On the Modified First Zagreb Index of Generalized Fuzzy Transformation Graphs

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Abstract

Let $G=(V,\omega,\rho)$ be a fuzzy graph with membership values ω and ρ of vertices and edges respectively. The modified first Zagreb index ($M_1^*(G)$) is the sum of degrees of every pair of adjacent vertices together with their corresponding membership values. In this paper, the modified first Zagreb index of fuzzy graphs is initiated and obtained upper bounds for $M_1^*(G^{x,y})$ generalized fuzzy transformation graphs in terms of elements of G.

Keywords: *First Zagreb index; Second Zagreb index; Generalized fuzzy transformation graph; .*

Subject Classification: 05C12, 05C76.

1 Introduction

Let $G=(V,\omega,\rho)$ be a fuzzy graph with vertex set V and the membership values for the vertices and edges are defined by $\omega: V\rightarrow[0,1]$ and $\rho: V\times V\rightarrow[0,1]$ respectively satisfying $\rho(u,v)\leq\omega(u)\wedge\omega(v)$ where $\wedge$ represents the minimum. $\omega(u)$ represents the membership value for the vertex $u\in V(G)$ and $\rho(e)$ represents the membership value of an edge $e\in E(G)$. The degree of a vertex $v\in V(G)$ is the sum of the membership values of the edges which are incident to a vertex v. i.e $deg_G(v)=\sum_{u\neq v}\rho(uv)$. It is obvious that $\sum_{v\in V(G)} deg_G(v)=\sum_{v\in V(G)}\sum_{u\neq v}\rho(uv)\leq 2m$ where $m=|E(G)|$. For undefined terminology in this paper refer [7,14].

A topological index is a number generated from a molecular structure (i.e., a graph) that indicates the essential structural properties of the proposed molecule. Indeed, it is an algebraic quantity connected with the chemical structure that correlates it with various physical characteristics. It is possible to determine several different properties, such as chemical activity, thermodynamic properties, physicochemical activity, and biological activity, using several topological indices. Zagreb indices[4] are one among them, these indices are used to calculate π-electron energy of a conjugate system.

Nowadays, due to various applications of fuzzy graph theory, a huge number of researchers are working on topological indices[2, 3, 5, 6, 9-11, 15, 16]. Kalathian et.al[12] have studied the first index for fuzzy graphs and is defined as follows:

$$M_1(G)=\sum_{i=1}^{n}\omega(u)deg_G(v_i)^2 \qquad (1)$$

To maintain the similarity of the first Zagreb index for crisp graphs, Islam and Pal[10] redefined the first Zagreb index in the following way.

$$M_1(G) = \sum_{i=1}^{n} [\omega(u)deg_G(v_i)]^2 \quad (2)$$

In this paper, the modified first Zagreb index is put forward by keeping (3) in mind. It is defined as:

$$M_1(G) = \sum_{uv\in E(G)} [\omega(u)deg_G + \omega(v)deg_G(v)] \quad (3)$$

In fact, (1) and (3) gives the same results for crisp graphs but (5) and (6) are not the same. The difference between these two definitions are explored by the following example.

Example 1. Consider a fuzzy graph with $V(G) = \{p, q, r, s\}$ and $E(G)\{pq, qr, qs, rs\}$ where $\omega(p) = 0.7$, $\omega(q) = 0.9$, $\omega(r) = 0.6$, $\omega(s) = 0.7$, $\rho(pq) = 0.5$, $\rho(qr) = 0.6$, $\rho(qs) = 0.5$ and $\rho(rs) = 0.4$. Then degrees of each vertex of G is given by $deg_G(p) = 0.5$, $deg_G(q) = 1.6$, $deg_G(r) = 1.0$ and $deg_G(s) = 0.9$. Therefore the first Zagreb index of G is given by:

$$M_1(G) = \sum_{u\in V(G)} [\omega(u)deg_G(u)]^2$$

$$= [(0.7)(0.5)]^2 + [(0.9)(1.6)]^2 + [(0.6)(1.0)]^2 + [(0.7)(0.9)]^2$$

$$= 4.0555.$$

Now consider the modified first Zagreb index for fuzzy graphs:

$$M_1^*(G) = \sum_{uv\in E(G)} [\omega(u)deg_G(u) + \omega(v)deg_G(v)]$$

$$= [\omega(p)deg_G(p) + \omega(q)deg_G(q)] + [\omega(q)deg_G(q) + \omega(r)deg_G(r)] + [\omega(q)deg_G(q) + \omega(s)deg_G(s)]$$

$$+[\omega(s)deg_G(s) + \omega(r)deg_G(r)]$$

$$= [(0.7)(0.5) + (0.9)(1.6)] + [(0.9)(1.6) + (0.6)(1.0)] + [(0.9)(1.6) + (0.7)(0.9)]$$

$$+[(0.7)(0.9) + (0.9)(1.0)]$$

$$= 7.13.$$

Clearly, for this graph $M_1^*(G) > M_1(G)$. Therefore, we have selected this parameter for the study.

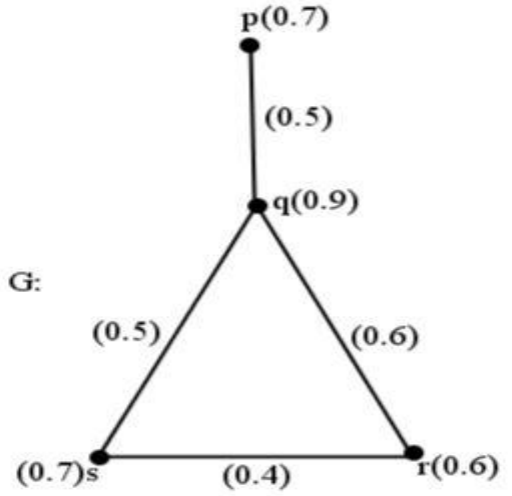

Fig 1. A fuzzy graph on 4-vertices.

