1,n}$ by an edge.

Theorem 2.12. The degree splitting graph of any graph G of order $n \geq 2$ is not a tree.

Proof. If a graph G contains a cycle, then degree splitting graph of G is also contains cycle. Therefore, the theorem is obvious. If G does not contain a cycle then G is called tree. Note that, a tree with $n(\geq 2)$ vertices has $n-1$ edges. It is known that, in any graph G with at least two vertices have the same degree. Suppose v_i and v_j in a graph G have the same degree vertices, then by the definition of degree splitting, introduce w to join the vertices v_i and v_j . This implies $v_i v_j w$ forms a cycle in $DS(G)$. Thus $DS(G)$ is not a tree.

Theorem 2.13. For $n \geq 2$, $\gamma_p(DS(B_{n,n})) = 1$.

Proof. Let $B_{n,n}$ be a bistar graph with $V = S_1 \cup S_2$ where $S_1 = \{u, v\}$ with u and v adjacent and $S_2 = \{u_i, v_i : 1 \leq i \leq n\}$. Note that u_i and v_i , $1 \leq i \leq n$, are pendant vertices with all u_i joined to u and all v_i joined to v . In order to obtain the degree splitting graph of $B_{n,n}$, we introduce new vertices w_1 and w_2 to S_1 and S_2 respectively. The vertex w_1 is joined to u and v while w_2 is joined to all the remaining vertices of $B(n, n)$. We claim that $D' = \{w_1\}$ is a γ_p -set for $DS(B_{n,n})$, because $V(DS(B_{n,n}))$ are power dominated by w_1 . Hence $\gamma_p(DS(B_{n,n})) = 1$.

On the other hand, while computing γ_p -set of $B_{n,n}$, add the vertices w_1 and w_2 to D . Hence $\gamma_p(B(n, n)) = 2$.

Definition 2.14. [4] The corona of two graphs G_1 and G_2 is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 where the i th vertex of G_1 is adjacent to every vertex in the i th copy of G_2 .

Theorem 2.15. Let G be a simple graph with $m(\geq 2)$ vertices and let P_n be path with $n(\geq 2)$ vertices.

$$\text{Then } \gamma_p(DS(G \circ P_n)) = \begin{cases} 2 & \text{if } n = 3, 4 \\ 1, & \text{otherwise} \end{cases}$$

Proof. Let G be a simple connected graph with $V(G) = \{u_1, u_2, \dots, u_m\}$ and let P_n , $n \geq 2$ be a path with $V(P_n) = \{v_1, v_2, \dots, v_n\}$ vertices. Consider the graph $G \circ P_n$, with the vertex set $V(G \circ P_n) = \{u_i : 1 \leq i \leq m\} \cup \{v_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$. This set can be rewritten as $V(G \circ P_n) = S_1 \cup S_2 \cup \dots \cup S_t \cup T$, where t is an integer ≥ 1 . Here $S_1, S_2, \dots, S_t$ consist of vertices of degrees $2, 3, \dots, t+1$ respectively and T is defined in the definition. In constructing the degree splitting graph $DS(G \circ P_n)$, new vertices $w_1, w_2, \dots, w_t$ are introduced to $S_1, S_2, \dots, S_t$ respectively, and made adjacent $\{w_i\}$, to every element in $\{S_i\}$, $1 \leq i \leq t$.

Let $DS(G \circ P_2)$ is a graph with $V(DS(G \circ P_2)) = \{S_i : 1 \leq i \leq t\} \cup \{w_i : 1 \leq i \leq t\} \cup T$. We claim that $D' = \{w_1\}$ is a power dominating set for $DS(G \circ P_2)$ because the vertices in the set S_1 are dominated initially and $V(DS(G \circ P_2)) - S_1$ are power dominated. Thus $\gamma_p(DS(G \circ P_n)) = 1$.

Case 1: When $n = 3, 4$

In this case, consider the subset $D' = \{w_1, w_2\}$ of $V(DS(G \circ P_n))$. It is note that the vertices w_1 and w_2 are adjacent to all the vertices of degrees 3 and 4 respectively in $DS(G \circ P_n)$. By the definition of power domination, in a graph $DS(G \circ P_n)$, the vertices in each $\{v_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ are dominated initially and the remaining vertices are power dominated by the elements of the set D' . Thus $\gamma_p(DS(G \circ P_n)) = 2$.

Case 2: When $n \geq 5$

In this case, consider the subset $D' = \{w_2\}$ of $V(DS(G \circ P_n))$. By the definition of power domination, in a graph $DS(G \circ P_n)$, the vertices $\{v_{ij} \mid \deg(v_{ij}) = 4\}$ are dominated and the remaining vertices are power dominated by the subset D' . Hence $\gamma_p(DS(G \circ P_n)) = 1$.

Theorem 2.16. Let G be a simple graph with $m(\geq 2)$ vertices and let H be n -regular graph with $n \geq 2$.

Then $\gamma_p(DS(G \circ H))$ is 1.

Proof. Let $u_1, u_2, \dots, u_m$ be the vertices of G , and $v_{i,j}$ be the vertices of H corresponding to u_i , where $i \in \{1, 2, \dots, m\}$ and $j \in \{1, 2, \dots, n\}$. The vertex set of the corona graph $G \circ H$ is $V(G \circ H) = \{u_i : 1 \leq i \leq m\} \cup \{v_{ij} : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$. This set can be rewritten as $V(G \circ H) = S_1 \cup S_2 \cup \dots \cup S_t \cup T$, where t is an integer ≥ 1 . Here $S_1, S_2, \dots, S_t$ consist of vertices of degrees $2, 3, \dots, t+1$ respectively and T is defined in the definition. In constructing the degree splitting graph $DS(G \circ H)$, new vertices $w_1, w_2, \dots, w_t$ are introduced to $S_1, S_2, \dots, S_t$ respectively, and made adjacent $\{w_i : 1 \leq i \leq t\}$, to every element in the corresponding $\{S_i : 1 \leq i \leq t\}$. We claim that $D' = \{w_1\}$ is a

minimum power dominating set for $DS(G \circ H)$ because the vertices in the subset S_1 are dominated initially and $V(DS(G \circ H)) - S_1$ are power dominated. Hence $\gamma_p(DS(G \circ H)) = 1$.

Corollary 2.17. Let G be a simple graph with $m(\geq 2)$ vertices. If $H \in \{C_n, K_n, Y_n\}$ then $\gamma_p(DS(G \circ H)) = 1$.

Proof. The proof can be given as done in Theorem 2.16.

3. Conclusion

A variant of domination called power domination on graphs is introduced in the problem of monitoring electric networks. In this paper we have established the minimum power domination number for the degree splitting graphs of certain kinds of graphs

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