2 Generalized Fuzzy Transformation Graphs

Basavanagoud et.al[1] studied the Zagreb indices of generalized transformation graphs. In same way the generalized fuzzy transformation graphs have been studied in [13]. In this paper, the modified first Zagreb index of generalized fuzzy transformation graphs are studied. Consider a fuzzy cycle C_5 depicted in Figure 2. The examples for generalized fuzzy transformation graphs of fuzzy cycle are given below:

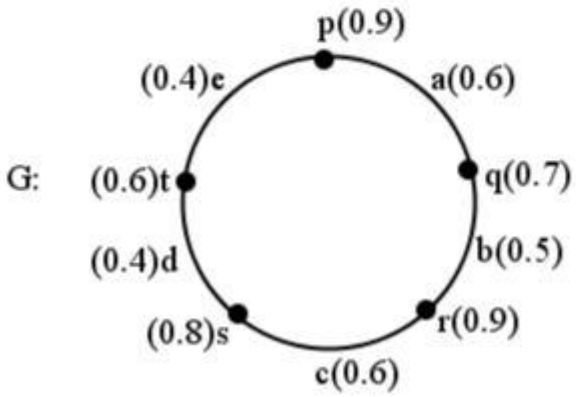

Fig 2. A fuzzy cycle graph on 5-vertices.

Example for G^{++}: Clearly, $V(G^{++}) = \{p, q, r, s, t, a, b, c, d, e\}$ with $\omega(p) = 0.9$, $\omega(q) = 0.7$, $\omega(r) = 0.9$, $\omega(s) = 0.8$, $\omega(t) = 0.6$, $\omega(a) = 0.6$, $\omega(b) = 0.5$, $\omega(c) = 0.6$, $\omega(d) = 0.4$ and $\omega(e) = 0.4$. Also $deg_{G^{++}}(p) = 2\rho(a) + 2\rho(e) = 2(0.6) + 2(0.4) = 2.0$. Similarly, $deg_{G^{++}}(q) = 2.2$, $deg_{G^{++}}(r) = 2.2$, $deg_{G^{++}}(s) = 2.0$, $deg_{G^{++}}(t) = 1.6$ and $deg_{G^{++}}(a) = 2\rho(a) = 1.2$. Similarly, $deg_{G^{++}}(b) = 1.0$, $deg_{G^{++}}(c) = 1.2$, $deg_{G^{++}}(d) = 0.8$ and $deg_{G^{++}}(e) = 0.8$. Now, the modified first Zagreb index of G^{++} is given by:

$$M_1^* = \sum_{uv\in E(G^{++})} [\omega_{G^{++}}(u)deg_{G^{++}}(u) + \omega_{G^{++}}(v)deg_{G^{++}}(v)]$$

$$= [\omega_{G^{++}}(p)deg_{G^{++}}(p) + \omega_{G^{++}}(q)deg_{G^{++}}(q)] + [\omega_{G^{++}}(p)deg_{G^{++}}(p) + \omega_{G^{++}}(t)deg_{G^{++}}(t)]$$

$$+[\omega_{G^{++}}(p)deg_{G^{++}}(p) + \omega_{G^{++}}(a)deg_{G^{++}}(a)] + [\omega_{G^{++}}(p)deg_{G^{++}}(p) + \omega_{G^{++}}(e)deg_{G^{++}}(e)]$$

$$+[\omega_{G^{++}}(q)deg_{G^{++}}(q) + \omega_{G^{++}}(r)deg_{G^{++}}(r)] + [\omega_{G^{++}}(q)deg_{G^{++}}(q) + \omega_{G^{++}}(a)deg_{G^{++}}(a)]$$

$$+[\omega_{G^{++}}(q)deg_{G^{++}}(q) + \omega_{G^{++}}(b)deg_{G^{++}}(b)] + [\omega_{G^{++}}(r)deg_{G^{++}}(r) + \omega_{G^{++}}(s)deg_{G^{++}}(s)]$$

$$+[\omega_{G^{++}}(r)deg_{G^{++}}(r) + \omega_{G^{++}}(b)deg_{G^{++}}(b)] + [\omega_{G^{++}}(r)deg_{G^{++}}(r) + \omega_{G^{++}}(c)deg_{G^{++}}(c)]$$

$$+[\omega_{G^{++}}(s)deg_{G^{++}}(s) + \omega_{G^{++}}(t)deg_{G^{++}}(t)] + [\omega_{G^{++}}(s)deg_{G^{++}}(s) + \omega_{G^{++}}(c)deg_{G^{++}}(c)]$$

$$+[\omega_{G^{++}}(s)deg_{G^{++}}(s) + \omega_{G^{++}}(d)deg_{G^{++}}(d)] + [\omega_{G^{++}}(t)deg_{G^{++}}(t) + \omega_{G^{++}}(d)deg_{G^{++}}(d)]$$

$$+[\omega_{G^{++}}(t)deg_{G^{++}}(t) + \omega_{G^{++}}(e)deg_{G^{++}}(e)]$$

$$= [(0.9)(2) + (0.7)(2.2)] + [(0.9)(2) + (0.6)(1.6)] + [(0.9)(2) + (0.6)(1.2)] + [(0.9)(2) + (0.4)(0.8)]$$

$$+[(0.7)(2.2) + (0.9)(2.2)] + [(0.7)(2.2) + (0.6)(1.2)] + [(0.7)(2.2) + (0.5)(1.0)]$$

$+[(0.9)(2.2)+(0.8)(2.0)]+[(0.9)(2.2)+(0.5)(1)]+[(0.9)(2.2)+(0.6)(1.2)]$
$+[(0.8)(2.0)+(0.6)(1.6)]+[(0.8)(2.0)+(0.6)(1.2)]+[(0.8)(2.0)+(0.4)(0.8)]$
$+[(0.6)(1.6)+(0.4)(0.8)]+[(0.6)(1.6)+(0.4)(0.8)]$
$=20.698.$

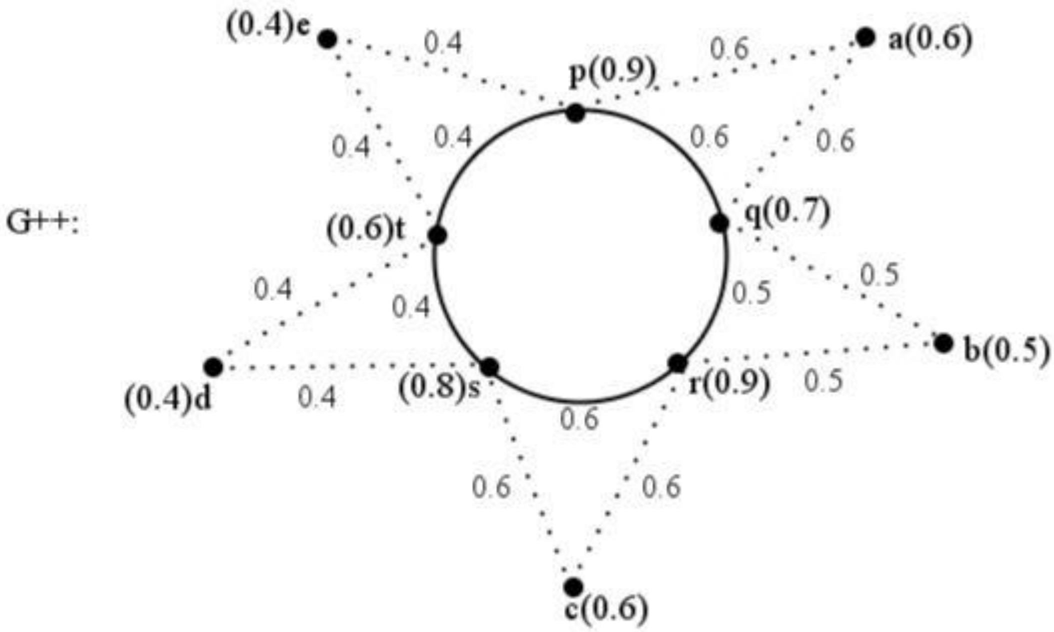

Fig 3. Generalized fuzzy transformation graph G++

Example for G^{+-}: Clearly, $V(G^{+-})=\{p,q,r,s,t,a,b,c,d,e\}$ with $\omega(p)=0.9$, $\omega(q)=0.7$, $\omega(r)=0.9$, $\omega(s)=0.8$, $\omega(t)=0.6$, $\omega(a)=0.6$, $\omega(b)=0.5$, $\omega(c)=0.6$, $\omega(d)=0.4$ and $\omega(e)=0.4$. Also $deg_{G^{+-}}(p)=deg_{G^{+-}}(q)=deg_{G^{+-}}(r)=deg_{G^{+-}}(s)=deg_{G^{+-}}(t)=2.5$ and $deg_{G^{+-}}(a)=1.8$, $deg_{G^{+-}}(b)=1.5$, $deg_{G^{+-}}(c)=1.8$, $deg_{G^{+-}}(d)=1.2$ and $deg_{G^{+-}}(e)=1.2$. Now, the modified first Zagreb index of G^{+-} is given by:

$$M_1^* = \sum_{uv\in E(G^{+-})}[\omega_{G^{+-}}(u)deg_{G^{+-}}(u)+\omega_{G^{+-}}(v)deg_{G^{+-}}(v)]$$

$=$
$[\omega_{G^{+-}}(p)deg_{G^{+-}}(p)+\omega_{G^{+-}}(q)deg_{G^{+-}}(q)]+[\omega_{G^{+-}}(p)deg_{G^{+-}}(p)+\omega_{G^{+-}}(t)deg_{G^{+-}}(t)]$
$+[\omega_{G^{+-}}(p)deg_{G^{+-}}(p)+\omega_{G^{+-}}(b)deg_{G^{+-}}(b)]+[\omega_{G^{+-}}(p)deg_{G^{+-}}(p)+\omega_{G^{+-}}(c)deg_{G^{+-}}(c)]$
$+[\omega_{G^{+-}}(p)deg_{G^{+-}}(p)+\omega_{G^{+-}}(d)deg_{G^{+-}}(d)]+[\omega_{G^{+-}}(q)deg_{G^{+-}}(q)+\omega_{G^{+-}}(r)deg_{G^{+-}}(r)]$
$+[\omega_{G^{+-}}(q)deg_{G^{+-}}(q)+\omega_{G^{+-}}(c)deg_{G^{+-}}(c)]+[\omega_{G^{+-}}(q)deg_{G^{+-}}(q)+\omega_{G^{+-}}(d)deg_{G^{+-}}(d)]$
$+[\omega_{G^{+-}}(q)deg_{G^{+-}}(q)+\omega_{G^{+-}}(e)deg_{G^{+-}}(e)]+[\omega_{G^{+-}}(r)deg_{G^{+-}}(r)+\omega_{G^{+-}}(s)deg_{G^{+-}}(s)]$
$+[\omega_{G^{+-}}(r)deg_{G^{+-}}(r)+\omega_{G^{+-}}(a)deg_{G^{+-}}(a)]+[\omega_{G^{+-}}(r)deg_{G^{+-}}(r)+\omega_{G^{+-}}(d)deg_{G^{+-}}(d)]$
$+[\omega_{G^{+-}}(r)deg_{G^{+-}}(r)+\omega_{G^{+-}}(e)deg_{G^{+-}}(e)]+[\omega_{G^{+-}}(s)deg_{G^{+-}}(s)+$

$\omega_{G^{+-}}(t)deg_{G^{+-}}(t)]$

$+[\omega_{G^{+-}}(s)deg_{G^{+-}}(s) + \omega_{G^{+-}}(a)deg_{G^{+-}}(a)] + [\omega_{G^{+-}}(s)deg_{G^{+-}}(b) + \omega_{G^{+-}}(t)deg_{G^{+-}}(b)]$

$+[\omega_{G^{+-}}(s)deg_{G^{+-}}(s) + \omega_{G^{+-}}(e)deg_{G^{+-}}(e)] + [\omega_{G^{+-}}(t)deg_{G^{+-}}(t) + \omega_{G^{+-}}(a)deg_{G^{+-}}(a)]$

$+[\omega_{G^{+-}}(t)deg_{G^{+-}}(t) + \omega_{G^{+-}}(b)deg_{G^{+-}}(b)] + [\omega_{G^{+-}}(t)deg_{G^{+-}}(t) + \omega_{G^{+-}}(c)deg_{G^{+-}}(c)]$

$= 41.7.$

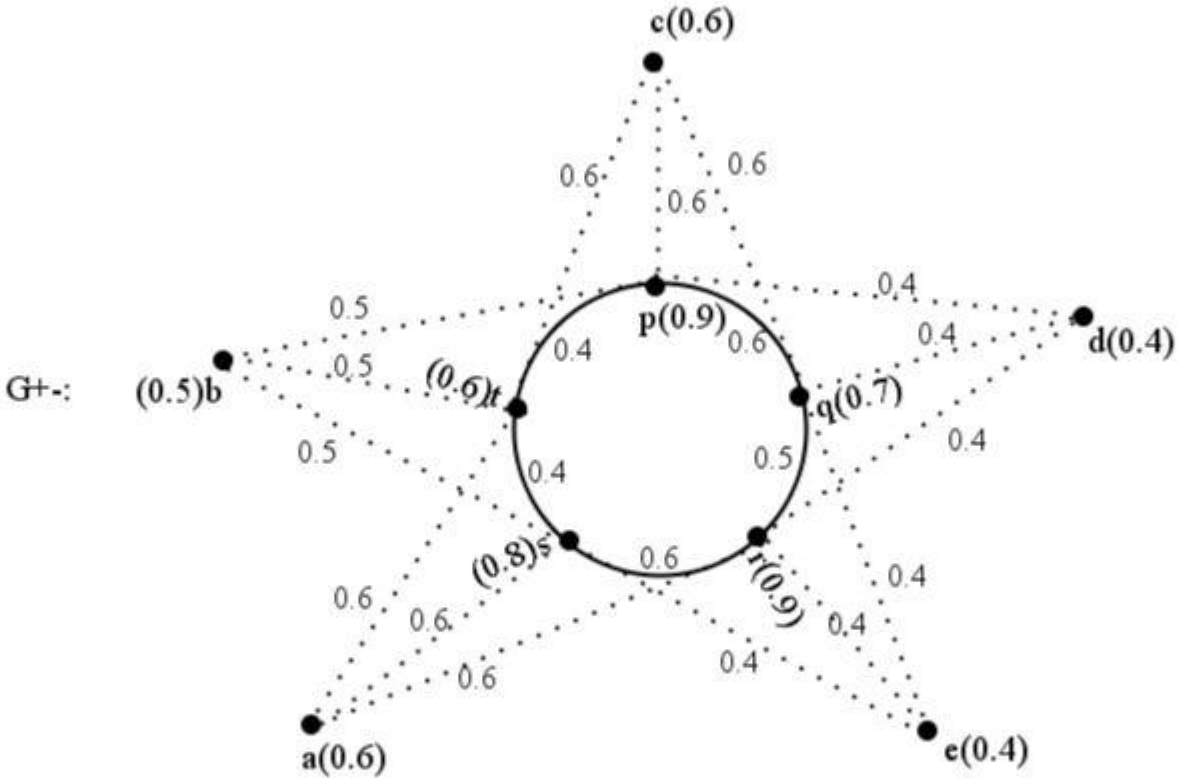

Fig 4. Generalized fuzzy transformation graph G+-

Example for G^{-+}: Clearly, $V(G^{-+}) = \{p, q, r, s, t, a, b, c, d, e\}$ with $\omega(p) = 0.9$, $\omega(q) = 0.7$, $\omega(r) = 0.9$, $\omega(s) = 0.8$, $\omega(t) = 0.6$, $\omega(a) = 0.6$, $\omega(b) = 0.5$, $\omega(c) = 0.6$, $\omega(d) = 0.4$ and $\omega(e) = 0.4$. Also $deg_{G^{-+}}(p) = 2.7 deg_{G^{-+}}(q) = 2.4 deg_{G^{-+}}(r) = 2.6 deg_{G^{-+}}(s) = 2.5 deg_{G^{-+}}(t) = 2.0$ and $deg_{G^{-+}}(a) = 1.2$, $deg_{G^{-+}}(b) = 1.0$, $deg_{G^{-+}}(c) = 1.2$, $deg_{G^{-+}}(d) = 0.8$ and $deg_{G^{-+}}(e) = 0.8$. Now, the modified first Zagreb index of G^{-+} is given by:

$$M_1^* = \sum_{uv \in E(G^{-+})} [\omega_{G^{-+}}(u)deg_{G^{-+}}(u) + \omega_{G^{-+}}(v)deg_{G^{-+}}(v)]$$

$=$

$[\omega_{G^{-+}}(p)deg_{G^{-+}}(p) + \omega_{G^{-+}}(r)deg_{G^{-+}}(r)] + [\omega_{G^{-+}}(p)deg_{G^{-+}}(p) + \omega_{G^{-+}}(s)deg_{G^{-+}}(s)]$

$+[\omega_{G^{-+}}(p)deg_{G^{-+}}(p) + \omega_{G^{-+}}(a)deg_{G^{-+}}(a)] + [\omega_{G^{-+}}(p)deg_{G^{-+}}(p) + \omega_{G^{-+}}(e)deg_{G^{-+}}(e)]$

$+[\omega_{G^{-+}}(q)deg_{G^{-+}}(q) + \omega_{G^{-+}}(s)deg_{G^{-+}}(s)] + [\omega_{G^{-+}}(q)deg_{G^{-+}}(q) +$

$\omega_{G^{-+}}(t)deg_{G^{-+}}(t)]$

$+[\omega_{G^{-+}}(q)deg_{G^{-+}}(q) + \omega_{G^{-+}}(a)deg_{G^{-+}}(a)] + [\omega_{G^{-+}}(q)deg_{G^{-+}}(q) + \omega_{G^{-+}}(b)deg_{G^{-+}}(b)]$

$+[\omega_{G^{-+}}(r)deg_{G^{-+}}(r) + \omega_{G^{-+}}(t)deg_{G^{-+}}(t)] + [\omega_{G^{-+}}(r)deg_{G^{-+}}(r) + \omega_{G^{-+}}(b)deg_{G^{-+}}(b)]$

$+[\omega_{G^{-+}}(r)deg_{G^{-+}}(r) + \omega_{G^{-+}}(c)deg_{G^{-+}}(c)] + [\omega_{G^{-+}}(s)deg_{G^{-+}}(s) + \omega_{G^{-+}}(c)deg_{G^{-+}}(c)]$

$+[\omega_{G^{-+}}(s)deg_{G^{-+}}(s) + \omega_{G^{-+}}(d)deg_{G^{-+}}(d)] + [\omega_{G^{-+}}(t)deg_{G^{-+}}(t) + \omega_{G^{-+}}(d)deg_{G^{-+}}(d)]$

$+[\omega_{G^{-+}}(t)deg_{G^{-+}}(t) + \omega_{G^{-+}}(e)deg_{G^{-+}}(e)]$

$= 29.0098.$

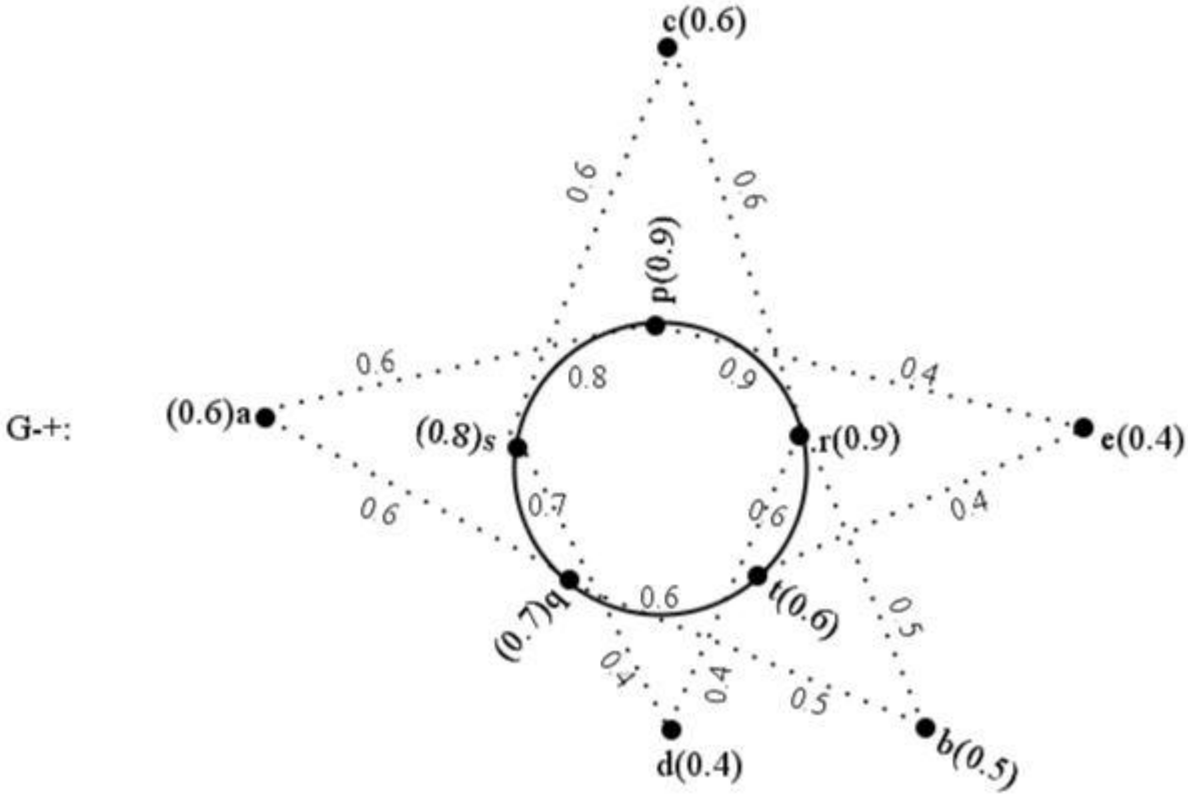

Fig 5. Generalized fuzzy transformation graph G-+

Example for G^{--}: Clearly, $V(G^{--}) = \{p, q, r, s, t, a, b, c, d, e\}$ with $\omega(p) = 0.9$, $\omega(q) = 0.7$, $\omega(r) = 0.9$, $\omega(s) = 0.8$, $\omega(t) = 0.6$, $\omega(a) = 0.6$, $\omega(b) = 0.5$, $\omega(c) = 0.6$, $\omega(d) = 0.4$ and $\omega(e) = 0.4$. Also $deg_{G^{--}}(p) = 3.2deg_{G^{--}}(q) = 2.7deg_{G^{--}}(r) = 2.9deg_{G^{--}}(s) = 3.0deg_{G^{--}}(t) = 2.9$ and $deg_{G^{--}}(a) = 1.8$, $deg_{G^{--}}(b) = 1.5$, $deg_{G^{--}}(c) = 1.8$, $deg_{G^{--}}(d) = 1.2$ and $deg_{G^{--}}(e) = 1.2$. Now, the modified first Zagreb index of G^{--} is given by:

$$M_1^* = \sum_{uv\in E(G^{--})} [\omega_{G^{--}}(u)deg_{G^{--}}(u) + \omega_{G^{--}}(v)deg_{G^{--}}(v)]$$

$=$

$[\omega_{G^{--}}(p)deg_{G^{--}}(p) + \omega_{G^{--}}(r)deg_{G^{--}}(r)] + [\omega_{G^{--}}(p)deg_{G^{--}}(p) + \omega_{G^{--}}(s)deg_{G^{--}}(s)]$

$$+[\omega_G{--}(p)deg_G{--}(p)+\omega_G{--}(b)deg_G{--}(b)]+[\omega_G{--}(p)deg_G{--}(p)+\omega_G{--}(c)deg_G{--}(c)]$$

$$+[\omega_G{--}(p)deg_G{--}(p)+\omega_G{--}(d)deg_G{--}(d)]+[\omega_G{--}(r)deg_G{--}(r)+\omega_G{--}(t)deg_G{--}(t)]$$

$$+[\omega_G{--}(r)deg_G{--}(r)+\omega_G{--}(a)deg_G{--}(a)]+[\omega_G{--}(r)deg_G{--}(r)+\omega_G{--}(d)deg_G{--}(d)]$$

$$+[\omega_G{--}(r)deg_G{--}(r)+\omega_G{--}(e)deg_G{--}(e)]+[\omega_G{--}(t)deg_G{--}(t)+\omega_G{--}(q)deg_G{--}(q)]$$

$$+[\omega_G{--}(t)deg_G{--}(t)+\omega_G{--}(a)deg_G{--}(a)]+[\omega_G{--}(t)deg_G{--}(t)+\omega_G{--}(b)deg_G{--}(b)]$$

$$+[\omega_G{--}(t)deg_G{--}(t)+\omega_G{--}(c)deg_G{--}(c)]+[\omega_G{--}(q)deg_G{--}(q)+\omega_G{--}(s)deg_G{--}(s)]$$

$$+[\omega_G{--}(q)deg_G{--}(q)+\omega_G{--}(c)deg_G{--}(c)]+[\omega_G{--}(q)deg_G{--}(q)+\omega_G{--}(d)deg_G{--}(d)]$$

$$+[\omega_G{--}(q)deg_G{--}(q)+\omega_G{--}(e)deg_G{--}(e)]+[\omega_G{--}(s)deg_G{--}(s)+\omega_G{--}(a)deg_G{--}(a)]$$

$$+[\omega_G{--}(s)deg_G{--}(s)+\omega_G{--}(b)deg_G{--}(b)]+[\omega_G{--}(s)deg_G{--}(s)+\omega_G{--}(e)deg_G{--}(e)]$$

$$=53.6302.$$

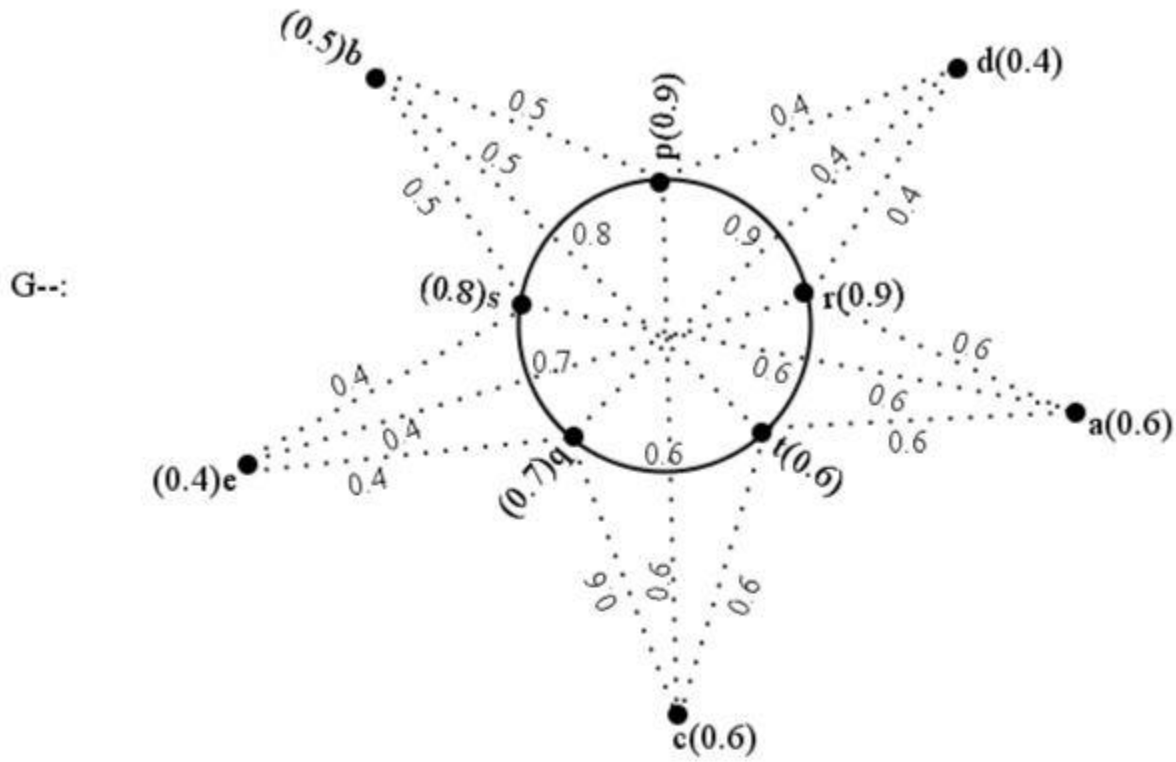

Fig 6. Generalized fuzzy transformation graph G--

3 Results

Observation 1 *Let* $G = (V, \omega, \rho)$ *be a fuzzy graph with* $v \in V(G)$ *and* $e \in E(G)$ *then*

$deg_{G^{++}}(v) = 2deg_G(v),\ deg_{G^{++}}(e') = 2\rho(e)$

$deg_{G^{+-}}(v) = \sum_{e \in E} \rho(e),\ deg_{G^{+-}}(e') = (n-2)\rho(e)$

$deg_{G^{-+}}(v) = deg_G(v) + \sum_{uv \notin E} [\omega(u) \wedge \omega(v)],\ deg_{G^{+-}}(e') = 2\rho(e)$

$deg_{G^{--}}(v) = \sum_{uv \notin E} [\omega(u) \wedge \omega(v)] + \sum_{v\text{-}e} \rho(e),\ deg_{G^{+-}}(e') = (n-2)\rho(e)$

where e' is the corresponding edge-vertex in G^{xy} and v-e denote the vertex v is not incident to an edge e.

Theorem 2 *Let* $G = (V, \omega, \rho)$ *be a fuzzy graph and* G^{++} *is the generalized fuzzy transformation graph of* G. *Thn*

$$M_1^*(G^{++}) \leq 2M_1^*(G) + 8m.$$

Proof. Let $G = (V, \omega, \rho)$ be a fuzzy graph with $\{\omega(v_1), \omega(v_2), \omega(v_3), \cdots, \omega(v_n)\}$ and $\{\rho(e_1), \rho(e_2), \rho(e_3), \cdots, \rho(e_m)\}$ be the membership values of vertices and edges of G respectively. Then $V(G^{++}) = V(G) \cup E(G)$ by definition of G^{++}. Therefore, the modified first Zagreb index of generalized transformation graph G^{++} is given by:

$$M_1^*(G^{++}) = \sum_{uv \in E(G^{++})} [\omega(u)deg_{G^{++}}(u) + \omega(v)deg_{G^{++}}(v)]$$

$$= \sum_{uv \in E(G^{++}) \cap E(G)} [\omega(u)deg_{G^{++}}(u) + \omega(v)deg_{G^{++}}(v)] + \sum_{uv \in E(G^{++})-E(G)} [\omega(u)deg_{G^{++}}(u) + \omega(v)deg_{G^{++}}(v)]$$

By Observation 1(a), we have $deg_{G^{++}}(v) = 2deg_G(v),\ deg_{G^{++}}(e') = 2\rho(e)$. Therefore,

$$M_1^*(G^{++}) = \sum_{uv \in E(G^{++}) \cap E(G)} [\omega(u)2deg_G(u) + \omega(v)2deg_G(v)] + \sum_{uv \in E(G^{++})-E(G)} [\omega(u)2deg_G(u) + \omega(v)\rho(uv)]$$

$$= 2\sum_{uv \in E(G^{++}) \cap E(G)} [\omega(u)deg_G(u) + \omega(v)deg_G(v)] + 2\sum_{uv \in E(G^{++})-E(G)} [\omega(u)deg_G(u) + \omega(v)\rho(uv)]$$

It is true that $0 \leq \omega \leq 1$ and $0 \leq \rho \leq 1$. Therefore,

$$M_1^*(G^{++}) \leq 2M_1^*(G) + 2[\sum_{u \in V} deg_G(u)] + 2\sum_{uv \in E(G^{++})-E(G)} \quad (1).$$

Since $\sum_{u \in V} deg_G(u) = \sum_{u \in V} \sum_{u \neq v} \rho(uv) \leq 2m$ and $|E(G^{++})| = 3m$. Therefore, $|E(G^{++}) - E(G)| = 3m - m = 2m$. Hence,

$$M_1^*(G^{++}) \leq 2M_1^*(G) + 2(2m) + 2(2m)$$

$$\leq 2M_1^*(G) + 8m.$$

Theorem 3 *Let* $G = (V, \omega, \rho)$ *be a fuzzy graph and* G^{+-} *is the generalized fuzzy transformation graph of* G. *Thn*

$$M_1^*(G^{+-}) \leq nm^2 + m(n-2)^2$$

Proof. Let $G = (V, \omega, \rho)$ be a fuzzy graph with $\{\omega(v_1), \omega(v_2), \omega(v_3), \cdots, \omega(v_n)\}$ and $\{\rho(e_1), \rho(e_2), \rho(e_3), \cdots, \rho(e_m)\}$ be the membership values of vertices and edges of G respectively. Then $V(G^{+-}) = V(G) \cup E(G)$ by definition of G^{+-}. Therefore, the modified first Zagreb index of generalized transformation graph G^{+-} is given by:

$$M_1^*(G^{+-}) = \sum_{uv\in E(G^{+-})} [\omega(u)deg_{G^{+-}}(u) + \omega(v)deg_{G^{+-}}(v)]$$

$$= \sum_{uv\in E(G^{+-})\cap E(G)} [\omega(u)deg_{G^{+-}}(u) + \omega(v)deg_{G^{+-}}(v)] + \sum_{uv\in E(G^{+-})-E(G)} [\omega(u)deg_{G^{+-}}(u) + \omega(v)deg_{G^{+-}}(v)]$$

By Observation 1(b), we have $deg_{G^{+-}}(v) = \sum_{e\in E} \rho(e)$, $deg_{G^{+-}}(e') = (n-2)\rho(e)$. Therefore,

$$M_1^*(G^{+-}) = \sum_{uv\in E(G^{\pm})\cap E(G)} [\omega(u)\sum_{e\in E} \rho(e) + \omega(v)\sum_{e\in E} \rho(e)] + \sum_{uv\in E(G^{+-})-E(G)} [\omega(u)\sum_{e\in E} \rho(e) + \omega(v)(n-2)\rho(e)]$$

It is true that $0 \le \omega \le 1$ and $0 \le \rho \le 1$. Therefore,

$$M_1^*(G^{+-}) = \sum_{uv\in E(G^{\pm})\cap E(G)} [(1)\sum_{e\in E} (1) + 1\sum_{e\in E} (1)] + \sum_{uv\in E(G^{+-})-E(G)} [1\sum_{e\in E} (1) + (1)(n-2)(1)]$$

Since, $|E(G^{+-})| = m(n-1)$. Therefore, $|E(G^{+-}) - E(G)| = m(n-1) - m = m(n-2)$. Hence,

$$M_1^*(G^{+-}) \le m(2m) + m(n-2)(m + (n-2))$$

$$\le nm^2 + m(n-2)^2.$$

Theorem 4 *Let* $G = (V, \omega, \rho)$ *be a fuzzy graph and* G^{-+} *is the generalized fuzzy transformation graph of* G. *Thn*

$$M_1^*(G^{-+}) \le \frac{n(n-1)^2}{2} + 2m(n-1) + n(n-1).$$

Proof. Let $G = (V, \omega, \rho)$ be a fuzzy graph with $\{\omega(v_1), \omega(v_2), \omega(v_3), \cdots, \omega(v_n)\}$ and $\{\rho(e_1), \rho(e_2), \rho(e_3), \cdots, \rho(e_m)\}$ be the membership values of vertices and edges of G respectively. Then $V(G^{-+}) = V(G) \cup E(G)$ by definition of G^{-+}. Therefore, the modified first Zagreb index of generalized transformation graph G^{-+} is given by:

$$M_1^*(G^{-+}) = \sum_{uv\in E(G^{-+})} [\omega(u)deg_{G^{-+}}(u) + \omega(v)deg_{G^{-+}}(v)]$$

$$= \sum_{uv\in E(G^{-+})\cap E(G)} [\omega(u)deg_{G^{-+}}(u) + \omega(v)deg_{G^{-+}}(v)] + \sum_{uv\in E(G^{-+})-E(G)} [\omega(u)deg_{G^{-+}}(u) + \omega(v)deg_{G^{-+}}(v)]$$

By Observation 1(c), we have $deg_{G^{-+}}(v) = deg_G(v) + \sum_{uv\notin E} [\omega(u) \wedge \omega(v)]$, $deg_{G^{+-}}(e') = 2\rho(e)$. Therefore,

$$M_1^*(G^{-+}) = \sum_{uv\in E(G^{\mp})\cap E(G)} [\omega(u)[deg_G(u) + \sum_{uv\notin E} [\omega(u)\wedge\omega(v)]] + \omega(v)[deg_G(v) + \sum_{uv\notin E} [\omega(u)\wedge\omega(v)]]]$$

$$+\sum_{uv\in E(G^{-+})-E(G)} [\omega(u)[deg_G(u) + \sum_{uv\notin E} [\omega(u)\wedge\omega(v)]] + \omega(v)2\rho(e)]$$

$$M_1^*(G^{-+}) = \sum_{uv\in E(G)} [\omega(u)deg_G(u) + \omega(v)deg_G(v)]$$

$$+\sum_{uv\in E(G)} [\omega(u)[\sum_{uv\notin E} [\omega(u)\wedge\omega(v)]] + \omega(v)[\sum_{uv\notin E} [\omega(u)\wedge \omega(v)]]]$$

$$+\sum_{u\in V(G)} [\omega(u)deg_G(u)] + \sum_{uv\in E(G^{++})-E(G)} [\omega(u)[\sum_{uv\notin E} [\omega(u)\wedge \omega(v)]]]$$

$$+2\sum_{uv\in E(G^{++})-E(G)} \rho(uv)$$

$$M_1^*(G^{-+}) \le M_1^*(G) + X + 2m + Y + n(n-1) \tag{4}$$

where

$$X = \sum_{uv \in E(G)} [\omega(u)[\sum_{uv \notin E} [\omega(u) \wedge \omega(v)]] + \omega(v)[\sum_{uv \notin E} [\omega(u) \wedge \omega(v)]]]$$

and

$$Y = \sum_{uv \in E(G^{++})-E(G)} [\omega(u)[\sum_{uv \notin E} [\omega(u) \wedge \omega(v)]]]$$

Now, we solve X and Y separately to get the required result:

$$X = \sum_{uv \in E(G)} [\omega(u)[\sum_{uv \notin E} [\omega(u) \wedge \omega(v)]] + \omega(v)[\sum_{uv \notin E} [\omega(u) \wedge \omega(v)]]]$$

$$\leq \sum_{uv \in E(G)} \omega(u)(n-1-deg_G(u))[\omega(u) \wedge \omega(v)] + \omega(v)(n-1- deg_G(v))[\omega(u) \wedge \omega(v)]$$

Since $0 \leq \omega(u) \wedge \omega(v) \leq 1$, therefore

$$X \leq (n-1)\sum_{uv \notin E} [\omega(u) + \omega(v)] - \sum_{uv \notin E} [\omega(u)deg_G(u) + \omega(v)deg_G(v)]$$

Since $0 \leq \omega(u) \leq 1$, therefore

$$X \leq 2m(n-1) - M_1^*(G). \quad (5)$$

Now consider,

$$Y = \sum_{uv \in E(G^{++})-E(G)} [\omega(u)[\sum_{uv \notin E} [\omega(u) \wedge \omega(v)]]]$$

$$\leq \sum_{uv \in E(G^{++})-E(G)} \omega(u)(n-1-deg_G(u))[\omega(u) \wedge \omega(v)]$$

$$= (n-1)\sum_{uv \in E(G^{++})-E(G)} \omega(u)[\omega(u) \wedge \omega(v)] + \sum_{uv \in E(G^{++})-E(G)} \omega(u)deg_G(u)[\omega(u) \wedge \omega(v)]$$

Since $0 \leq \omega(u) \leq 1$ and $0 \leq \omega(u) \wedge \omega(v) \leq 1$, therefore

$$Y \leq \frac{n(n-1)^2}{2} - 2m. \quad (6)$$

Putting (5) and (6) in (4) we get the required result.

Theorem 5 *Let $G = (V, \omega, \rho)$ be a fuzzy graph and G^{--} is the generalized fuzzy transformation graph of G. Thn*

$$M_1^*(G^{--}) \leq 2m(m+n-1) + \frac{1}{2}[(2n+m-3)[(2m(n-4)) + n(n-1)]] - 4m - 2M_1^*(G)$$

Proof. Let $G = (V, \omega, \rho)$ be a fuzzy graph with $\{\omega(v_1), \omega(v_2), \omega(v_3), \cdots, \omega(v_n)\}$ and $\{\rho(e_1), \rho(e_2), \rho(e_3), \cdots, \rho(e_m)\}$ be the membership values of vertices and edges of G respectively. Then $V(G^{--}) = V(G) \cup E(G)$ by definition of G^{--}. Therefore, the modified first Zagreb index of generalized transformation graph G^{--} is given by:

$$M_1^*(G^{--}) = \sum_{uv \in E(G^{--})} [\omega(u)deg_{G^{--}}(u) + \omega(v)deg_{G^{--}}(v)]$$

$$= \sum_{uv \in E(G^{--}) \cap E(G)} [\omega(u)deg_{G^{--}}(u) + \omega(v)deg_{G^{--}}(v)] + \sum_{uv \in E(G^{--})-E(G)} [\omega(u)deg_{G^{--}}(u) + \omega(v)deg_{G^{--}}(v)]$$

By Observation 1(d), we have $deg_{G^{--}}(v) = \sum_{uv \notin E}[\omega(u) \wedge \omega(v)] + \sum_{v-e} \rho(e)$, $deg_{G^{+-}}(e') = (n-2)\rho(e)$. Therefore,

$$M_1^*(G^{--}) = \sum_{uv \in E(G^{--}) \cap E(G)} [\omega(u)[\sum_{v-e} \rho(e) + \sum_{uv \notin E}[\omega(u) \wedge \omega(v)]] + \omega(v)[\sum_{v-e} \rho(e) + \sum_{uv \notin E}[\omega(u) \wedge \omega(v)]]]$$

$$+ \sum_{uv \in E(G^{--}) - E(G)} [\omega(u)[\sum_{v-e} \rho(e) + \sum_{uv \notin E}[\omega(u) \wedge \omega(v)]] + \omega(v)(n-2)\rho(e)]$$

$$M_1^*(G^{--}) \leq X + Y \quad (7)$$

where

$$X = \sum_{uv \in E(G^{--}) \cap E(G)} [\omega(u)[\sum_{v-e} \rho(e) + \sum_{uv \notin E}[\omega(u) \wedge \omega(v)]] + \omega(v)[\sum_{v-e} \rho(e) + \sum_{uv \notin E}[\omega(u) \wedge \omega(v)]]]$$

and

$$Y = \sum_{uv \in E(G^{--}) - E(G)} [\omega(u)[\sum_{v-e} \rho(e) + \sum_{uv \notin E}[\omega(u) \wedge \omega(v)]] + \omega(v)(n-2)\rho(e)]$$

On solving X and Y bearing in mind the following facts :

- $0 \leq \omega \leq 1$ and $0 \leq \rho \leq 1$
- $0 \leq \omega(u) \wedge \omega(v) \leq 1$
- $|E(G^{--})| = \frac{n(n-1)}{2} + m(n-4)$
- $\sum_{u-e} \rho(e) \leq [m - deg_G(u)]\rho(e)$
- $\sum_{uv \notin E}[\omega(u) \wedge \omega(v)] \leq (n-1-deg_G(u))[\omega(u) \wedge \omega(v)]$

we get $X \leq 2m(m+n-1) - 2M_1^*(G)$ and

$$Y = \frac{1}{2}[(2n+m-3)[2m(n-4) + n(n-1)]] - 4m$$

Put the values of X and Y in (7) to get the result.

4 Conclusion:

The modified first Zagreb index plays an important role in obtaining the bounds for generalized fuzzy transformation graphs. The QSPR-study on this parameter can be done by the researchers to validate its applications in chemistry.

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